DOI: 10.17516/1997-1397-2020-13-1-114-130
УДК 517.55+512.761

Analytic Continuation for Solutions to the System of Trinomial Algebraic Equations

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Received 05.10.2019, received in revised form 09.11.2019, accepted 24.12.2019

Abstract. In the paper, we deal with the problem of getting analytic continuations for the monomial function associated with a solution to the reduced trinomial algebraic system. In particular, we develop the idea of applying the Mellin-Barnes integral representation of the monomial function for solving the extension problem and demonstrate how to achieve the same result following the fact that the solution to the universal trinomial system is polyhomogeneous. As a main result, we construct Puiseux expansions (centred at the origin) representing analytic continuations of the monomial function.

Keywords: algebraic equation, analytic continuation, Puiseux series, discriminant locus, Mellin–Barnes integral.

Citation: I.A.Antipova, E.A. Kleshkova, V.R.Kulikov, Analytic Continuation for Solutions to the System of Trinomial Algebraic Equations, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 114–130.
DOI: 10.17516/1997-1397-2020-13-1-114-130.

Introduction

We consider a system of n trinomial algebraic equations of the form

$$\sum_{\alpha\in A^{(i)}} a_\alpha^{(i)} y^\alpha = 0,\ i = 1,\dots,n, \tag{1}$$

with the unknown $y = (y_1,\dots,y_n) \in (\mathbb{C}\setminus 0)^n$ and variable coefficients $a_\alpha^{(i)}$, where $A^{(i)} \subset \mathbb{Z}^n$ are fixed three-element subsets and $y^\alpha = y_1^{\alpha_1}\cdot\ldots\cdot y_n^{\alpha_n}$ is a monomial. Without loss of generality we assume, that all sets $A^{(i)}$ contain the zero element $\bar{0}$ (this may be achieved by dividing the ith equation in (1) by a monomial with the exponent in $A^{(i)}$, see the system (2) below). We call (1) the *universal trinomial system* since any trinomial algebraic system is a result of the substitution of polynomials in new variables for coefficients $a_\alpha^{(i)}$.

When $n = 1$, the system (1) is a scalar trinomial equation. It has a special place in the centuries-old history of algebraic equations. As early as 1786, Bring proved that every quintic polynomial could be reduced to the trinomial form $y^5 + ay + b$ using the Tschirnhaus transformation. At the turn of the XIX–XX centuries, the dependence of norms of roots on coefficients

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of the trinomial equation with fixed support was actively studied. Although algebraic characterisation of the mentioned dependence was given by Bohl already in 1908, the geometric view on the problem has been formed much later. In the recent study by Theobald and de Wolff [15], a geometrical and topological characterisation for the space of univariate trinomials was provided by reinterpreting the problem in terms of the amoeba theory.

Of particular interest is the *reduced system* of n trinomial equations

$$y^{\omega^{(i)}} + x_i y^{\sigma^{(i)}} - 1 = 0, \ i = 1, \dots, n, \tag{2}$$

with the unknown $y = (y_1, \dots, y_n)$, equation supports $A^{(i)} := \{\omega^{(i)}, \sigma^{(i)}, \overline{0}\} \subset \mathbb{Z}^n_{\geqslant}$ and variable complex coefficients $x = (x_1, \dots, x_n)$. It is assumed that a matrix ω composed of column vectors $\omega^{(1)}, \dots, \omega^{(n)}$ is nondegenerate.

Let $y(x) = (y_1(x), \dots, y_n(x))$ be a multivalued algebraic vector-function of solutions to the system (2). We call a branch of $y(x)$ defined by conditions $y_i(0) = 1, \ i = 1, \dots, n$ the *principal solution* to the system (2). Having determined the principal solution $y(x)$, we consider the following monomial function

$$y^d(x) := y_1^{d_1}(x) \cdot \ldots \cdot y_n^{d_n}(x), \ d = (d_1, \dots, d_n) \in \mathbb{Z}^n_+. \tag{3}$$

Our goal is to obtain Puiseux expansions (centred at the origin) representing analytic continuations of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (2). Puiseux type parameterizations of an algebraic variety via the amoeba of the discriminant locus of the variety canonical projection were studied in [6]. The existence of such parameterizations for plane curves was proved by Puiseux [12]: this fact is known as the Newton-Puiseux theorem which states that one can find local parameterizations of the form $x = t^k$, $y = \varphi(t)$, where φ is a convergent power series. We aim at investigating Puiseux expansions for analytic continuations of (3) which may fail to "recognize" some pieces of the discriminant set. It means that the convergence domain G of a series projects onto the domain $\mathrm{Log}(G)$ containing a certain collection of connected components of the discriminant amoeba complement. An example in Section 1 illustrates how the series converging in the preimage $\mathrm{Log}^{-1}(E_0)$ of the component E_0 of the amoeba complement admits an analytic continuation to the domain G for which $\mathrm{Log}(G)$ covers components E_1, E_2 and an amoeba tentacle separating them, see Fig. 1. This analytic continuation is given by another series expansion.

When $n = 1$, analytic continuations for the Taylor series of the principal solution to the universal algebraic equation (not necessarily a trinomial) were found in [3], where the Mellin-Barnes integral representation for the solution was used as a tool of the analytic continuation. This integral, with indicating the convergence region of it, was wholly studied in [2]. While a power series converges in a polycircular domain, a Mellin-Barnes integral converges in a sectorial domain which is defined only by conditions for arguments $\arg x_i$ of variables x_i. Remark that the intersection of these domains is always nonempty. Consequently, a series expansion of the solution to the equation admits an analytic continuation into the sectorial domain by means of the integral. Of course, we may follow this approach to getting analytic extensions for the monomial (3) in a case when the corresponding Mellin-Barnes integral represents it. Herein we can obtain analytic continuations of the Taylor series in the form of Puiseux series via the multidimensional residues technique.

However, we can get the same series following the fact that the solution $y(a)$ to the system (1) is polyhomogeneous. This means that via some monomial transformation of coefficients the system (1) can be reduced to the form (2) or to another system which, similarly, has only one variable coefficient in each equation. We perceive any reduced system of equations as the general (homogeneous) system (1) written in suitable coordinates. The transition from one reduced system to another enables us to obtain series continuations for monomials of coordinates of solutions to these systems.

The paper is organized as follows. In Section 1 we review the technique of the calculation of multidimensional Mellin-Barnes integrals which is based on the separating cycle principle formulated in [16] (see also [17]). We present an example which illustrates what computational issues can arise in this way of getting analytic extensions. In Section 2 we discuss the procedure of the dehomogenization (reduction) of the system (1) and obtain the Taylor series expansions for the monomials of the principal solutions to all reduced systems associated with the system (1). Theorem 1 gives these series as a result of the application of the logarithmic residue formula [5] and the linearization procedure for each reduced system. The idea of using the logarithmic residue formula for getting the Taylor expansions was developed in [8], where the special instance of the reduced polynomial system with the diagonal matrix ω was considered. In Section 3 we use Taylor expansions derived in Theorem 1 and appropriate monomial transformations to obtain the desired Puiseux series which are supposed to be the analytic continuations of the Taylor series for the monomial $y^d(x)$ of the principal solution to (2) (Theorem 2). Finally, we discuss the example from Section 1 again in terms of the result of Theorem 2.

