Series of rational moduli components of stable rank two vector bundles on $\mathbb{P}^3$

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Abstract

We study the problem of rationality of an infinite series of components, the so-called Ein components, of the Gieseker-Maruyama moduli space $M(e,n)$ of rank 2 stable vector bundles with the first Chern class $e=0$ or -1 and all possible values of the second Chern class n on the projective space $\mathbb{P}^3$. We show that, in a wide range of cases, the Ein components are rational, and in the remaining cases they are at least stably rational. As a consequence, the union of the spaces $M(e,n)$ over all $n \geq 1$ contains new series of rational components in the case $e=0$, exteding and improving previously known results of V.Vedernikov (1985) on series of rational families of bundles, and a first known infinite series of rational components in the case $e=-1$. Explicit constructions of rationality (stable rationality) of Ein components are given. Our approach is based on the study of a correspondence between generalized null correlation bundles constituting open subsets of Ein components and certain rank 2 reflexive sheaves on $\mathbb{P}^3$. This correspondence is obtained via elementary transformations along surfaces. We apply the technique of Quot-schemes and universal spaces of extensions of sheaves to relate the parameter spaces of these two types of sheaves. In

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the case of rationality, we construct universal families of generalized null correlation bundles over certain open subsets of Ein components showing that these subsets are fine moduli spaces. As a by-product of our construction, for $c_1 = 0$ and n even, they provide, perhaps the first known, examples of fine moduli spaces not satisfying the condition "n is odd", which is a usual sufficient condition for fineness.
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1 Introduction

For $e \in \{-1, 0\}$ and $n \in \mathbb{Z}_+$, let $M(e,n)$ be the Gieseker-Maruyama moduli space of stable rank 2 algebraic vector bundles with Chern classes $c_1 = e,\ c_2 = n$ on the projective space $\mathbb{P}^3$. R. Hartshorne [12] showed that $M(e,n)$ is a quasi-projective scheme, nonempty for arbitrary $n \geq 1$ in the case $e = 0$ and, respectively, for even $n \geq 2$ in the case $e = -1$, and the deformation theory predicts that each irreducible component of $M(e,n)$ has dimension at least $8n - 3 + 2e$.

In this paper we study the problem of rationality of irreducible components of $M(e,n)$. Since 70ies not so much has been known about it. In particular, in the case $e = 0$, it is known (see [12], [10], [5], [7], [24], [25]) that the scheme $M(0,n)$ contains an irreducible component I_n of the expected dimension $8n-3$, and this component is the closure of the open subset of $M(0,n)$ constituted by the so-called mathematical instanton vector bundles. Furthermore, according to the recent result of [26, Theorem 3], $M(0,n)$ contains, besides I_n, at least one more irreducible component for any $n \geq 146$. Next, $M(0,n)$ is irreducible (hence coincides with I_n) and rational for $n = 1, 2$ [12]. The rationality of I_3 and of I_5 was proved in [10] and [18], respectively, and for $n = 4$ and $n \geq 6$ the rationality of I_n is still a challenging open question. Note that $M(0,n)$ is reducible for $n \geq 3$, and the exact number of irreducible components of $M(0,n)$ is nowadays known only up to $n = 5$ [1]. We list these components in Section 9.

In the case $e = -1$, for each $n \geq 1$, the space $M(-1,2n)$ contains at least one irreducible component Y_{2n} of the expected dimension $16n - 5$ [12]. In particular, $M(-1,2) = Y_2$ is a rational variety of the expected dimension 11 by [15]. The space $M(-1,4)$ is also known – it contains, besides the rational component Y_4 of the expected dimension 27, one more rational component of dimension 28. For $n \geq 6$ the exact number of irreducible components of $M(0,n)$ is still unknown (see details in Section 9).

In 1978 W. Barth and K. Hulek [6] found, for each integer $k \geq 1$, a rational $(3k^2 + 10k + 8)$-dimensional family $\tilde{Q}_k$ of vector bundles from $M(0, 2k+1)$, and G. Ellingsrud and S. A. Strømme in [10, (4.6)–(4.7)] showed that the image of $\tilde{Q}_k$ under the modular morphism $\tilde{Q}_k \to M(0, 2k+1)$ is an open subset of an irreducible component Q_k distinct from the instanton component I_{2k+1}. Besides, from the definition of Q_k, it follows that it is (at least) unirational. Later in 1984, V. K. Vedernikov [28] constructed, for $1 \leq l \leq k$, a family $V_1(k,l)$ of bundles from $M(0,n_1)$; for $1 \le 2l \le k$, a family $V_2(k,l)$ of bundles from $M(0,n_2)$; for $1 \le 2l \le k+2$, a family $V_3(k,l)$ of bundles from $M(-1,n_3)$, where n_1,n_2,n_3 are certain polynomials on k,l. In his subsequent paper [29], one more family $V_4(k)$ of bundles from $M(0,(k+1)^2)$ was found for $k \ge 1$. In [28], [29], the constructions of stable rationality of $V_1(k,l)$ and of rationality of $V_2(k,l)$ and $V_4(k)$ were given, respectively (see Remark 8.4 below for details). Besides, the author asserted that these families are open subsets of irreducible components of $M(e,n)$, though the proofs for these statements were not given. A more general series of rank 2 bundles depending on triples of integers a,b,c, appeared in 1984 in the paper of A. Prabhakar Rao [23] (cf. Remark 8.5). Soon after that, in 1988, L. Ein [9] independently studied these bundles (called in his paper the generalized null correlation bundles) and proved that they constitute open subsets of irreducible components of $M(e,n)$ (called below Ein components). Surprisingly, Ein components contain Vedernikov's families $V_1(k,l)$ and $V_4(k)$, respectively, $V_2(k,l)$ and $V_3(k,l)$ as their open subsets in special cases when $e=a=0$, respectively, $a=b$ (see details in Remark 8.4). Moreover, when $e=a=0,\ b=k\ge 1,\ c=k+1$, the closure of Vedernikov's family $V_1(k,1)$ coincides with the component Q_k of Ellingsrud-Strømme, i. e. Q_k is also an Ein component.

The problem of rationality of Ein components is the main subject of this paper. We will prove their rationality in a wide range of parameters a,b,c when $(e,a)\ne(0,0),\ c>2a+b-e$, and their (at least) stable rationality in the remaining cases. In particular, we show that our results cover Vedernikov's results in the case of $e=0,\ a=b>0,\ c>3a$ and improve them in the case of $e=a=0,\ b>0$ (see Remark 8.4). Together with the remaining Vedernikov's results, this gives a complete solution to the problem of rationality or, otherwise, (at least) stable rationality of Ein components for all possible values of e,a,b,c. Before proceeding to precise formulations, we recall briefly the definition of generalized null correlation bundles.

For integers a,b,c with $b\ge a\ge 0,\ c>a+b$, consider the monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c+e) \to \mathcal{H} \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0, \tag{1.1}$$

where

$$\mathcal{H} = \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(-a+e) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \oplus \mathcal{O}_{\mathbb{P}^3}(-b+e), \tag{1.2}$$

such that the cohomology sheaf E of this monad is locally free. According to [23, Prop. 3.1] (see also [9, Prop. 1.2(a)]), such monads exist and their cohomology rank 2 vector bundle E is stable. We call E a *generalized null correlation bundle* and denote by $N_{\rm nc}$ the set of all generalized null correlation bundles for the above integers e,a,b,c. Ein shows in [9] that $N_{\rm nc}$ is a dense Zariski open subset of an irreducible component $N(e,a,b,c)$ of the space $M(e,n)$, where $n=c^2-a^2-b^2-e(c-a-b)$. We therefore call these moduli components $N(e,a,b,c)$ the *Ein components* of $M(e,n)$.

We give now a sketch of the contents of the paper. In Section 2, we begin the study of the Ein component $N(e,a,b,c)$ for any admissible e,a,b,c. We first describe a certain dense open subset N specified by the behaviour of restrictions of generalized null correlation bundles from N onto surfaces S of the linear series $\mathrm{P} = |\mathcal{O}_{\mathbb{P}^3}(c-b)|$. (The precise definition of N is given in (2.31)). Using Quot-schemes, we then conststruct a certain principal PGL-bundle $Y \to N$ together with a family of generalized null correlation bundles over Y, and, respectively, a variety X with a surjection $\theta : X \to N$ which is an open subfibration of some explicitely described projective fibration over N. These data yield a family $\mathbf{E}$ of generalized null correlation bundles over the variety $\mathbf{X} = X \times_N Y$ induced by the aforementioned family. In Section 3, we relate to $\mathbf{E}$ a family $\mathbf{F}$ of rank-2 reflexive sheaves. These sheaves F are obtained from bundles E of the family $\mathbf{E}$ by elementary transformations $E \rightsquigarrow F$ along specially chosen surfaces S of degree $c-b$. This is an analogue of the so-called reduction step procedure of R. Hartshorne (cf. Remark 3.2(i)).