1. Mellin-Barnes integral as a tool of analytic continuation

Traditionally, the Mellin-Barnes integrals are regarded as the inverse Mellin transform for special meromorphic functions, which are rations of products of a finite number of superpositions of gamma functions with affine functions. Their role in the theory of algebraic equations was revealed first by Mellin in [9], where he wrote down without any proof the integral representation for the solution to the universal algebraic equation later investigated in [2]. In our study we consider such integrals in the extended sense, having in mind the presence of a polynomial factor in the integrand besides gamma-functions.

The Mellin integral transform for monomials of a solution to the reduced polynomial system was studied in [1] and [14]. Following [14], we associate the Mellin-Barnes integral

$$\frac{1}{(2\pi i)^n}\int\limits_{\gamma+i\mathbb{R}^n}\prod_{j=1}^{n}\frac{\Gamma(z_j)\Gamma\Big(\frac{d_j}{\omega_j}-\frac{1}{\omega_j}\langle\sigma_j,z\rangle\Big)}{\Gamma\Big(\frac{d_j}{\omega_j}-\frac{1}{\omega_j}\langle\sigma_j,z\rangle+z_j+1\Big)}Q(z)x^{-z}\,dz \tag{4}$$

with the monomial $y^d(x)$. In (4) x^{-z} denotes the product $x_1^{-z_1}\cdot\ldots\cdot x_n^{-z_n}$, σ_j is the jth row of the matrix σ composed of column vectors $\sigma^{(1)},\ldots,\sigma^{(n)}$, γ belongs to the domain

$$U=\{u\in\mathbb{R}^n_+:\langle\sigma_j,u\rangle<d_j,\ j=1,\ldots,n\},$$

and $Q(z)$ is a polynomial represented by the determinant

$$Q(z)=\frac{1}{\det\omega}\det\left|\left|\delta_i^j(d_j-\langle\sigma_j,z\rangle)+\sigma_j^{(i)}z_i\right|\right|_{i,j=1}^{n},$$

where δ_i^j is the Kronecker symbol. Here it is assumed that ω is a diagonal matrix with elements $\omega_1,\ldots,\omega_n$ on the diagonal.

Remark that the integral (4) can have the empty convergence domain. It follows from [7] that its convergence domain is nonempty if and only if all the diagonal minors of the matrix σ

are positive. In this case, the integral (4) represents the monomial $y^d(x)$ of the principal solution to the system (2), and it can be used as a tool of the constructive analytic continuation of power series.

Let us show how to calculate the integral (4). A method of the calculation is based on the separating cycle principle formulated in [16] and developed in [17]. This principle deals with the calculation of the Grothendieck-type integrals

$$\frac{1}{(2\pi i)^n} \int\limits_{\Delta_g} \frac{h(z)dz}{f_1(z)\dots f_n(z)}, \tag{5}$$

where the integration set Δ_g is the skeleton of the polyhedron Π_g associated with the holomorphic proper mapping $g : (g_1, \dots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$, and the integrand has poles on divisors $D_j = \{z : f_j(z) = 0\}$, $j = 1, \dots, n$. The polyhedron Π_g is the preimage $g^{-1}(G)$ of the domain $G = G_1 \times \dots \times G_n$, where each G_j is a domain on the complex plane with the piecewise smooth boundary. We associate a facet $\sigma_j = \{z : g_j(z) \in \partial G_j,\ g_k(z) \in G_k,\ k \neq j\}$ of the polyhedron Π_g with $j \in \{1, \dots, n\}$.

Definition. *A polyhedron Π_g is said to be compatible with the set of divisors $\{D_j\}$, if for each $j = 1, \dots, n$ the corresponding facet σ_j of the polyhedron Π_g does not intersect the divisor D_j .*

Assume further that the intersection $Z = D_1 \cap \dots \cap D_n$ is discrete. The local residue with respect to the family of divisors $\{D_j\}$ at each point $a \in Z$ (the Grothendieck residue) is defined by the integral (see [16])

$$\operatorname{res}_{f,a}\Omega = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_a(f)} \Omega,$$

where Ω is the integrand in (5), and $\Gamma_a(f)$ is a cycle given in the neighborhood U_a of the point a as follows

$$\Gamma_a(f) = \{z \in U_a : |f_1(z)| = \varepsilon_1, \dots, |f_n(z)| = \varepsilon_n\}, \quad \varepsilon_j << 1.$$

If a is a simple zero of the mapping f, i.e. the Jacobian $J_f = \partial f / \partial z$ is nonzero at the point a, then the local residue is calculated by the formula

$$\operatorname{res}_{f,a}\Omega = \frac{h(a)}{J_f(a)}. \tag{6}$$

Theorem 1 (principle of separating cycles)**.** *If the polyhedron Π_g is bounded and compatible with the family of polar divisors $\{D_j\}$, then the integral (5) equals to the sum of Grothendieck residues in the domain Π_g.*

One can reduce the integral (4) to the canonical form (5) in the following way. We interpret the vertical integration subspace $\gamma + i\mathbb{R}^n$ as the skeleton of some polyhedron. For instance, in the case $n = 1$, it can be the skeleton of only two polyhedra: the right and left halfplanes with the separating line $\gamma + i\mathbb{R}$. For $n > 1$ this subspace may serve as the skeleton of an infinite number of polyhedra. Our objective is to divide all the set of $2n$ families of polar hyperplanes of the integral (4)

$$\begin{aligned} &L_j : z_j = -\nu, \\ &L_{n+j} : \frac{d_j}{\omega_j} - \frac{1}{\omega_j}\langle \sigma_j, z \rangle = -\nu, \quad j = 1, \dots, n, \quad \nu \in \mathbb{Z}_{\geqslant} \end{aligned}$$

into n divisors and construct a polyhedron compatible with this family of divisors. We consider polyhedra of the type

$$\Pi_g = \{z \in \mathbb{C}^n : \mathrm{Re} g_j(z) < r_j,\ j = 1, \dots, n\},$$

where $g_j(z)$ are linear functions with real coefficients. It is clear that $\Pi_g = \pi + i\mathbb{R}^n$ where π is a simplicial n-dimensional cone in the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$. Remark that in the case of an unbounded polyhedron, besides the compatibility condition of the polyhedron and polar divisors, one should require a sufficiently rapid decrease of the integrand Ω in the polyhedron Π_g. For the integral (4) the nonconfluence property provides the decrease of the integrand, see [10] and [17]. We recall that the nonconfluence property for the hypergeometric Mellin-Barnes integral means that sums of coefficients of the variable z_j over all gamma-factors in the numerator and the denominator are equal.

Now, applying the technique discussed above, we construct analytic continuations for the solution to the following system of equations

$$\begin{cases} y_1^4 + x_1 y_1^2 y_2 - 1 = 0, \\ y_2^4 + x_2 y_1 y_2^2 - 1 = 0. \end{cases} \tag{7}$$

For the description of the convergence domains of power series and Mellin-Barnes integrals we introduce the following mappings from $(\mathbb{C} \setminus 0)^n$ into $\mathbb{R}^n$:

$$\mathrm{Log} : (x_1, \dots, x_n) \longrightarrow (\log|x_1|, \dots, \log|x_n|)\,,$$

$$\mathrm{Arg} : (x_1, \dots, x_n) \longrightarrow (\arg x_1, \dots, \arg x_n)\,.$$

The monomial $y_1(x) \cdot y_2(x)$ of the principle solution to the system (7) admits the Taylor series representation

$$\sum_{|k|\geqslant 0} \frac{(-1)^{|k|}}{k!} \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}k_1 + \frac{1}{4}k_2\right)\Gamma\left(\frac{1}{4} + \frac{1}{4}k_1 + \frac{1}{2}k_2\right)}{\Gamma\left(\frac{5}{4} - \frac{1}{2}k_1 + \frac{1}{4}k_2\right)\Gamma\left(\frac{5}{4} + \frac{1}{4}k_1 - \frac{1}{2}k_2\right)} \frac{1}{16}(1 + k_1 + k_2) x_1^{k_1} x_2^{k_2}, \tag{8}$$

which converges in some neighborhood of the origin, see Theorem 1 below. In turn, the Mellin-Barnes integral of the form

$$\frac{1}{(2\pi i)^2} \int\limits_{\gamma + i\mathbb{R}^2} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma\left(\frac{1}{4} - \frac{1}{2}z_1 - \frac{1}{4}z_2\right)\Gamma\left(\frac{1}{4} - \frac{1}{4}z_1 - \frac{1}{2}z_2\right)}{\Gamma\left(\frac{5}{4} + \frac{1}{2}z_1 - \frac{1}{4}z_2\right)\Gamma\left(\frac{5}{4} - \frac{1}{4}z_1 + \frac{1}{2}z_2\right)} \frac{(1 - z_1 - z_2)}{16} x^{-z} dz, \tag{9}$$

where γ is a point in the open quadrangle

$$U = \left\{u \in \mathbb{R}_+^2 : 2u_1 + u_2 < 1,\ u_1 + 2u_2 < 1\right\},$$

represents the monomial $y_1(x) \cdot y_2(x)$ in a sectorial domain $\mathrm{Arg}^{-1}(\Theta)$ determined by

$$\Theta = \left\{(\theta_1, \theta_2) \in \mathbb{R}^2 :\ |\theta_1| < \frac{\pi}{2},\ |\theta_2| < \frac{\pi}{2}, |2\theta_2 - \theta_1| < \frac{3\pi}{4}, |\theta_2 - 2\theta_1| < \frac{3\pi}{4}\right\}, \tag{10}$$

here $\theta_1 = \arg x_1, \theta_2 = \arg x_2$. Fig. 2 shows the domain Θ which is the interior of the convex octagon. The general description of convergence domains of multiple Mellin-Barnes integrals gives Theorem 4.4.25 in the book [13]. Thus, the integral (9) gives the analytic continuation of the series (8) into the sectorial domain $\mathrm{Arg}^{-1}(\Theta)$.

We next calculate the integral (9) using the principle of separating cycles. It admits a representation as a sum of local residues of the integrand

$$\Omega = \frac{\Gamma(z_1)\Gamma(z_2)\Gamma\left(\frac{1}{4}-\frac{1}{2}z_1-\frac{1}{4}z_2\right)\Gamma\left(\frac{1}{4}-\frac{1}{4}z_1-\frac{1}{2}z_2\right)}{\Gamma\left(\frac{5}{4}+\frac{1}{2}z_1-\frac{1}{4}z_2\right)\Gamma\left(\frac{5}{4}-\frac{1}{4}z_1+\frac{1}{2}z_2\right)}\frac{(1-z_1-z_2)}{16}x_1^{-z_1}x_2^{-z_2}dz_1\,dz_2 \tag{11}$$

in some polyhedron, which contains the vertical imagine integration subspace $\gamma + i\mathbb{R}^2$ as the skeleton. Furthermore, the polyhedron and polar divisors of Ω should satisfy the compatibility conditions.

The form Ω has four families of polar complex lines:

$$\begin{aligned}
&L_1 : z_1 = -\nu,\\
&L_2 : z_2 = -\nu,\\
&L_3 : \frac{1}{4} - \frac{1}{4}(2z_1 + z_2) = -\nu,\\
&L_4 : \frac{1}{4} - \frac{1}{4}(z_1 + 2z_2) = -\nu, \ \ \nu \in \mathbb{Z}_{\geq}.
\end{aligned} \tag{12}$$

Figs. 3 and 4 show the intersection of the real subspace with families (12), and also with

$$\begin{aligned}
&L_5 : \frac{5}{4} + \frac{1}{2}z_1 - \frac{1}{4}z_2 = -\nu,\\
&L_6 : \frac{5}{4} - \frac{1}{4}z_1 + \frac{1}{2}z_2 = -\nu,
\end{aligned}$$

which are polar sets of gamma-functions in the denominator of the form (11). The quadrangle U, to which the point γ belongs, is coloured in grey.

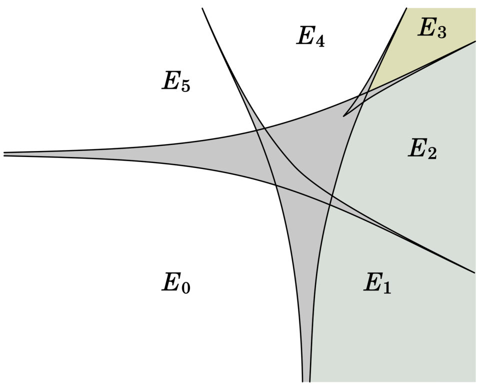

Fig. 1. The discriminant amoeba of the system (7) and its complement components E_ν

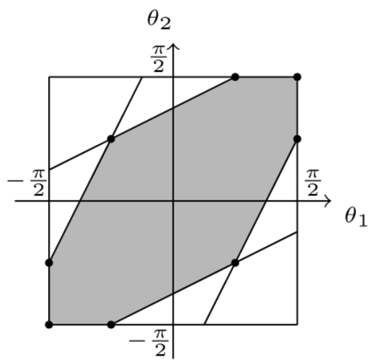

Fig. 2. The domain Θ

First, given the set of all polar lines of the integrand Ω, we form two divisors $D_1 = \{L_2, L_3\}$ and $D_2 = \{L_1, L_4\}$. We next construct a polyhedron $\Pi_1 = \pi_1 + i\mathbb{R}^2$ compatible with this set of divisors, with the skeleton $\gamma + i\mathbb{R}^2$. Fig. 3 shows a two-dimensional cone (sector) $\pi_1 \subset \mathbb{R}^2$ generated by rays which are parallel to the real sections of L_3 and L_4. It forms the polyhedron Π_1. Second, we consider divisors $D_1^{'} = \{L_3, L_4\}$ and $D_2^{'} = \{L_2\}$. A cone π_2 generated by rays which are parallel to the real sections of L_3 and L_2 forms a polyhedron $\Pi_2 = \pi_2 + i\mathbb{R}^2$, compatible with the set of divisors $D_1^{'}$, $D_2^{'}$, see Fig. 4.