In Section 4, we provide a detailed enough plan of the proof of the main result of the paper — Theorem 8.1 which states that $N(e,a,b,c)$ is at least a stably rational variety for all admissible values of e,a,b,c, and, moreover, if $c > 2a+b-e,\ b > a,\ (e,a) \neq (0,0)$, then $N(e,a,b,c)$ is rational and its open subset N is a fine moduli space. The idea is to construct and then relate two diagrams of varieties and projections:

$$W \xrightarrow{\pi} X \xrightarrow{\theta} N \quad \text{and} \quad V \xrightarrow{\lambda} T \xrightarrow{\mu} \mathrm{R} \xrightarrow{\mathrm{r}} \mathrm{P}. \tag{1.3}$$

In these diagrams all the projections are open subfibrations of some locally trivial projective fibrations (see diagrams (4.2) and (4.13) for details). In particular, V is rational and W is birational to $N \times \mathbb{P}^k$ for certain $k \geq 0$. We then relate the two diagrams in (1.3) by constructing an isomorphism

$$f : W \xrightarrow{\sim} V \tag{1.4}$$

and its inverse morphism $h = f^{-1} : V \xrightarrow{\sim} W$. On the level of sets the maps f and h are given by explicit formulas (4.14) and (4.15). In a sense, these are just the above mentioned elementary transformation $E \rightsquigarrow F$ and its dual $F \rightsquigarrow E$. The isomorphism (1.4) then immediately yields Theorem 8.1: the condition $c > 2a+b-e,\ b > a,\ (e,a) \neq (0,0)$ by the dimension count leads to the isomorphism $\theta \circ \pi : W \xrightarrow{\sim} N$, so that $N \simeq V$ is rational; respectively, it is stably rational otherwise.

Our plan described in Section 4 consists of four steps, which are developed in full detail in the subsequent Sections 5–8. In steps 1–3 which are performed in Sections 5, 6, and 7, we construct the varieties and the projections, respectively, $W \xrightarrow{\pi} X$, T, and $V \xrightarrow{\lambda} T \xrightarrow{\mu} \mathrm{R} \xrightarrow{\mathrm{r}} \mathrm{P}$ involved in (1.3). The key technical result here is Theorem 6.4 stating the existence of the variety T, and the proof of this Theorem is based on the use of one specific property of the $H^0_*(\mathcal{O}_{\mathbb{P}^3})$-module $H^1_*(E)$ of a generalized null correlation bundle E. Besides, we build new families $\underline{\mathbf{E}}$ and $\underline{\mathbf{F}}$ of generalized null correlation bundles and, respectively, reflexive sheaves. The interplay between the two pairs of families $\mathbf{E}$, $\mathbf{F}$ and $\underline{\mathbf{E}}$, $\underline{\mathbf{F}}$ leads to the final step 4 of the proof of Theorem 8.1 which is completed in Section 8. Thus, the union of the spaces $M(e,n)$ over all $n \geq 1$ contains an infinite series of rational components (see Corollary 8.2). As a by-product of Theorem 8.1, we show that, for $c_1 = 0$ and n even, the open subsets N of Ein components $N(e,a,b,c)$ provide, perhaps, the first known examples of fine moduli components of rank 2 stable bundles not satisfying the condition "n is odd" – a usual sufficient condition for fineness (see Remark 8.3). As another application of Theorem 8.1, in Section 9 we give a list of known irreducible components of $M(e,n)$, including Ein components, for small values of n, up to $n = 20$, specify those of Ein components which are rational, respectively, stably rational, for both $e = 0$ and $e = -1$, and give their dimensions.

Conventions and notation.

- Everywhere in this paper we work over the base field $\mathbf{k} = \bar{\mathbf{k}}$ of characteristic 0.
- $\mathbb{P}^3$ is the projective 3-space over $\mathbf{k}$.
- Given a morphism of schemes $f : X \to Y$ and a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^3 \times Y$, set
$$\mathcal{F}_X := (\mathrm{id}_{\mathbb{P}^3} \times f)^* \mathcal{F}.$$
This notation will be systematically used throughout the paper.
- For any coherent sheaf $\mathcal{G}$ on a scheme X, we set $\mathbb{P}(\mathcal{G}) := \mathrm{Proj}(S^{\cdot}_{\mathcal{O}_X} \mathcal{G})$. Also, $\mathcal{O}_Y(1)$ denotes the Grothendieck invertible sheaf on $Y = \mathbb{P}(\mathcal{G})$.
- Given $m, n \in \mathbb{Z}$, $\mathbf{P}$ a projective space an arbitrary dimension, X a scheme, and $\mathcal{A}$ a coherent sheaf on $\mathbf{P} \times \mathbb{P}^3 \times X$, set
$$\mathcal{A}(m,n) := \mathcal{A} \otimes \mathcal{O}_{\mathbf{P}}(n) \boxtimes \mathcal{O}_{\mathbb{P}^3}(m) \boxtimes \mathcal{O}_X, \quad \mathcal{A}(m) := \mathcal{A}(m,0). \tag{1.5}$$
- $M(e,n)$ is the Gieseker-Maruyama moduli space of stable rank 2 algebraic vector bundles on $\mathbb{P}^3$, with Chern classes $c_1 = e \in \{-1, 0\},\ c_2 = n \in \mathbb{Z}_+$ for $e = 0$, respectively, $\in 2\mathbb{Z}_+$ for $e = -1$.
- $N(e,a,b,c)$ is the Ein component of the moduli space $M(e,n),\ b \geq a \geq 0,\ c > a + b,\ n = c^2 - a^2 - b^2 - e(c - a - b)$.
- N_{nc} is the open dense subset of $N(e,a,b,c)$ consisting of generalized null correlation bundles.
- N is the open dense subset of N_{nc} defined in (2.31).
- For a stable rank 2 vector bundle E with $c_1(E) = e,\ c_2(E) = n$ on $\mathbb{P}^3$, we denote by $[E]$ its isomorphism class in $M(e,n)$.

- For a projective $\mathbb{P}^m$-fibration $p : X \to Y$, by its *open subfibration* we mean an open subset U of X, together with the projection $p|_U : U \to Y$.

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2 Ein component $N(e, a, b, c)$ and its dense open subset N

In this Section, for an arbitrary Ein moduli component $N(e, a, b, c)$ we introduce a certain dense open subset N of $N(e, a, b, c)$ which will be the main object of our study. We then construct a family $\mathbf{E}$ of generalized null correlation bundles on $\mathbb{P}^3$ with base $\mathbf{X}$ mapping surjectively onto N under the modular morphism $\mathbf{X} \to N,\ \mathbf{x} \mapsto \left[\mathbf{E}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}\right]$ (see Theorem 2.2). This family $\mathbf{E}$ will be used in subsequent sections.

Given integers e, a, b, c with $e \in \{-1, 0\}$ and $b \geq a \geq 0,\ c > a + b$, consider the Ein component $N(e, a, b, c)$ of $M(e, n),\ n = c^2 - a^2 - b^2 - e(c - a - b)$. As it is known from [9, (2.2.B) and Section 3] (see also [4, Section 5]), $\dim N(e, a, b, c) = h^0(\mathcal{H}(c - e)) - h^0(S^2\mathcal{H}(-e)) - 1$. Substituting here $\mathcal{H}$ from (1.2), we obtain:

$$\begin{aligned} \dim N(e, a, b, c) &= \binom{c + a - e + 3}{3} + \binom{c + b - e + 3}{3} + \binom{c - a + 3}{3} \\ &+ \binom{c - b + 3}{3} - \binom{a + b - e + 3}{3} - \binom{b - a + 3}{3} - \binom{2a - e + 3}{3} \\ &- \binom{2b - e + 3}{3} - 3 - t(e, a, b), \end{aligned} \tag{2.1}$$

where

$$t(0, a, b) = \begin{cases} 4, & \text{if } a = b = 0, \\ 1, & \text{if } 0 = a < b \text{ or } a = b > 0, \\ 0, & \text{otherwise.} \end{cases} \qquad t(-1, a, b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Consider the open dense subset N_{nc} of $N(e, a, b, c)$ consisting of generalized null correlation bundles. From (1.1)–(1.2), we have

$$h^1(E(m)) = h^0(\mathcal{O}_{\mathbb{P}^3}(c + m)) - h^0(\mathcal{O}_{\mathbb{P}^3}(a + m)) - h^0(\mathcal{O}_{\mathbb{P}^3}(b + m)), \quad m \leq -1$$

for any bundle $[E]\in N_{\rm nc}$. In particular,

$$h^1(E(-c))=1, \tag{2.3}$$

$$h^1(E(-b))=\binom{c-b+3}{3}-1, \quad b>0. \tag{2.4}$$

Consider more closely the monad (1.1) with cohomology bundle $[E]\in N_{\rm nc}$:

$$\begin{gathered} 0\to\mathcal{O}_{\mathbb{P}^3}(-c+e)\xrightarrow{\lambda}\mathcal{H}\xrightarrow{\mu}\mathcal{O}_{\mathbb{P}^3}(c)\to 0, \quad \textit{where} \\ \lambda=(f_2,-f_1,f_4,-f_3)^t, \quad \mu=(f_1,f_2,f_3,f_4), \quad \mu\circ\lambda=0, \end{gathered} \tag{2.5}$$

$$\begin{gathered} f_1\in V_1:=H^0(\mathcal{O}_{\mathbb{P}^3}(c-a)), \quad f_2\in V_2:=H^0(\mathcal{O}_{\mathbb{P}^3}(c+a-e)), \\ f_3\in V_3:=H^0(\mathcal{O}_{\mathbb{P}^3}(c-b)), \quad f_4\in V_4:=H^0(\mathcal{O}_{\mathbb{P}^3}(c+b-e)). \end{gathered} \tag{2.6}$$