We can see in Fig. 3 that families L_5, L_6 as well as L_1, L_2, L_3, L_4 come into the polyhedron Π_1, so in the cone π_1 there are points at which two, three and even four lines intersect. However, the form Ω has nonzero residues only at points $z(k) = (z_1(k), z_2(k))$ with coordinates

$$\begin{aligned} z_1(k) &= \frac{1}{3} + \frac{8}{3}k_1 - \frac{4}{3}k_2, \\ z_2(k) &= \frac{1}{3} - \frac{4}{3}k_1 + \frac{8}{3}k_2, \ k = (k_1, k_2) \in \mathbb{Z}^2_{\geqslant}. \end{aligned} \tag{13}$$

The intersection points (13) of lines L_3, L_4 are indicated in Fig. 3 by a black colour. Hence, the sum of local residues at points $z(k)$ yields the Puiseux series

$$P_1(x) = \frac{1}{x_1^{1/3} x_2^{1/3}} \sum_{k \in \mathbb{Z}^2_{\geqslant}} c_k x_1^{-8/3k_1 + 4/3k_2} x_2^{4/3k_1 - 8/3k_2} \tag{14}$$

with coefficients

$$c_k = \frac{(-1)^{|k|}}{k!} \frac{\Gamma\big(\frac{1}{3} + \frac{8}{3}k_1 - \frac{4}{3}k_2\big)\Gamma\big(\frac{1}{3} - \frac{4}{3}k_1 + \frac{8}{3}k_2\big)}{\Gamma\big(\frac{4}{3} + \frac{5}{3}k_1 - \frac{4}{3}k_2\big)\Gamma\big(\frac{4}{3} - \frac{4}{3}k_1 + \frac{5}{3}k_2\big)} \frac{1}{9}(1 - 4k_1 - 4k_2). \tag{15}$$

Four families of lines L_2, L_3, L_4 and L_5 come into the polyhedron Π_2, see Fig. 2. However, the form Ω has nonzero residues only at points $z(k) = (z_1(k), z_2(k))$ with coordinates

$$\begin{aligned} z_1(k) &= \frac{1}{2} + 2k_1 + \frac{1}{2}k_2, \\ z_2(k) &= -k_2, \ k = (k_1, k_2) \in \mathbb{Z}^2_{\geqslant}. \end{aligned} \tag{16}$$

Points (16) are black in Fig. 4, where lines L_2, L_3 intersect. The sum of residues at $z(k)$ yields the Puiseux series

$$P_2(x) = \frac{1}{x_1^{1/2}} \sum_{k \in \mathbb{Z}^2_{\geqslant}} c_k x_1^{-2k_1 - 1/2k_2} x_2^{k_2} \tag{17}$$

with coefficients

$$c_k = \frac{(-1)^{|k|}}{k!} \frac{\Gamma\big(\frac{1}{2} + 2k_1 + \frac{1}{2}k_2\big)\Gamma\big(\frac{1}{8} - \frac{1}{2}k_1 + \frac{3}{8}k_2\big)}{\Gamma\big(\frac{3}{2} + k_1 + \frac{1}{2}k_2\big)\Gamma\big(\frac{7}{8} - \frac{1}{2}k_1 - \frac{5}{8}k_2\big)} \frac{1}{16}(1 - 4k_1 + k_2). \tag{18}$$

We remark that arguments of Γ–functions in coefficients of the series (8) and also in (15) and (18) can be real nonpositive numbers, which are poles for the function Γ. So, by a ration of two Γ–functions we mean a meromorphic function with removable singularities at those points. For instance, we mean

$$\frac{\Gamma(-1)}{\Gamma(0)} = \frac{\Gamma(-1)}{-\Gamma(-1)} = -1.$$

So, series (14) and (17) are analytic extensions of the series (8).

We now characterize domains of convergence of Puiseux series obtained above in the logarithmic scale. According to the two-sided Abel lemma for hypergeometric series [10], there exists a relationship between the structure of the convergence domain of this series and its support. Since series (14) and (17) represent branches of the multivalued algebraic function $y_1(x) \cdot y_2(x)$ with singularities on the discriminant set of the system (7), projections of convergence domains of such series on the space of variables $\log|x_1|$, $\log|x_2|$ are unions of several components of the discriminant amoeba complement, see Fig. 1. We recall that the amoeba of the algebraic set

$V \subset \mathbb{C}^n$ is defined to be the image of V under the mapping Log. In this way, the series (14) converges in the domain $G_1 = \mathrm{Log}^{-1}(E_3)$, where E_3 is an amoeba complement component. The projection $\mathrm{Log}(G_2)$ of the convergence domain G_2 of the series (17) covers two components E_1, E_2 and an amoeba tentacle separating them, see Fig. 1.

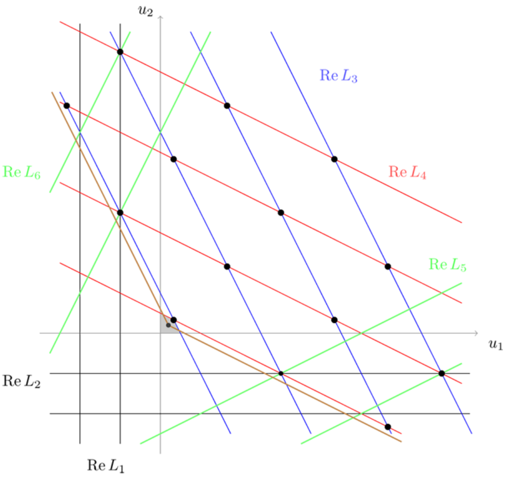

Fig. 3. The real section of polar divisors. The cone π_1

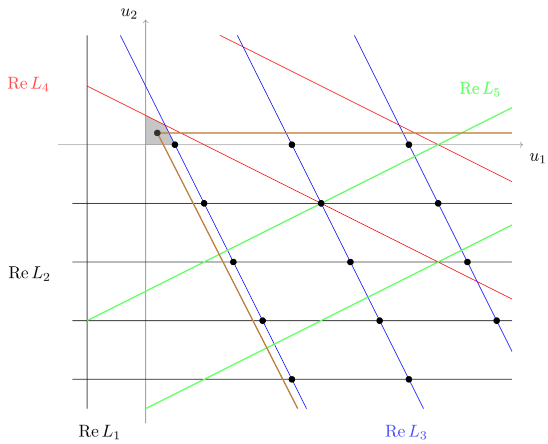

Fig. 4. The real section of polar divisors. The cone π_2

2. Taylor series for monomials of solutions to reduced systems

We consider the system of n trinomials (1) with unknowns $y_1, \dots, y_n$, variable coefficients $a = (\dots, a_\alpha^{(i)}, \dots)$ and the set of supports $A^{(1)}, \dots, A^{(n)}$, the same as the system (2) has.

Let us denote by A the disjunctive union of sets $A^{(i)}$. It consists of $3n$ elements, and we interpret it as the $(n \times 3n)$ – matrix

$$A = \left(A^{(1)}, \dots, A^{(n)}\right) = (\alpha^1, \dots, \alpha^{3n}),$$

with columns α^k which are exponents of monomials of the system (1). We order elements $\alpha \in A$, and, correspondingly, coefficients $a_\alpha^{(i)}$, $\alpha \in A$ of the system (1). The set of coefficients $a = (a_\alpha)$ is a vector space $\mathbb{C}^A \simeq \mathbb{C}^{3n}$.