Moreover, since μ is surjective, it follows that the subset $\cap_{i=1}^4\{f_i(x)=0\}$ of $\mathbb{P}^3$ is empty. In particular, polynomials f_1 and f_3 do not have common factors of positive degree. This implies, in particular, that the surfaces

$$S:=\{f_3(x)=0\} \quad \text{and} \quad S':=\{f_1(x)=0\} \tag{2.7}$$

intersect in a curve

$$C_0:=S\cap S'. \tag{2.8}$$

Note that, for the surface S defined in (2.7), the equality

$$h^0(E(-b)|_S)>0. \tag{2.9}$$

holds. Indeed, the sheaf $K:=\ker(\mu)(-b)|_S$ satisfies the exact triple

$$0\to\mathcal{O}_S(-c-b+e)\to K\to E|_S\to 0.$$

By the definition of S the composition $\mathcal{O}_S\overset{i}{\hookrightarrow}\mathcal{H}\xrightarrow{\mu(-b)|_S}\mathcal{O}_S(c-b)$ is the zero morphism. Hence i factors through a non-zero morphism $\mathcal{O}_S\to K$, i. e. $h^0(K)\neq 0$. Therefore, passing to sections in the above triple and using the vanishing of $h^0(\mathcal{O}_S(-c-b+e))$, we obtain (2.9).

Now consider the space $M=M(e,a,b,c)$ of monads (1.1):

$$M:=\{(f_1,f_2,f_3,f_4)\in\Pi_{i=1}^4V_i \mid (2.5) \text{ is true and } \cap_{i=1}^4\{f_i(x)=0\}=\emptyset\}. \tag{2.10}$$

There is a well-defined modular morphism

$$\rho:\ M\twoheadrightarrow N_{\rm nc}, \quad (f_1,f_2,f_3,f_4)\mapsto[\ker(\mu)/\mathrm{im}(\lambda)].$$

Clearly, M is an open subset of the affine space $\Pi_{i=1}^4V_i$, hence it is irreducible. Consider its dense open subset

$$\begin{gathered} M_s=\{(f_1,\dots,f_4)\in M \mid \text{surface } S=\{f_3(x)=0\} \text{ in (2.7)} \\ \text{and curve } C_0=\{f_1(x)=f_3(x)=0\} \text{ in (2.8) are smooth}\}. \end{gathered} \tag{2.11}$$

Since $N_{\rm nc}$ is irreducible, there exists a dense open subset N_s of $N_{\rm nc}$ contained in $\rho(M_s)$:

$$\begin{array}{ccc} & \rho(M_s) & \\ \nearrow & & \searrow \\ N_s & \xrightarrow{\text{dense open}} & N_{\rm nc} \end{array} \tag{2.12}$$

Remark 2.1. The choice of the subset N_s satisfying (2.12) is not unique. From now on we, therefore, assume that, for each collection of admissible values of e, a, b, c, N_s is a maximal (with respect to inclusion) such subset.

Next, there exists a big enough positive integer m such that all bundles from $N_{\rm nc}$ are m-regular in the sense of Mumford-Castelnuovo [17, Section 4.3]. Let $\mathcal{P} \in \mathbb{Q}[x]$ be the Hilbert polynomial $\mathcal{P}(k) = \chi(E(k))$, $[E] \in N_{\rm nc}$, and let $\boldsymbol{\mathcal{B}} := \mathbf{k}^{N_m} \otimes \mathcal{O}_{\mathbb{P}^3}(-m)$, where $N_m := \mathcal{P}(m)$. Consider the Quot-scheme $Q := \mathrm{Quot}_{\mathbb{P}^3}(\boldsymbol{\mathcal{B}}, \mathcal{P})$, together with the universal quotient morphism $\boldsymbol{\mathcal{B}} \boxtimes \mathcal{O}_Q \twoheadrightarrow \mathbb{E}$. Then, the scheme

$$\mathcal{Y} = \Big\{ y \in Q \mid \ \big[\mathbb{E}|_{\mathbb{P}^3 \times \{y\}}\big] \in N_s \Big\}$$

is an open subscheme of Q, together with a family

$$\mathbb{E}_{\mathcal{Y}} = \mathbb{E}|_{\mathbb{P}^3 \times \mathcal{Y}}$$

of generalized null correlation bundles over $\mathcal{Y}$. Since all bundles from N_s are stable, then, according to the GIT-construction [17, Section 4.3] of N_s, the modular morphism

$$\varphi : \ \mathcal{Y} \to N_s = \mathcal{Y}//G, \quad y \mapsto \big[\mathbb{E}_{\mathcal{Y}}|_{\mathbb{P}^3 \times \{y\}}\big], \qquad G = PGL(N_m), \tag{2.13}$$

is a geometric G-quotient and a principal G-bundle.

Since by Serre duality, for any $[E] \in N_s$ one has $h^2(E(c-e-4)) = h^1(E(-c))$, $h^2(E(b-e-4)) = h^1(E(-b))$, using (2.3), (2.4), and the base change we obtain that the sheaves

$$L = R^2 p_{2*} \mathbb{E}_{\mathcal{Y}}(c-e-4), \quad L' = R^2 p_{2*} \mathbb{E}_{\mathcal{Y}}(b-e-4), \tag{2.14}$$

where $p_2 : \mathbb{P}^3 \times \mathcal{Y} \to \mathcal{Y}$ is the projection, are locally free $\mathcal{O}_{\mathcal{Y}}$-sheaves of ranks

$$\mathrm{rk}\, L = 1, \quad \mathbf{r} := \mathrm{rk}\, L' = \binom{c-b+3}{3} - 1. \tag{2.15}$$

Consider the linear series

$$\mathbf{P} := |\mathcal{O}_{\mathbb{P}^3}(c-b)| \tag{2.16}$$

and its dense open subset

$$\mathrm{P} := \{S \in \mathbf{P} \mid S \text{ is a smooth surface}\}. \tag{2.17}$$

Let

$$\Gamma = \big\{(S,x) \in \mathbf{P} \times \mathbb{P}^3 \mid x \in S\big\} \tag{2.18}$$

be the universal family of surfaces of degree $c-b$ in $\mathbb{P}^3$. There is an exact triple on $\mathbf{P} \times \mathbb{P}^3 \times \mathcal{Y}$:

$$0 \to \mathcal{O}_{\mathbf{P}}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(b-c) \boxtimes \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathbf{P}} \boxtimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\Gamma\times\mathcal{Y}} \to 0. \tag{2.19}$$

Tensoring it with the sheaf $\mathbb{E}_{\mathcal{Y}}(c-e-4) \boxtimes \mathcal{O}_{\mathbf{P}}$ and applying to the resulting exact triple the functor $R^i \operatorname{pr}_{13*}$, where $\operatorname{pr}_{13} : \mathbf{P} \times \mathbb{P}^3 \times \mathcal{Y} \to \mathbf{P} \times \mathcal{Y}$ is a projection, in view of the base change and the equalities $h^3\big(E(b-e-4)\big) = 0$ we obtain an exact triple

$$\mathcal{O}_{\mathbf{P}}(-1) \boxtimes L' \xrightarrow{\psi} \mathcal{O}_{\mathbf{P}} \boxtimes L \to R^2 \operatorname{pr}_{13*} (\mathcal{O}_{\mathbf{P}} \boxtimes \mathbb{E}_{\mathcal{Y}}(c-e-4)|_{\Gamma\times\mathcal{Y}}) \to 0. \tag{2.20}$$

Now take an arbitrary point $y \in \mathcal{Y}$ and denote

$$[E_y] := \varphi(y).$$

Resricting the triple (2.20) onto $\mathbf{P} \times \{y\}$ and using (2.15) and the base change, we obtain an exact triple

$$\mathbf{r}\mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\psi\otimes\mathbf{k}(y)} \mathcal{O}_{\mathbf{P}} \to \operatorname{coker}\big(\psi \otimes \mathbf{k}(y)\big) \to 0, \tag{2.21}$$

where by the base change we have for any surface $S \in \mathbf{P}$:

$$\begin{aligned} \operatorname{coker}\big(\psi \otimes \mathbf{k}(y)|_{\{(S,y)\}}\big) &= R^2 \operatorname{pr}_{23*} (\mathbb{E}(c-e-4) \boxtimes \mathcal{O}_{\mathbf{P}}|_{\Gamma\times Y})\,|_{\{(S,y)\}} \\ &= H^2\big(E_y(c-e-4)|_S\big). \end{aligned} \tag{2.22}$$