The system (1) can be reduced by an appropriate change of coefficients in such a way that only one variable coefficient remains in each equation, and the other ones will be constant as in the system (2). Herein, supports $A^{(1)}, \dots, A^{(n)}$ remain the same, and the solution to the system (1) can be restored by the solution to any reduced system. On the whole, the reduction procedure (dehomogenization) of the system is based on the polyhomogeniety property of the solution $y(a) = (y_1(a), \dots, y_n(a))$, which can be expressed as follows:

$$y\Big(\dots \lambda_0^{(i)} \lambda^\alpha a_\alpha^{(i)} \dots\Big) = \Big(\lambda_1^{-1} y_1\big(..a_\alpha^{(i)}..\big), \dots, \lambda_n^{-1} y_n\big(..a_\alpha^{(i)}..\big)\Big), \tag{19}$$

where $\lambda_0 = \Big(\lambda_0^{(1)}, \dots, \lambda_0^{(n)}\Big)$, $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C} \setminus 0)^n$, see [4].

In each set $A^{(i)}$ we fix a pair of elements $\mu^{(i)}$, $\nu^{(i)}$ and form the $n \times n$-matrix

$$\varkappa := \left(\mu^{(1)} - \nu^{(1)}, \ldots, \mu^{(n)} - \nu^{(n)}\right) \tag{20}$$

with columns $\mu^{(i)} - \nu^{(i)}$. The matrix $\varkappa$ is assumed to be nondegenerate. Each fixed set of n pairs $\mu^{(i)}$, $\nu^{(i)}$ corresponds to the reduced system of trinomials

$$r^{(i)}_{\beta^{(i)}} y^{\beta^{(i)}} + y^{\mu^{(i)}} - y^{\nu^{(i)}} = 0, \;\; i = 1, \ldots, n, \tag{21}$$

with new unknown $y = (y_1, \ldots, y_n)$, variable coefficients $r = \left(r^{(i)}_{\beta^{(i)}}\right) \in \mathbb{C}^n$ and $\beta^{(i)} \in A^{(i)}$. In each set $A^{(i)}$, we can choose an unordered pair $\mu^{(i)}, \nu^{(i)}$ in three ways. Hence, we consider at most 3^n ways of the reduction of the system (1) to the form (21). If $\mu^{(i)} = \omega^{(i)}$, $\nu^{(i)} = \overline{0}$ and $\beta^{(i)} = \sigma^{(i)}$ for all $i \in \{1, \ldots, n\}$, then we get the system (2).

Consider a branch of the solution to the system (21) under condition $y_i(0) = 1$ and call it the principle solution. For the vector $d = (d_1, \ldots, d_n) \in \mathbb{R}^n_+$ we introduce the monomial function $y^d(r) := y_1^{d_1}(r) \cdot \ldots \cdot y_n^{d_n}(r)$ of coordinates of the principal solution to the system (21). Concerning the system (21), we use the following notations: β is the matrix formed of columns $\beta^{(i)}$ and $\overline{\beta}$ is the matrix with columns $\left(\beta^{(i)} - \nu^{(i)}\right)$. Moreover, the symbol $\Gamma(b)$ we will use for the short writing of the product $\prod\limits_{k=1}^{n} \Gamma(b_k)$, where $b = (b_1, \ldots, b_n)$ is a vector. The diagonal matrix with components of the vector b on the main diagonal we denote by diag$[b]$ and the I denotes the vector with unit coordinates.

Theorem 1. *The monomial $y^d(r)$ of the principle solution to the system (21) admits the Taylor series representation with coefficients*

$$c_k = \frac{(-1)^{|k|}}{k!} \frac{\Gamma\left(\varkappa^{-1} d + \varkappa^{-1} \overline{\beta} k\right)}{\Gamma\left(\varkappa^{-1} d + \varkappa^{-1} \overline{\beta} k - k + I\right)} Q(k), \;\; k \in \mathbb{Z}^n_{\geqslant}, \tag{22}$$

where $Q(k)$ is the determinant of the matrix $\left(\mathrm{diag}\left[\varkappa^{-1} d + \varkappa^{-1} \overline{\beta} k\right] - \varkappa^{-1} \overline{\beta}\, \mathrm{diag}\,[k]\right)$, $k! := k_1! \cdot \ldots \cdot k_n!$ and $\mid k \mid := k_1 + \ldots + k_n$.

Proof. Following [4], we carry out the linearization of the system (21). For that we regard (21) as a system of equations in the space $\mathbb{C}^n_r \times \mathbb{C}^n_y$ with coordinates $r = (r^{(i)}_{\alpha^{(i)}})$, $y = (y_1, \ldots, y_n)$, and introduce in $\mathbb{C}^n \times \mathbb{C}^n$ the change of variables $(\xi, W) \to (r, y)$ by setting

$$y = W^{-\varkappa^{-1}}, \;\; r = \xi \odot W^{\varkappa^{-1} \bar{\beta} - E}, \tag{23}$$

where $\xi = (\xi_1, \ldots, \xi_n)$, $W = (W_1, \ldots, W_n)$, $\odot$ denotes the Hadamard (coordinate-wise) product and E is the unit matrix. As a result of this change of variables, the system (21) can be written in the vector form as follows

$$W = \xi + I. \tag{24}$$

Equations of the system (24) are linear, so the change of variables (23) is called the linearization. Coordinates of the solution to the system (21) in new variables $\xi = (\xi_1, \ldots, \xi_n)$, $W = (W_1, \ldots, W_n)$ take the form

$$y_j(r(\xi)) = (W_1, \ldots, W_n)^{-\left(\varkappa^{-1}\right)^{(j)}},$$

where $W_i = 1 + \xi_i$, $\left(\varkappa^{-1}\right)^{(j)}$ is the jth column of the inverse matrix $\varkappa^{-1}$ for the matrix $\varkappa$.

We represent the inversion $\xi(r)$ of the linearization (23) as an implicit mapping given by the following set of equations

$$F(\xi, r) = (F_1(\xi, r), \ldots, F_n(\xi, r)) = \xi \odot W^{\varkappa^{-1}\bar{\beta}-E} - r = 0. \tag{25}$$

Calculate the vector $y(\xi)$ at the value of the mapping $\xi(r)$. To this end, following the idea implemented in [8] for a system of polynomials with a diagonal matrix ω, we apply the logarithmic residue formula, see [5, Th. 20.1, 20.2]. It yields the following integral

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{y^d(\xi)\Delta(\xi)d\xi}{F(\xi, r)},$$

where $\Gamma_\varepsilon = \{\xi \in \mathbb{C}^n : |\xi_j| = \varepsilon, \, j = 1, \ldots, n\}$, $\Delta(\xi)$ is the Jacobian of the mapping (25) with respect to ξ and $F(\xi, r)$ denotes the product $F_1(\xi, r) \cdot \ldots \cdot F_n(\xi, r)$. The radius ε we choose in such a way that the corresponding polycylinder lies outside the zero set of the Jacobian $\Delta(\xi)$.

Lemma 1. *The Jacobian of the mapping $F(\xi, r)$ with respect to ξ is*

$$\Delta(\xi) = W^{(\varkappa^{-1}\bar{\beta})I-2I} \det\left(E + \mathrm{diag}[\xi]\varkappa^{-1}\bar{\beta}\right).$$

Proof. The jth component of the mapping $F(\xi, r)$ has the following form:

$$F_j = F_j(\xi, r) = \xi_j \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta}-E)_k^{(j)}} - r_j.$$

The calculation of the derivative of F_j with respect to ξ_j looks as follows:

$$\begin{aligned}\frac{\partial F_j}{\partial \xi_j} &= \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta}-E)_k^{(j)}} + \xi_j(\varkappa^{-1}\bar{\beta} - E)_j^{(j)} \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta}-E)_k^{(j)}-\delta_k^j} = \\ &= (1 + \xi_j(\varkappa^{-1}\bar{\beta})_j^{(j)}) \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta})_k^{(j)}-2\delta_k^j},\end{aligned}$$

and the derivative with respect to ξ_i, when $i \neq j$, is equal to

$$\frac{\partial F_j}{\partial \xi_i} = \xi_j(\varkappa^{-1}\bar{\beta} - E)_i^{(j)} \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta})_k^{(j)}-\delta_k^j-\delta_k^i},$$

where δ_k^j, δ_k^i denote the Kronecker symbols.