From the triple (2.21), it follows that

$$h^2\big(E_y(c-e-4)|_S\big) \le 1. \tag{2.23}$$

On the other hand, the Grothendieck-Serre duality for a locally free $\mathcal{O}_S$-sheaf $E_y|_S$ yields

$$h^2\big(E_y(c-e-4)|_S\big) = h^0\big(E_y(-b)|_S\big). \tag{2.24}$$

Next, the triple (2.21) shows that

$$\mathbf{P}(y) := \operatorname{Supp}(\operatorname{coker}(\psi \otimes \mathbf{k}(y))) \tag{2.25}$$

is a linear subspace of codimension at most $\mathbf{r} = \dim \mathbf{P}$ in $\mathbf{P}$. Hence this subspace $\mathbf{P}(y)$ is always nonempty, (2.22)–(2.24) give the following explicit description of $\mathbf{P}(y)$:

$$\mathbf{P}(y) = \{S \in \mathbf{P} |\ h^0(E_y(-b)|_S) = 1\}. \tag{2.26}$$

Set $\tau(y) = \dim \mathbf{P}(y)$ and let

$$\tau := \min_{y\in\mathcal{Y}} \tau(y), \quad Y := \{y \in \mathcal{Y}|\ \tau(y) = \tau\}. \tag{2.27}$$

Since $\mathcal{Y}$ is irreducible, the semicontinuity yields that Y is a dense open subset of $\mathcal{Y}$. Moreover, from (2.13) it follows that there exists a dense open subset N of N_s, hence also of $N_{\rm nc}$ and of $N(e,a,b,c)$:

$$N \xhookrightarrow{\text{dense open}} N_s \tag{2.28}$$

defined by the fact that

$$Y = \varphi^{-1}\big(N\big) = \mathcal{Y} \times_{N_s} N \quad \text{and} \quad \varphi : Y \to N \text{ is a principal } G\text{-bundle}. \tag{2.29}$$

The set N is explicitly described as follows. For any point $[E] \in N_{\rm nc}$, consider the exact triple

$$0 \to E(b-c) \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \to E \boxtimes \mathcal{O}_{\mathbf{P}} \to E \boxtimes \mathcal{O}_{\mathbf{P}}|_{\Gamma} \to 0$$

and apply to it the functor $R^i\,{\rm pr}_{2*}$, where ${\rm pr}_2 : \mathbb{P}^3 \times \mathbf{P} \to \mathbf{P}$ is the projection. Then, similar to (2.21), we obtain an exact triple

$$\mathbf{r}\mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\psi_E} \mathcal{O}_{\mathbf{P}} \to \operatorname{coker}\psi_E \to 0.$$

Similar to the above, set $\mathbf{P}\big([E]\big) := {\rm Supp}(\operatorname{coker}\psi_E)$, $\tau_E := \dim \mathbf{P}([E])$. Then, as in (2.26)–(2.27), we have

$$\mathbf{P}\big([E]\big) = \{S \in \mathbf{P}|\ h^0(E(-b)|_S) = 1\}, \qquad \min_{[E]\in N_s} \tau_E = \tau, \tag{2.30}$$

and

$$N = \{[E] \in N_s \mid \tau_E = \tau\}. \tag{2.31}$$

Denote

$$\mathrm{P}([E]) = \mathrm{P} \cap \mathbf{P}\big([E]\big), \tag{2.32}$$

where P was defined in (2.17). From (2.9), (2.30), and the definition of N_s, it follows that $\mathrm{P}([E])$ is a nonempty, hence dense open subset of $\mathbf{P}\big([E]\big)$:

$$\mathrm{P}([E]) \xhookrightarrow{\text{dense open}} \mathbf{P}\big([E]\big)\ , \qquad [E] \in N_s. \tag{2.33}$$

Now consider the subscheme $\boldsymbol{\mathcal{X}}$ of $\mathbf{P} \times \mathcal{Y}$ together with the projection $\boldsymbol{\theta}$: $\boldsymbol{\mathcal{X}} \to \mathcal{Y}$, defined as

$$\boldsymbol{\mathcal{X}} := \{\mathbf{x} = (S,y) \in \mathbf{P} \times \mathcal{Y} \mid S \in \mathbf{P}(y)\}, \quad \boldsymbol{\theta} : \boldsymbol{\mathcal{X}} \to \mathcal{Y},\ (S,y) \mapsto y. \tag{2.34}$$

Remark that, as ${\rm rk}\, L = 1$ by (2.15), the triple (2.20) twisted by $\mathcal{O}_{\mathbf{P}}(1) \boxtimes L^\vee$ can be rewritten as

$$\mathcal{O}_{\mathbf{P}} \boxtimes (L' \otimes L^\vee) \xrightarrow{\psi} \mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathcal{Y}} \to \mathbb{B} \to 0, \tag{2.35}$$

where

$$\mathbb{B} := \mathcal{O}_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathcal{Y}}|_{\boldsymbol{\mathcal{X}}}$$

is a line bundle on $\boldsymbol{\mathcal{X}}$. In view of (2.26) and (2.34) the fibre of $\mathbb{B}$ over an arbitrary point $\mathbf{x} = (S, y) \in \boldsymbol{\mathcal{X}}$ has the description

$$\mathbb{B} \otimes \mathbf{k}(\mathbf{x}) = H^0(E_y(-b)|_S). \tag{2.36}$$

Applying to (2.35) the functor p_{2*}, where $p_2 : \mathbf{P} \times \mathcal{Y} \to \mathcal{Y}$ is the projection, we obtain an exact triple

$$L' \otimes L^\vee \xrightarrow{f} S^{c-b}\mathcal{V} \otimes \mathcal{O}_{\mathcal{Y}} \to \mathbb{U} \to 0, \quad \mathbb{U} = p_{2*}\mathbb{B}, \tag{2.37}$$

where $\mathcal{V} = H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$, $f = p_{2*}\psi$ and

$$\boldsymbol{\mathcal{X}} = \mathbb{P}(\mathbb{U}). \tag{2.38}$$

In addition, $\mathbb{B} = \mathcal{O}_{\mathbb{P}(\mathbb{U})}(1)$, and there is the canonical epimorphism

$$p_2^*\mathbb{U} \twoheadrightarrow \mathbb{B}. \tag{2.39}$$

Remark that, since $\mathbb{E}$ has a natural $GL(N_m)$-linearization as a sheaf over Q, the sheaf $L' \otimes L^\vee$ has an induced $GL(N_m)$-linearization, and the sheaf $S^{c-b}V \otimes \mathcal{O}_{\mathcal{Y}}$ also has a (trivial) $GL(N_m)$-linearization. Hence by (2.37) the sheaf $\mathbb{U}$ also inherits $GL(N_m)$-linearization. It follows that $\boldsymbol{\mathcal{X}}$ inherits G-action such that $\boldsymbol{\theta} : \boldsymbol{\mathcal{X}} \to \mathcal{Y}$ is a G-equivariant morphism. Hence the geometric quotient

$$\mathcal{X} := \boldsymbol{\mathcal{X}}//G$$

is well-defined, and the canonical projection

$$\Phi : \ \boldsymbol{\mathcal{X}} \to \mathcal{X}$$

is a principal G-bundle.

Furthermore, comparing (2.26) with (2.30), we see that, for any $[E] \in N_{\mathrm{nc}}$ and any $y \in \varphi^{-1}\big([E]\big)$ the fibre $\boldsymbol{\theta}^{-1}(y) = \mathbf{P}(y)$ as a subspace of $\mathbf{P}$ coincides with a subspace $\mathbf{P}\big([E]\big)$ of $\mathbf{P}$, and hence depends only on $[E]$. This implies that: (i) $\boldsymbol{\theta}$ is a G-equivariant morphism and therefore induces a morphism of categorical quotients $\theta_s : \mathcal{X} \to N_s$; (ii) a fibre $\theta_s^{-1}\big([E]\big)$ is a subspace $\mathbf{P}\big([E]\big)$ of $\mathbf{P}$. Thus $\theta_s : \mathcal{X} \to N_s$ is a $\mathbb{P}^\tau$-subfibration of the trivial fibration $\mathrm{P} \times N_s \to N_s$. Hence it is locally trivial.

Next, since $f = p_{2*}\psi$, we can rewrite (2.25) as

$$\boldsymbol{\theta}^{-1}(y) = \mathbf{P}(y) = P(\mathrm{coker}(\psi \otimes \mathbf{k}(y))), \quad y \in \mathcal{Y}.$$

Set

$$X := \theta_s^{-1}(N) \cap P \times N, \qquad \theta := \theta_s|_X : X \to N. \tag{2.40}$$

By definition, $\theta : X \to N$ is a morphism with a fibre $\theta^{-1}([E])$ over an arbitrary point $[E] \in N$ being an open dense subset $\mathrm{P}([E])$ of subspace $\mathbf{P}([E]) \simeq \mathbb{P}^\tau$ of P (see (2.33)). Hence $\theta : X \to N$ is an open subfibration of the locally trivial $\mathbb{P}^\tau$-fibration $\theta_s : \mathcal{X} \to N_s$. Hence θ is also locally trivial. Furthermore, since N is irreducible, it follows that X is also irreducible.

We now arrive at the following result.