Extracting common factors in the rows and columns of the obtained determinant, we get the assertion of the lemma. □

Remark that at the origin the Jacobi matrix for the mapping $F(\xi, r)$ is the unit matrix. Hence, the Jacobian $\Delta(\xi)$ does not vanish in the neighborhood of the origin and conditions of Theorems 20.1, 20.2 from [5] hold.

The monomial $y^d(r)$ after the change of variables takes the following form:

$$y^d(\xi) = W^{-\varkappa^{-1}d}.$$

Consequently, application of the logarithmic residue formula yields the integral representation:

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d+(\varkappa^{-1}\bar{\beta})I-2I}}{F(\xi, r)} \det\left(E + \mathrm{diag}[\xi]\varkappa^{-1}\bar{\beta}\right) d\xi. \tag{26}$$

Expand the kernel of the integral (26) into a multiple geometric series. To this end, we use the coordinate notations:

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d+(\varkappa^{-1}\bar\beta)I-2I}}{\prod\limits_{j=1}^{n}\left(\xi_j \prod\limits_{k=1}^{n} W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}} - r_j\right)} \det\left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi =$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d+(\varkappa^{-1}\bar\beta)I-2I}}{W^{(\varkappa^{-1}\bar\beta)I-I}\prod\limits_{j=1}^{n}\xi_j\left(1 - \frac{r_j}{\xi_j \prod\limits_{k=1}^{n} W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}}}\right)} \det\left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi =$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d-I}}{\prod\limits_{j=1}^{n}\xi_j\left(1 - \frac{r_j}{\xi_j \prod\limits_{k=1}^{n} W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}}}\right)} \det\left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi.$$

Since there exists such a number δ that for all $\xi \in \Gamma_\varepsilon$ and $\|r\| < \delta$ the inequality

$$\frac{r_j}{\xi_j \prod\limits_{k=1}^{n} W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}}} < 1$$

is valid, the integral (26) admits the following representation:

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d-I} \det\left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)}{\prod\limits_{j=1}^{n}\xi_j} \left(\sum_{k\in\mathbb{Z}^n_\geqslant} \prod_{j=1}^{n} \left(\frac{r_j}{\xi_j W^{(\varkappa^{-1}\bar\beta-E)^{(j)}}}\right)^{k_j}\right) d\xi.$$

Changing the order of summation and integration in the last integral, we get the series

$$y^d(r) = \sum_{k\in\mathbb{Z}^n_\geqslant} \left(\frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}(d+\bar\beta k)+k-I}}{\xi^{k+I}} \det\left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi\right) r^k.$$

The coefficient c_k of the series is determined by the expression in parentheses. It can be calculated by the Cauchy integral formula. As a result, we get:

$$c_k = \frac{1}{k!}\frac{\partial^k}{\partial \xi^k}\left(W^{-\varkappa^{-1}(d+\bar\beta k)+k-I} \det\left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)\right)\Big|_{\xi=0}.$$

We bring the factor $W^{-\varkappa^{-1}(d+\bar\beta k)+k-I}$ into the determinant in such a way that each row of it still to depend on one variable ξ_j. We obtain

$$c_k = \frac{1}{k!}\frac{\partial^k}{\partial \xi^k} \det\left(\operatorname{diag}\left[W^{\operatorname{diag}\left[-\varkappa^{-1}(d+\bar\beta k)+k-I\right]}\right] \times \left(E + \operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)\right)\Big|_{\xi=0}.$$

We next use the multilinearity property of the determinant and the fact that each row depends only on one variable ξ_j. As a result, we have

$$c_k = \frac{1}{k!} \det\left\| \frac{\partial^{k_j}}{\partial \xi_j^{k_j}} W_j^{\left(-\varkappa^{-1}(d+\bar\beta k)\right)_j+k_j-1}\left(\delta_i^j + \xi_j(\varkappa^{-1}\bar\beta)_j^{(i)}\right)\Big|_{\xi_j=0}\right\|_{i,j=1}^{n}.$$

Finally, we perform calculations in the above determinant:

$$\frac{\partial^{k_j}}{\partial \xi_j^{k_j}} W_j^{\left(-\varkappa^{-1}(d+\bar{\beta}k)\right)_j + k_j - 1} \left(\delta_i^j + \xi_j (\varkappa^{-1}\bar{\beta})_j^{(i)}\right)\Bigg|_{\xi_j=0} =$$

$$=(-1)^{k_j}\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j \delta_i^j - k_j(\varkappa^{-1}\bar{\beta})_j^{(i)}\right)\prod_{m=1}^{k_j-1}\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j - k_j + m\right) =$$

$$=(-1)^{k_j}\frac{\Gamma\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j\right)}{\Gamma\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j - k_j + 1\right)}\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j \delta_i^j - k_j(\varkappa^{-1}\bar{\beta})_j^{(i)}\right).$$

Taking out the common factor in each row of the determinant and taking into account the factor $\frac{1}{k!}$, we get the view of the coefficient c_k declared in formula (22). $\square$

Coefficients of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (2) one can find by formula (22) setting $\varkappa = \omega$, $\overline{\beta} = \sigma$. Thus, the series is as follows:

$$y^d(x) = \sum_{k\in\mathbb{Z}^n_{\geqslant}} \frac{(-1)^{|k|}}{k!}\frac{\Gamma(\omega^{-1}d + \omega^{-1}\sigma k)}{\Gamma(\omega^{-1}d+\omega^{-1}\sigma k - k + I)} P(k) x^k, \tag{27}$$

where $P(k) = \det\left(\operatorname{diag}\left[\omega^{-1}d + \omega^{-1}\sigma k\right] - \omega^{-1}\sigma \operatorname{diag}[k]\right)$.

3. Puiseux series

We fix n couples $\mu^{(i)},\ \nu^{(i)} \in A^{(i)}$ of exponents of the system (2) and compose the matrix

$$\varkappa = \left(\varkappa_j^{(i)}\right) = \left(\mu_j^{(i)} - \nu_j^{(i)}\right),$$

assuming that it is nondegenerate. In accordance with the choice of the set of pairs $\mu^{(i)}$, $\nu^{(i)}$, let us devide the set $\{1,\ldots,n\}$ on three disjoint subsets:

$$\begin{aligned} J &= \{j : \nu^{(j)} = \overline{0},\ \mu^{(j)} = \omega^{(j)}\},\\ L &= \{l : \nu^{(l)} = \overline{0},\ \mu^{(l)} = \sigma^{(l)}\},\\ T &= \{t : \nu^{(t)} = \sigma^{(t)},\ \mu^{(t)} = \omega^{(t)}\}. \end{aligned} \tag{28}$$

We introduce two matrices

$$\Phi := \varkappa^{-1}\cdot\sigma,\ \ \Psi := \varkappa^{-1}\cdot\omega,$$

with rows $\varphi_1,\ldots,\varphi_n$ and $\psi_1,\ldots,\psi_n$ respectively. Moreover, we consider truncated rows

$$\varphi_l^J,\ \psi_l^L,\ \psi_l^T,\ l\in L,$$

$$\varphi_t^J,\ \psi_t^L,\ \psi_t^T,\ t\in T,$$

which consist of entries of rows φ_l, ψ_l, $l\in L$ and φ_t, ψ_t, $t\in T$ indexed by elements of sets J, L and T. Respectively, we introduce truncated vectors k^J, k^L, k^T for the vector $k=(k_1,\ldots,k_n)$. The scalar product of vectors we denote as follows $\langle\cdot,\cdot\rangle$.