Theorem 2.2. *(i) Let X be defined in* (2.40)*. There is an open subfibration $\theta : X \to N$ of a locally trivial $\mathbb{P}^\tau$-fibration, and a fibre $\mathrm{P}([E]) = \theta^{-1}([E])$ over an arbitrary point $[E] \in N$ is given by* (2.32)*. In other words, the set of closed points of the scheme X is described as*

$$X = \{(S, [E]) \in \mathrm{P} \times N \mid h^0(E(-b)|_S) = 1\}. \tag{2.41}$$

In particular,

$$\dim X = \dim N(e, a, b, c) + \tau, \tag{2.42}$$

where $\dim N(e, a, b, c)$ is given by formula (2.1).
(ii) Set $Y = \mathcal{Y} \times_{N_s} N$. There are cartesian diagrams

$$\begin{array}{ccc} \boldsymbol{\mathcal{X}} & \xrightarrow{\Phi} & \mathcal{X} \\ {\scriptstyle\boldsymbol{\theta}}\downarrow & & \downarrow{\scriptstyle\theta_s} \\ \mathcal{Y} & \xrightarrow{\varphi} & N_s, \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \xrightarrow{\Phi} & X \\ {\scriptstyle\boldsymbol{\theta}}\downarrow & & \downarrow{\scriptstyle\theta} \\ Y & \xrightarrow{\varphi} & N, \end{array} \tag{2.43}$$

in which horizontal maps are principal G-bundles. Here the second diagram is obtained from the first via the commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\text{dense open}} & X \\ {\scriptstyle\theta_s}\downarrow & & \downarrow{\scriptstyle\theta} \\ N_s & \xleftarrow{\text{dense open}} & N. \end{array} \tag{2.44}$$

Furthermore, vertical maps in the second diagram are open subfabrations of locally trivial $\mathbb{P}^\tau$-fibrations.
(iii) The composition $\mathbf{X} \overset{\text{open}}{\hookrightarrow} \boldsymbol{\mathcal{X}} \overset{\boldsymbol{\theta}}{\to} \mathcal{Y} \hookrightarrow Q$ induces a family

$$\mathbf{E} = \mathbb{E}_{\mathbf{X}} \tag{2.45}$$

of generalized null correlation bundles on $\mathbb{P}^3$ with base $\mathbf{X}$, where $\mathbb{E}$ is the universal quotient sheaf on $\mathbb{P}^3 \times Q$.

3 Family $\mathbf{E}$ of generalized null correlation bundles and related family of reflexive sheaves $\mathbf{F}$ on $\mathbb{P}^3$

In the first part of this section, we study more closely generalized null correlation bundles E of the family $\mathbf{E}$ introduced in Theorem 2.2(iii). In the second part, we associate to $\mathbf{E}$ a family $\mathbf{F}$ of reflexive rank 2 sheaves on $\mathbb{P}^3$. These two families will play the main role in subsequent constructions.

Consider an arbitrary sheaf $[E] \in N$. By definition (see (1.1)–(1.2)), the sheaf E is the cohomology sheaf of the monad (2.5) with the data (2.6). From the definition of N (see (2.11)–(2.12) and (2.27)–(2.29)), it follows that the monad (2.5) can be chosen in such a way that the related surface S and the curve C_0 defined by (2.7) and (2.8) are both smooth (hence irreducible). In particular, C_0 is a smooth irreducible complete intersection curve with the conormal sheaf $N^{\vee}_{C_0/\mathbb{P}^3} \simeq \mathcal{O}_{C_0}(a-c) \oplus \mathcal{O}_{C_0}(b-c)$. Besides, (2.5)–(2.8) yield:

$$\mathcal{O}_S(C_0) \simeq \mathcal{O}_{\mathbb{P}^3}(S')|_S \simeq \mathcal{O}_S(c-a). \tag{3.1}$$

Furthermore, by [23, Example 3.3], there is a well defined quotient sheaf $\mathcal{O}_{C_0}(a+b-e)$ of $N^{\vee}_{C_0/\mathbb{P}^3}$,

$$N^{\vee}_{C_0/\mathbb{P}^3} = \mathcal{O}_{C_0}(a-c) \oplus \mathcal{O}_{C_0}(b-c) \twoheadrightarrow \mathcal{O}_{C_0}(a+b-e), \tag{3.2}$$

which determines a double scheme structure $\overline{C}_0$ on C_0 with the following properties:

(i) the curve $\overline{C}_0$ is a locally complete intersection curve satisfying the exact triple

$$0 \to \mathcal{O}_{C_0}(a+b-e) \to \mathcal{O}_{\overline{C}_0} \to \mathcal{O}_{C_0} \to 0; \tag{3.3}$$

(ii) $\overline{C}_0$ is the zero-scheme of some section of the sheaf $E(c-a-b)$:

$$\overline{C}_0 = (s)_0, \quad 0 \neq s \in H^0(E(c-a-b)). \tag{3.4}$$

Remark that (3.4) implies an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+b-c) \xrightarrow{s} E \xrightarrow{\alpha} \mathcal{I}_{\overline{C}_0}(c-a-b+e) \to 0. \tag{3.5}$$

Note first that, since $c-a-e > 0$, it follows that, in (3.2), the quotient sheaf $\mathcal{O}_{C_0}(a+b-e)$ does not coincide with the direct summand $\mathcal{O}_{C_0}(b-c)$ of the conormal sheaf $N^{\vee}_{C_0/\mathbb{P}^3}$, so that the curve $\overline{C}_0$ defined in (3.2)–(3.4) is not a subscheme of the surface S. Therefore, the sheaf $\kappa = \ker(\mathcal{O}_{C_0'} \twoheadrightarrow \mathcal{O}_{C_0})$, where the scheme C_0' is defined as the scheme-theoretic intersection $C_0' = \overline{C}_0 \cap S$, has dimension at most zero:

$$0 \to \kappa \to \mathcal{O}_{C_0'} \to \mathcal{O}_{C_0} \to 0, \qquad \dim \kappa \leq 0. \tag{3.6}$$

(Here, the inequality $\dim\kappa \le 0$ is provided by smoothness and irreducibility of the curve C_0.) This together with (3.1) implies an exact triple

$$0 \to \mathcal{I}_{Z,S}(e-b) \to \mathcal{O}_S(c-a-b+e) \to \mathcal{O}_{C_0'}(c-a-b+e) \to 0 \qquad (3.7)$$

and a relation $\kappa \simeq \mathcal{O}_Z$ for some subscheme Z of S of dimension at most zero:

$$\dim Z \le 0. \qquad (3.8)$$

The exact triples

$$0 \to \mathcal{O}_{\mathbb{P}^3}(e-a) \xrightarrow{\cdot S} \mathcal{O}_{\mathbb{P}^3}(c-a-b+e) \to \mathcal{O}_S(c-a-b+e) \to 0,$$

$$0 \to \mathcal{I}_{\overline{C}_0}(c-a-b+e) \to \mathcal{O}_{\mathbb{P}^3}(c-a-b+e) \to \mathcal{O}_{\overline{C}_0}(c-a-b+e) \to 0$$

together with (3.7) extend to a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{C_0}(e-a) & \to & \mathcal{I}_{\overline{C}_0}(c-a-b+e) & \xrightarrow{\beta} & \mathcal{I}_{Z,S}(e-b) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^3}(e-a) & \xrightarrow{\cdot S} & \mathcal{O}_{\mathbb{P}^3}(c-a-b+e) & \to & \mathcal{O}_S(c-a-b+e) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{C_0}(e-a) & \to & \mathcal{O}_{\overline{C}_0}(c-a-b+e) & \to & \mathcal{O}_{C_0'}(c-a-b+e) & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (3.9)$$

Now, the composition of morphisms $\beta\circ\alpha$, where α is taken from (3.5) and β is defined in this diagram, decomposes as

$$\beta\circ\alpha:\ E \overset{\otimes\mathcal{O}_S}{\twoheadrightarrow} E|_S \overset{\gamma}{\twoheadrightarrow} \mathcal{I}_{Z,S}(e-b) \qquad (3.10)$$

for some epimorphism $\gamma:\ E|_S \overset{\gamma}{\twoheadrightarrow} \mathcal{I}_{Z,S}(e-b)$.

Note that (3.8) implies the equalities $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z,\mathcal{O}_{\mathbb{P}^3}) = 0,\ i=1,2,$ which together with the exact sequence

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z,\mathcal{O}_{\mathbb{P}^3}) \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z,\mathcal{O}_S) \to \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z,\mathcal{O}_{\mathbb{P}^3}(b-c))$$

obtained from the exact triple $0 \to \mathcal{O}_{\mathbb{P}^3}(b-c) \xrightarrow{\cdot S} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$ yield

$$\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{O}_Z,\mathcal{O}_S) = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{O}_Z,\mathcal{O}_S) = 0. \qquad (3.11)$$

Applying the functor $\mathcal{E}xt^{\cdot}_{\mathcal{O}_S}(-,\mathcal{O}_S)$ to the exact triple $0\to\mathcal{I}_{Z,S}\to\mathcal{O}_S\to\mathcal{O}_Z\to 0$ and using (3.11), we obtain

$$\mathcal{I}_{Z,S}^{\vee}\simeq\mathcal{O}_S. \tag{3.12}$$

Dualizing the morphism γ in (3.10) and using (3.12) and the isomorphism $(E|_S)^{\vee}\simeq(E|_S)(-e)$, after twisting it by $\mathcal{O}_S(e-b)$, we obtain the morphism $\mathbf{s}=(\gamma)^{\vee}(e-b):\mathcal{O}_S\to E(-b)|_S$, i. e., a section $0\ne\mathbf{s}\in H^0(E(-b)|_S)$. This section is a subbundle morphism on $S\setminus Z$, hence, in view of (3.8), it extends to the Koszul exact triple

$$0\to\mathcal{O}_S\overset{\mathbf{s}}{\to}E(-b)|_S\xrightarrow{\mathbf{s}^{\vee}\otimes\wedge^2\mathbf{s}}\mathcal{I}_{Z,S}(e-2b)\to 0.$$

This triple shows that

$$\gamma=\mathbf{s}^{\vee}\quad\text{and}\quad Z=(\mathbf{s})_0. \tag{3.13}$$

A standard computation using (3.8) and (3.13) shows that

$$l(Z)=c_2(E(-b)|_S)=(c-a)(c-b)(c+a-e)>0, \tag{3.14}$$

hence (3.8) yields

$$\dim Z=\dim(\mathbf{s})_0=0. \tag{3.15}$$

Besides, the equality

$$h^0(E(-b)|_S)=1 \tag{3.16}$$

follows from (2.23) and (2.24) (or, equivalently, from (2.41), since by assumption $(S,[E])\in X$). Hence, $H^0(E(-b)|_S)$ is spanned by $\mathbf{s}$.

From (3.15)–(3.16), it follows

Theorem 3.1. *For any point $(S,[E])\in X$, the equality $h^0(E(-b)|_S)=1$ holds, and $\dim(s)_0=0$ for any $0\ne s\in H^0(E(-b)|_S)$.*

Consider the incidence variety Γ introduced in (2.18). Using the embedding $\mathbf{X}\hookrightarrow P\times Y$ (cf. Theorem 2.2), set

$$\mathbf{\Gamma}=(\Gamma\times Y)\times_{\mathbf{P}\times Y}\mathbf{X}$$

and let $\rho:\mathbf{\Gamma}\to\mathbf{X}$ be the natural projection. Set

$$\mathbf{L}=L_{\mathbf{X}},$$

where the invertible $\mathcal{O}_{\mathcal{Y}}$-sheaf L was defined in (2.14). Consider the family of generalized null correlation bundles $\mathbf{E}$ defined in Theorem 2.2(iii). The first equality in (2.14) and the base change imply

$$\mathbf{L}=R^2\rho_*(\mathbf{E}|_{\mathbf{\Gamma}}(c-e-4))$$

(here and below we use the convention (1.5) on notation), so that the relative Serre duality for the projection ρ yields

$$\mathbf{L}^{\vee}\simeq\rho_*(\mathbf{E}|_{\mathbf{\Gamma}}(-b)). \tag{3.17}$$

Respectively, for an arbitrary point $\mathbf{x} \in \mathbf{X}$ and a surface $S_{\mathbf{x}} := \mathbf{\Gamma} \times_{\mathbf{X}} \{\mathbf{x}\} \subset \mathbb{P}^3$, we have

$$\mathbf{L}^\vee \otimes_{\mathcal{O}_{\mathbf{X}}} \mathbf{k}(\mathbf{x}) = H^0(\mathbf{E}(-b)|_{S_{\mathbf{x}}}). \tag{3.18}$$

Following our convention on notation, denote $\mathbf{L}_{\mathbf{\Gamma}} = (\mathrm{id}_{\mathbb{P}^3} \times \rho)^* \mathbf{L}$, $\mathbf{E}_{\mathbf{\Gamma}} = (\mathrm{id}_{\mathbb{P}^3} \times \rho)^* \mathbf{E}$. The isomorphism (3.17) induces a section $s_{\mathbf{\Gamma}} \in H^0(\mathbf{E}_{\mathbf{\Gamma}}(-b) \otimes \mathbf{L}_{\mathbf{\Gamma}})$ defined as

$$s_{\mathbf{\Gamma}} : \ \mathcal{O}_{\mathbf{\Gamma}} = \rho^* \rho_* (\mathbf{E}_{\mathbf{\Gamma}}(-b)) \otimes \mathbf{L}_{\mathbf{\Gamma}} \xrightarrow{ev} \mathbf{E}_{\mathbf{\Gamma}}(-b) \otimes \mathbf{L}_{\mathbf{\Gamma}}. \tag{3.19}$$

Let

$$\mathcal{Z} = (s_{\mathbf{\Gamma}})_0$$

be the zero scheme of this section. By the base change for any $\mathbf{x} \in \mathbf{X}$ the scheme

$$Z_{\mathbf{x}} = \mathcal{Z} \cap S_{\mathbf{x}} \tag{3.20}$$

is the zero set of the section $s_{\mathbf{\Gamma}}|_{S_{\mathbf{x}}} \in H^0(\mathbf{E}(-b)|_{S_{\mathbf{x}}})$, hence from Theorem 3.1 and the definition of $\mathbf{X}$ we have $2 = \mathrm{codim}_{S_{\mathbf{x}}} Z_{\mathbf{x}} = \mathrm{codim}_{\mathbf{\Gamma}} \mathcal{Z}$, so that

$$\mathrm{codim}_{\mathbb{P}^3 \times \mathbf{X}} \mathcal{Z} = \mathrm{codim}_{\mathbb{P}^3 \times \{\mathbf{x}\}} Z_{\mathbf{x}} = 3. \tag{3.21}$$

Use (3.19) and the relation

$$\mathbf{E}^\vee \simeq \mathbf{E}(-e), \tag{3.22}$$

and consider the composition $\boldsymbol{\varepsilon} : \ \mathbf{E} \overset{\otimes \mathcal{O}_{\mathbf{\Gamma}}}{\twoheadrightarrow} \mathbf{E}|_{\mathbf{\Gamma}} \overset{s_{\mathbf{\Gamma}}^\vee}{\twoheadrightarrow} \mathcal{I}_{\mathcal{Z},\mathbf{\Gamma}}(e-b) \otimes \mathbf{L}_{\mathbf{\Gamma}}$. Setting

$$\mathbf{F} := \ker \boldsymbol{\varepsilon}, \tag{3.23}$$

we obtain an exact triple

$$0 \to \mathbf{F} \to \mathbf{E} \xrightarrow{\boldsymbol{\varepsilon}} \mathcal{I}_{\mathcal{Z},\mathbf{\Gamma}}(e-b) \otimes \mathbf{L}_{\mathbf{\Gamma}} \to 0. \tag{3.24}$$

Remark 3.2. (i) Take any point $\mathbf{x} \in \mathbf{X}$ and restrict the last triple onto $\mathbb{P}^3 \times \{\mathbf{x}\}$. We will obtain the triple

$$0 \to F \to E \xrightarrow{\varepsilon} \mathcal{I}_{Z,S}(e-b) \to 0, \tag{3.25}$$

where $E = \mathbf{E}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}$ is a generalized null correlation bundle, $S = S_{\mathbf{x}}$ is a smooth surface from the linear series $\mathbf{P}$ defined by the point $\mathbf{x}$ (namely, $(S, [E]) = \Phi(\mathbf{x})$), $Z = Z_{\mathbf{x}}$, $\mathcal{I}_{Z,S}$ is the ideal sheaf of Z in S, and $F = \ker \varepsilon$. This triple is an analogue of the so-called reduction step in the sense of Hartshorne [13, Prop. 9.1], hence F, and therefore also $\mathbf{F}$, is a reflexive sheaf.
(ii) In (3.25), Z is the zero-set of the section $s = \varepsilon^\vee$ of the bundle $E(-b)|_S$. Therefore, a standard computation using (3.14) and the relations

$$c_1(E) = e, \quad c_2(E) = c^2 - a^2 - b^2 - e(a-b-c), \quad \deg S = c - b,$$

shows that

$$
\begin{aligned}
&c_1(F) = e + b - c, \\
&c_2(F) = c^2 - a^2 - bc - e(c - a - b), \\
&c_3(F) = l(Z) = c_2(E(-b)|_S) = (c - a)(c - b)(c + a - e).
\end{aligned} \tag{3.26}
$$

Since the sheaf F is determined uniquely up to an isomorphism by the pair $x = ([E], S) \in X$ as the kernel of an epimorphism ε in (3.25), we will also use the following notation for F:

$$F = F(x) = F(E, S). \tag{3.27}$$

(iii) From (3.24) it follows that the sheaf $\mathbf{F}$ is determined by the sheaf $\mathbf{E}$ uniquely up to an isomorphism. Hence, since $\mathbf{E}$ inherits a $GL(N_m)$-linearization as a quotient sheaf over (an open subset of) the Quot-scheme, the sheaf $\mathbf{F}$ also inherits a $GL(N_m)$-linearization.