Theorem 2. *For any collection of n couples $\mu^{(i)}$, $\nu^{(i)} \in A^{(i)}$ with the nondegeneracy condition of the corresponding matrix $\varkappa$ there exists an analytic continuation of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (2) in the form of the Puiseux series*

$$\sum_{k\in\mathbb{Z}^n_{\geqslant}} \tilde{c}_k x^{m(k)},$$

which has the support consisting of points $m(k) = (m_1(k), \ldots, m_n(k))$ with coordinates

$$\begin{aligned}
&m_j(k) = k_j,\ j \in J,\\
&m_l(k) = -\langle \varphi_l^J, k^J\rangle - \langle \psi_l^L, k^L\rangle + \langle \psi_l^T, k^T\rangle - \langle d, \varkappa_l^{-1}\rangle,\ l \in L,\\
&m_t(k) = \langle \varphi_t^J, k^J\rangle + \langle \psi_t^L, k^L\rangle - \langle \psi_t^T, k^T\rangle + \langle d, \varkappa_t^{-1}\rangle,\ t \in T,
\end{aligned}$$

and coefficients $\tilde{c}_k$ expressed in terms of coefficients (22) as follows

$$\tilde{c}_k = e^{i\pi \sum\limits_{t\in T}(k_t + m_t(k))} c_k.$$

Proof. We start the proof with finding the monomial change of variables $r = r(a)$ reducing the system (1) to the form (21). To this end, we get the Smith normal form S_q for the matrix $\varkappa$, multiplying it on the left and right by unimodular matrices C and F as follows:

$$C\varkappa F = S_q, \tag{29}$$

here the S_q is a diagonal matrix with integers $q_1, \ldots, q_n$ on the diagonal, and $q_j \mid q_{j+1},\ 1 \leqslant j \leqslant n-1$, see [11]. It follows from (29) that the inverse matrix $\varkappa^{-1}$ admits the representation

$$\varkappa^{-1} = F S_q^{-1} C. \tag{30}$$

As it was mentioned above, the solution $y(a)$ of the system (1) is polyhomogeneous. We find the polyhomogeneity parameters $\lambda_0^{(i)}$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that

$$\begin{gathered}
\lambda_0^{(i)} \lambda^{\mu^{(i)}} a^{(i)}_{\mu^{(i)}} = 1,\\
\lambda_0^{(i)} \lambda^{\nu^{(i)}} a^{(i)}_{\nu^{(i)}} = -1,
\end{gathered} \tag{31}$$

for $i = 1, \ldots, n$. For that, we solve the following system of equations:

$$\lambda^{\varkappa^{(i)}} = g_i,\ i = 1, \ldots, n, \tag{32}$$

where

$$g_i = -\frac{a^{(i)}_{\nu^{(i)}}}{a^{(i)}_{\mu^{(i)}}}.$$

Using the relation (30), we can write the solution of the system (32) in the matrix form as follows

$$\lambda = g^{\varkappa^{-1}} = g^{FS_q^{-1}C} = \left(\left(g^{f^{(1)}}\right)^{\frac{1}{q_1}}, \ldots, \left(g^{f^{(n)}}\right)^{\frac{1}{q_n}}\right)^C,$$

where the vector g has coordinates g_i, and $f^{(1)}, \ldots, f^{(n)}$ are columns of the matrix F. By choosing for each i all q_i values of the radical $\left(g^{f^{(i)}}\right)^{\frac{1}{q_i}}$, we yield all branchers of the matrix radical $g^{\varkappa^{-1}}$. There are $\mid \det \varkappa \mid = q_1 \cdot \ldots \cdot q_n$ of them.

For each $i \in \{1, \dots, n\}$ we find the parameter $\lambda_0^{(i)}$, using one of relations (31). If $\nu^{(i)} = \overline{0}$, then $\lambda_0^{(i)} = -\dfrac{1}{a_0^{(i)}}$. For $\mu^{(i)} = \omega^{(i)}$ we get

$$\lambda_0^{(i)} = \frac{1}{a_{\omega^{(i)}}^{(i)}} \cdot \left(\left(g^{f^{(1)}}\right)^{\frac{1}{q_1}}, \dots, \left(g^{f^{(n)}}\right)^{\frac{1}{q_n}} \right)^{-C\omega^{(i)}}.$$

If $i \in J$, then the coefficient $r_{\sigma^{(i)}}^{(i)}$ of the system (21) can be expressed in terms of coefficients a of the system (1) in two ways:

$$\begin{aligned} r_{\sigma^{(i)}}^{(i)} &= -\frac{a_{\sigma^{(i)}}^{(i)}}{a_0^{(i)}} \cdot \left(g^{f^{(1)}}\right)^{\frac{\langle c_1, \sigma^{(i)} \rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{\frac{\langle c_n, \sigma^{(i)} \rangle}{q_n}}, \\ r_{\sigma^{(i)}}^{(i)} &= \frac{a_{\sigma^{(i)}}^{(i)}}{a_{\omega^{(i)}}^{(i)}} \cdot \left(g^{f^{(1)}}\right)^{\frac{\langle c_1, \sigma^{(i)} - \omega^{(i)} \rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{\frac{\langle c_n, \sigma^{(i)} - \omega^{(i)} \rangle}{q_n}}. \end{aligned} \tag{33}$$

If $i \in L$, then the coefficient $r_{\omega^{(i)}}^{(i)}$ of the system (21) can be expressed in terms of coefficients a of the system (1) as follows

$$r_{\omega^{(i)}}^{(i)} = -\frac{a_{\omega^{(i)}}^{(i)}}{a_0^{(i)}} \cdot \left(g^{f^{(1)}}\right)^{\frac{\langle c_1, \omega^{(i)} \rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{\frac{\langle c_n, \omega^{(i)} \rangle}{q_n}}. \tag{34}$$

For $i \in T$ the relation is as follows

$$r_{\overline{0}}^{(i)} = \frac{a_{\overline{0}}^{(i)}}{a_{\omega^{(i)}}^{(i)}} \cdot \left(g^{f^{(1)}}\right)^{-\frac{\langle c_1, \omega^{(i)} \rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{-\frac{\langle c_n, \omega^{(i)} \rangle}{q_n}}. \tag{35}$$

In formulae (33)–(35) vectors $c_1, \dots, c_n$ are rows of the matrix C.