Since by construction

$$F = \mathbf{F}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}, \quad \mathbf{x} \in \mathbf{X}, \tag{3.28}$$

and $\det \mathbf{E} \simeq \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}(e)$, it follows from (3.26) that

$$\det \mathbf{F} \simeq \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}(e + b - c). \tag{3.29}$$

As $\mathbf{F}$ is a rank 2 reflexive sheaf on $\mathbb{P}^3 \times \mathbf{X}$ by Remark 3.2(i), (3.29) implies

$$\mathbf{F}^\vee = \mathbf{F}(c - e - b). \tag{3.30}$$

Next, from (2.19) follows the relation $N_{\mathbf{\Gamma}/\mathbb{P}^3 \times \mathbf{X}} \simeq \mathcal{O}_{\mathbf{\Gamma}}(c - b, 1)$, and (3.21) implies

$$\mathcal{E}xt^1(\mathcal{I}_{\mathcal{Z},\mathbf{\Gamma}}(e - b) \otimes \mathbf{L}_{\mathbf{\Gamma}}, \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}) = \mathcal{E}xt^1(\mathbf{L}_{\mathbf{\Gamma}}(e - b), \mathcal{O}_{\mathbb{P}^3 \times \mathbf{X}}) = \mathbf{L}_{\mathbf{\Gamma}}^\vee(c - e, 1).$$

Thus, dualizing the triple (3.24) and using (3.22) and (3.30) we obtain an exact triple

$$0 \to \mathbf{E}(b - c) \to \mathbf{F} \xrightarrow{\boldsymbol{\psi}} \mathbf{L}_{\mathbf{\Gamma}}^\vee(b, 1) \to 0. \tag{3.31}$$

Note that the restriction of (3.31) onto $\mathbb{P}^3 \times \{\mathbf{x}\}$ for any $\mathbf{x} \in \mathbf{X}$ yields an exact triple

$$0 \to E(b - c) \to F \xrightarrow{\psi} \mathcal{O}_S(b) \to 0, \quad E = \mathbf{E}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}, \quad F = \mathbf{F}|_{\mathbb{P}^3 \times \{\mathbf{x}\}}. \tag{3.32}$$

4 Outline of the proof of the main result

In this section we will give the plan for the proof of the main result of the paper — Theorem 8.1. It consists of four steps.

Step 1. This step is described in detail in Section 5. We consider the set

$$\begin{aligned} &W = \{([E], S, C) \mid ([E], S) \in X,\ C = (s)_0 \text{ is a smooth curve,} \\ &\text{where } 0 \neq s \in H^0(F(E,S)(c-a-b))\}. \end{aligned} \tag{4.1}$$

(Remind that here we use the notation $F(E,S)$ introduced in (3.27) for a reflexive sheaf F determined by the point $x = ([E], S) \in X$, see Remark 3.2(i-ii).) It is proved in Corollary 5.3 that this set underlies a variety W with a projection $\pi : W \to X$ which is an open subfibration of a locally trivial $\mathbb{P}^m$-fibration over X, where m is given by (5.13). We thus have a diagram of cartesian squares extending the right diagram (2.43):

$$\begin{array}{rccc} \mathbf{E_W}, \mathbf{F_W} & \mathbf{W} & \xrightarrow{\tilde{\Phi}} & W \ni ([E,]S, C) \\ & \downarrow{\scriptstyle\boldsymbol{\pi}} & & \downarrow{\scriptstyle\pi} \\ \mathbf{E}, \mathbf{F} & \mathbf{X} & \xrightarrow{\Phi} & X \\ & \downarrow{\scriptstyle\boldsymbol{\theta}} & & \downarrow{\scriptstyle\theta} \\ & Y & \xrightarrow{\varphi} & N, \end{array} \tag{4.2}$$

in which horizontal maps are principal G-bundles. Here $\mathbf{E}$ and $\mathbf{F}$ are the families of $\mathcal{O}_{\mathbb{P}^3}$-sheaves with the base $\mathbf{X}$ introduced in (2.45) and (3.23), and $\mathbf{E_W}$ and $\mathbf{F_W}$ are their lifts onto $\mathbb{P}^3 \times \mathbf{W}$.

Step 2. At this step, which is descibed in detail in Section 6, we construct a new family $\underline{\mathbf{F}}$ of reflexive sheaves on $\mathbb{P}^3$, of the type described in Remark 3.2, and with a rational base $\mathbf{T}$. These data $\mathbf{T}$ and $\underline{\mathbf{F}}$ are explicitely described in (4.7) and (4.9) below. We then restrict our consideration to a certain dense open subset T of $\mathbf{T}$ which will be essential for our subsequent arguments.

We start with the linear series $\mathbf{P} = |\mathcal{O}_{\mathbb{P}^3}(c-b)|$ introduced in (2.16) and consider its dense open subset P of smooth surfaces - see (2.17). Set

$$\mathrm{R} := \{(S, C) \in \mathrm{P} \times \mathrm{Hilb}_{\mathbb{P}^3} \mid C \in |\mathcal{O}_S(c-a)| \text{ is a smooth curve}\}, \tag{4.3}$$

together with a natural projection $\mathrm{r} : \mathrm{R} \to \mathrm{P},\ (S, C) \mapsto S$.

Remark 4.1. Since any $S \in \mathrm{P}$ is a smooth (hence irreducible) surface, it follows from the cohomology of the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(b-a) \to \mathcal{O}_{\mathbb{P}^3}(c-a) \to \mathcal{O}_S(c-a) \to 0$$

that
(i) the fibre $\mathrm{r}^{-1}(S)$ is an open dense subset of the linear series $|\mathcal{O}_S(c-a)|$ consisting of smooth curves and

$$\dim \mathrm{r}^{-1}(S) = \binom{c-a+3}{3} - \binom{b-a+3}{3} - 1,$$

and all the curves of this linear series are complete intersections of the form

$$C = S \cap S', \quad S' \in |\mathcal{O}_{\mathbb{P}^3}(c-a)|; \tag{4.4}$$

(ii) the projection $\mathrm{r} : \mathrm{R} \to \mathrm{P}$ is an open subfibration of a locally trivial projective fibration with fibre $|\mathcal{O}_S(c-a)|$ over a point $S \in \mathrm{P}$; hence, since P is rational, R is rational as well; moreover,

$$\begin{aligned} \dim \mathrm{R} &= \dim \mathrm{P} + \dim \mathrm{r}^{-1}(S) \\ &= \binom{c-b+3}{3} + \binom{c-a+3}{3} - \binom{b-a+3}{3} - 2. \end{aligned} \tag{4.5}$$

Take an arbiitrary point $(S, C) \in \mathrm{R}$ and consider the group

$$\mathrm{Ext}^i(x) := \mathrm{Ext}^i(\mathcal{I}_C(c-2a-b+e), \mathcal{O}_{\mathbb{P}^3}). \tag{4.6}$$

In Section 6 we prove that the dimension of this group does not depend on the point x and is given by the formula (6.6). This implies that the set

$$\mathbf{T} = \{t = (x, \xi) \mid x = (S, C) \in \mathrm{R},\ \xi \in P(\mathrm{Ext}^1(x))\} \tag{4.7}$$

is the set of closed points of the variety (denoted below by the same letter $\mathbf{T}$) of dimension given by the formula (6.8), and the projection

$$\boldsymbol{\mu} : \ \mathbf{T} \to \mathrm{R}, \quad (x, \xi) \mapsto x \tag{4.8}$$

is a locally trivial projective fibration. In particular, since R is rational, $\mathbf{T}$ is also rational.

Furthermore, in Theorem 6.3 we state that on $\mathbb{P}^3 \times \mathbf{T}$ there is a sheaf $\underline{\mathbf{F}}$ defined as the universal extension sheaf

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+b-c) \boxtimes \mathcal{O}_{\mathbf{T}}(1) \to \underline{\mathbf{F}} \to \mathcal{I}_{\mathbf{\Sigma}, \mathbb{P}^3 \times \mathbf{T}}(e-a), \to 0, \quad \mathbf{\Sigma} = \Sigma \times_{\mathrm{R}} \mathbf{T}, \tag{4.9}$$

where Σ is the incidence subvariety of $\mathbb{P}^3 \times \mathrm{R}$ defined as

$$\Sigma := \{(x, S, C) \in \mathbb{P}^3 \times \mathrm{R} \mid x \in C\}. \tag{4.10}$$

Here $\underline{\mathbf{F}}$ may be considered as a family of reflexive $\mathcal{O}_{\mathbb{P}^3}$-sheaves with base $\mathbf{T}$ and with Chern classes given by (3.26), see Remark 6.2.

In the last part of Section 6, we prove one technical result about reflexive sheaves of the family $\underline{\mathbf{F}}$ with base $\mathbf{T}$. It shows that, if for $t = (S, C, \xi)$, a sheaf F_t

of the family $\underline{\mathbf{F}}$ has an epimorphism onto an invertible $\mathcal{O}_S$-sheaf $\mathcal{O}_S(b)$, then the kernel of this morphism is a generalized null correlation bundle twisted by $\mathcal{O}_{\mathbb{P}^3}(b-c)$, just as in the exact triple (3.32) in which we put $F = F_t$. It is proved in Theorem 6.4. A principal technical point used in the proof is the following specific property of any generalized null correlation bundle E: it has the cohomology $H^0_*(\mathcal{O}_{\mathbb{P}^3})$-module $H^1_*(E)$ with one generator. This theorem is crucial for further constructions.