In particular, if for all $i \in \{1, \dots, n\}$ we choose $\mu^{(i)} = \omega^{(i)}$, $\nu^{(i)} = \overline{0}$, then $L = \varnothing$, $T = \varnothing$ and $\varkappa = \omega$. The matrix ω is nondegenerate by assumption and the system (21) coincides with the system (2). In this case, we get the change of variables $x = x(a)$. It can be written in two ways:

$$\begin{aligned} x_i &= -\frac{a_{\sigma^{(i)}}^{(i)}}{a_{\overline{0}}^{(i)}} \cdot \left(h^{v^{(1)}}\right)^{\frac{\langle u_1, \sigma^{(i)} \rangle}{p_1}} \cdot \ldots \cdot \left(h^{v^{(n)}}\right)^{\frac{\langle u_n, \sigma^{(i)} \rangle}{p_n}}, \\ x_i &= \frac{a_{\sigma^{(i)}}^{(i)}}{a_{\omega^{(i)}}^{(i)}} \cdot \left(h^{v^{(1)}}\right)^{\frac{\langle u_1, \sigma^{(i)} - \omega^{(i)} \rangle}{p_1}} \cdot \ldots \cdot \left(h^{v^{(n)}}\right)^{\frac{\langle u_n, \sigma^{(i)} - \omega^{(i)} \rangle}{p_n}}. \end{aligned} \tag{36}$$

In formulae (36) the vector h has coordinates $h_i = -\dfrac{a_{\overline{0}}^{(i)}}{a_{\omega^{(i)}}^{(i)}}$, vectors $u_1, \dots, u_n$ are rows of the unimodular matrix U, in turn, vectors $v^{(1)}, \dots, v^{(n)}$ are columns of the unimodular matrix V such that $\omega = US_pV$, where $S_p = \mathrm{diag}[p_1, \dots, p_n]$, $p_j \mid p_{j+1}$, $1 \leqslant j \leqslant n-1$.

Remark that $g_i = h_i$ for $i \in J$. Furthermore, if $i \in L$ then $g_i = -\dfrac{a_{\overline{0}}^{(i)}}{a_{\sigma^{(i)}}^{(i)}}$, and for $i \in T$ we have $g_i = -\dfrac{a_{\sigma^{(i)}}^{(i)}}{a_{\omega^{(i)}}^{(i)}}$. Getting these ratios from (36), we substitute the expressions for g_i into (33)–(35).

As a result, we get coordinates of the monomial transformation $r = r(x)$ for the transition from the system (21) to the system (2):

$$\begin{aligned}
r^{(j)}_{\sigma^{(j)}} &= x_j \prod_{l\in L} x_l^{-\varphi_l^{(j)}} \cdot \prod_{t\in T} (-x_t)^{\varphi_t^{(j)}},\ j \in J,\\
r^{(j)}_{\omega^{(j)}} &= \prod_{l\in L} x_l^{-\psi_l^{(j)}} \cdot \prod_{t\in T} (-x_t)^{\psi_t^{(j)}},\ j \in L,\\
r^{(j)}_{\overline{0}} &= -\prod_{l\in L} x_l^{\psi_l^{(j)}} \cdot \prod_{t\in T} (-x_t)^{-\psi_t^{(j)}},\ j \in T.
\end{aligned} \tag{37}$$

According to the polyhomogeneity property (19), the division of the jth coordinate of the solution to the system (1) on $\lambda_j \neq 0$ is compensated by the multiplication of the coefficient $a^{(i)}_\alpha$ on λ^α. So taking into account (32) we obtain the relationship between monomials $y^d(x)$ and $y^d(r)$ of the following form:

$$y^d(x) = \prod_{j=1}^{n} \frac{g_j^{\langle d,\, \varkappa_j^{-1}\rangle}}{h_j^{\langle d,\, \omega_j^{-1}\rangle}} y^d(r), \tag{38}$$

where $\varkappa_j^{-1}$, ω_j^{-1} are jth rows of matrices $\varkappa^{-1}$ and ω^{-1} correspondingly. Using relations (36), and the fact that $g_j = h_j$ for $j \in J$, we write (38) as follows:

$$y^d(x) = \prod_{l\in L} x_l^{-\langle d,\, \varkappa_l^{-1}\rangle} \prod_{t\in T} \left(e^{i\pi} x_t\right)^{\langle d,\, \varkappa_t^{-1}\rangle} y^d(r). \tag{39}$$

Hence, making the substitution (37) in the expansion (22) and taking into account the relation (39), we conclude, that the support S of the required Puiseux series consists of points $m(k) = (m_1(k), \ldots, m_n(k))$ with coordinates

$$\begin{aligned}
m_j(k) &= k_j,\ j \in J,\\
m_l(k) &= -\langle \varphi_l^J, k^J\rangle - \langle \psi_l^L, k^L\rangle + \langle \psi_l^T, k^T\rangle - \langle d, \varkappa_l^{-1}\rangle,\ l \in L,\\
m_t(k) &= \langle \varphi_t^J, k^J\rangle + \langle \psi_t^L, k^L\rangle - \langle \psi_t^T, k^T\rangle + \langle d, \varkappa_t^{-1}\rangle,\ t \in T.
\end{aligned}$$

The coefficient $\tilde{c}_k$ of the Puiseux series is expressed in terms of the coefficient (22) by the following formula

$$\tilde{c}_k = e^{i\pi \sum\limits_{t\in T} (k_t + m_t(k))} c_k.$$

□

As mentioned in Section 1, by the two-sided Abel lemma for hypergeometric series [10] the cone of the support S of the series defines the logarithmic image $\mathrm{Log}(G)$ of the convergence domain G of the series. This means that the geometry of the domain G is closely related to the structure of the amoeba $\mathcal{A}$ of the discriminant hypersurface ∇ of the system (2). The amoeba $\mathcal{A}$ can be obtained from the amoeba $\mathcal{A}'$ of the discriminant set of the system (21) via an affine transform associated with the change of variables $r = r(x)$. Consequently, the recession cone of the set $\mathrm{Log}(G)$ for the Puiseux series of the monomial $y^d(x)$ is the image of the negative orthant $-\mathbb{R}^n_+$ under an affine transform.

In conclusion, we return to the example from Section 1 to make the following remark. By Theorem 2 we associate the Puiseux series (14) with couples of exponents:

$$(2, 1), (0, 0) \in A^{(1)}, \quad (1, 2), (0, 0) \in A^{(2)},$$

and, accordingly, the series (17) with the set

$$(2, 1), (0, 0) \in A^{(1)}, \quad (0, 4), (0, 0) \in A^{(2)}.$$

The first author was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS"(no. 18-1-7-60-1). The second author was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS"(no. 18-1-7-60-2). The third author was supported by the grant of the Ministry of Education and Science of the Russian Federation (no. 1.2604.2017/PCh).

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DOI: 10.1007/BF02677509

Аналитические продолжения решений систем триномиальных алгебраических уравнений

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Аннотация. Статья посвящена исследованию аналитических продолжений мономиальной функции координат решения приведенной триномиальной алгебраической системы. В частности, показано, как техника интегральных представлений Меллина-Барнса и свойство полиоднородности решения универсальной триномиальной системы применяются для разрешения задачи аналитического продолжения. Таким образом, получены разложения Пюизо (с центром в нуле), представляющие аналитические продолжения ряда Тейлора указанной мономиальной функции.

Ключевые слова: алгебраическое уравнение, аналитическое продолжение, ряд Пюизо, дискриминант, интеграл Меллина-Барнса.