Step 3. At this step which is worked out in detail in Section 7, we use the above family of reflexive sheaves $\underline{\mathbf{F}}$ to construct a family $\underline{\mathbf{E}}$ of generalized null correlation bundles with rational base V. For this, we first construct a locally trivial projective bundle $\boldsymbol{\lambda}: \ \mathbf{U} \to \mathbf{T}$ with fibre $\boldsymbol{\lambda}^{-1}(t)$ over an arbitrary point $t = (S, C, \xi) \in T$ equal to the projectivized vector space $\mathrm{Hom}(F_t, \mathcal{O}_S(b))$. The local triviality of this projective fibration is a consequence of Theorem (7.1) which, in particular, states that the dimension of the above space $\mathrm{Hom}(F_t, \mathcal{O}_S(b))$ does not depend on t. As a corollary of Theorems 6.4 and 7.1 we then find dense open subsets T of $\mathbf{T}$ and V of $\mathbf{U}$ such that (i) $\lambda = \boldsymbol{\lambda}|_V : V \to T$ is a surjection and, for $(t, \mathbf{k}\psi) \in V$, and (ii) the morphism $\psi : F_t \to \mathcal{O}_S(b)$ is surjective. More precisely, V is set-theoretically defined as the set of data

$$\begin{aligned} V = \{(S, C, \xi, \mathbf{k}\psi) \mid\ & (S, C, \xi, \mathbf{k}\psi) \text{ satisfies the above conditions (i)-(ii)} \\ & \text{and the open condition } [\ker(\psi)(c-b)] \in N\} \end{aligned} \tag{4.11}$$

(the precice definition of V is given in (7.15)). As a consequence, we obtain a family $\underline{\mathbf{E}}$ of generalized null correlation bundles related to the family $\underline{\mathbf{F}}$ via the exact triple

$$0 \to \underline{\mathbf{E}}(b-c) \to \underline{\mathbf{F}}_V \xrightarrow{\Psi} \mathcal{O}_{\Gamma_V}(b) \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_V(1) \to 0 \tag{4.12}$$

(see Remark 7.2). Here $\Gamma_V \subset \mathbb{P}^3 \times V$ is the graph of the family of surfaces S and $\mathcal{O}_V(1)$ is the restricted onto V Grothendieck sheaf of the above mentioned projective fibration - see (7.11). This triple is the relativized over V version of the exact triple (3.32). As a result of the constructions of Steps 2 and 3, we obtain the following diagram of morphisms:

$$\underline{\mathbf{F}}_V,\ \underline{\mathbf{E}} \qquad \begin{array}{ccccc} V & \xhookrightarrow{\text{open}} & \mathbf{U} & & \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\lambda}} & & \\ T & \xhookrightarrow{\text{open}} & \mathbf{T} & \longleftarrow & \boldsymbol{\Sigma} \\ {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\boldsymbol{\mu}} & & \swarrow & \\ \mathrm{R} & \longleftarrow & \Sigma & & \\ {\scriptstyle\mathrm{r}}\downarrow & & & & \\ \mathrm{P}, & & & & \end{array} \tag{4.13}$$

together with the family of $\mathcal{O}_{\mathbb{P}^3}$-sheaves $\underline{\mathbf{F}}$ with base $\mathbf{T}$ and the induced families of $\mathcal{O}_{\mathbb{P}^3}$-sheaves $\underline{\mathbf{F}}_V$, $\underline{\mathbf{E}}$ with base V. Remind that, in this diagram, varieties R, T, Σ and $\mathbf{\Sigma}$ were defined in (4.3), (4.7), (4.10) and (4.9), respectively.

Step 4. At this final step performed in Section 8, we show that there is an isomorphism $f: W \xrightarrow{\cong} V$. Set-theoretically the map $f: W \to V$ on closed points is given as follows.

For any $([E], S, C) \in W$ consider the exact triple (3.25). Dualizing it, we obtain a) the exact triple (3.32) with $F = F(E, S)$ and an epimorphism $\psi: F \to \mathcal{O}_S(b)$, and b) an extension class $\xi \in P(\mathrm{Ext}^1(\mathcal{I}_C(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3}))$ given by the exact triple (5.3) with $F = F(E, S)$. Then, we define f as

$$f([E], S, C) := (S, C, \xi, \mathbf{k}\psi). \tag{4.14}$$

From the description of V given in Step 3, it follows that the point $f([E], S, C)$ belongs to V.

Respectively, the inverse $h = f^{-1}: V \to W$ of f is set-theoretically described as

$$h(S, C, \xi, \mathbf{k}\psi) := ([E], S, C), \quad \text{where} \quad E = \ker(\psi)(c - b). \tag{4.15}$$

In Theorem 8.1(i), we prove that the map f, respectively, its inverse h is the underlying map of an isomorphism between W and V. The idea is to relate the diagrams (4.2) and (4.13). We first construct a G-invariant morphism $\mathbf{f}_V: \mathbf{W} \to V$ which descends to the desired morphism $f: W \to V$ satisfying the relation $\mathbf{f}_V = f \circ \tilde{\Phi}$ since $\tilde{\Phi}: \mathbf{W} \to W$ is categorical quotient.

Next, we construct a principal $PGL(N_m)$-bundle $\mathbf{\Phi}: \mathbf{V} \to V$ and a morphism $\mathbf{f}: \mathbf{W} \to \mathbf{V}$ making the diagram

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{\mathbf{f}} & \mathbf{V} \\ {\scriptstyle \tilde{\Phi}}\downarrow & & \downarrow{\scriptstyle \mathbf{\Phi}} \\ W & \xrightarrow{f} & V \end{array} \tag{4.16}$$

commutative (see (8.7)-(8.8) and (8.12)-(8.13) for details).

Last, we construct the morphisms $\mathbf{h}: \mathbf{V} \to \mathbf{W}$ and $h: V \to W$ making the diagram

$$\begin{array}{ccc} \mathbf{W} & \xleftarrow{\mathbf{h}} & \mathbf{V} \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \mathbf{\Phi}} \\ W & \xleftarrow{h} & V \end{array}$$

commutative and show that $\mathbf{h}$ and h are inverse, respectively, to $\mathbf{f}$ and f (see (8.18)–(8.24) for details).

Technical aspects of the proof are based on the universal properties of Quot-schemes, Hilbert schemes and projectivized spaces of extensions involved in the constructions of the families $\mathbf{E_W}, \mathbf{F_W}$ in diagram (4.2) and the families $\underline{\mathbf{F}}_V$, $\underline{\mathbf{E}}$ in diagram (4.13).

In Theorem 8.1(ii)–(iii), we obtain the main result of the paper, the stable rationality of the space $N(e,a,b,c)$ and, respectively, its rationality for $(e,a) \neq (0,0)$, $c > 2a+b-e$, and $b > a$, as a quick consequence of the statement (i) of this Theorem.

5 Properties of reflexive sheaves of the family F

In this section, we study more closely reflexive sheaves F of the family $\mathbf{F}$ – see (3.28). Note that an arbitrary sheaf F is obtained from a generalized null correlation bundle $[E] \in N$ by the triple (3.25).

This consideration leads to the following theorem.

Theorem 5.1. *For any $[E] \in N$ the following statements hold.*
(i) There exists a surface $S \in \theta^{-1}\big([E]\big)$ such that the reflexive sheaf F defined by the pair $\big(S,[E]\big)$ as in Remark 3.2 satisfies the conditions

$$h^0(F(c-a-b)) = \begin{cases} 1, & \text{if } (e,a) \neq (0,0), \\ 2, & \text{if } e=a=0,\ b>0, \\ 3, & \text{if } e=a=b=0, \end{cases} \tag{5.1}$$

$$h^1(F(c-a-b)) = 0. \tag{5.2}$$

(ii) For any $0 \neq s \in H^0(F(c-a-b))$ there an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} F(c-a-b) \to \mathcal{I}_C(c-2a-b+e) \to 0, \tag{5.3}$$

where $C = (s)_0$ is a complete intersection curve $C = S \cap S'$, where S' is certain surface of degree $c-a$ in $\mathbb{P}^3$. In addition, $C=(s)_0$ is smooth for a general $s \in H^0(F(c-a-b))$.
(iii) In the case $h^0(F(c-a-b)) \leq 2$, the space $P(H^0(F(c-a-b)))$ is naturally identified with a linear subspace of the linear series $|\mathcal{O}_S(C)| = |\mathcal{O}_S(c-a)|$. In the case $h^0(F(c-a-b)) = 3$, the space $P(H^0(F(c-a-b))^\vee)$ is naturally identified with a linear subspace of the linear series $|\mathcal{O}_{\mathbb{P}^3}(c-b)|$.

Proof. (i) Consider the generalized null correlation bundle $[E] \in N$ and the corresponding monad (1.1) with the cohomology sheaf E. From the description (2.5)–(2.8) of this monad it follows that there is a smooth complete intersection curve C_0 defined in (2.8) having the properties (3.1)–(3.2). Besides, there is a well-defined double scheme structure $\overline{C}_0$ on C_0 satisfying the exact triple (3.5), and another nonreduced scheme structure C_0' on C_0 together with a zero-dimensional subscheme Z of C_0, and these schemes fit in the commutative diagram (3.9). By (3.5), the composition $\beta\circ\alpha\circ s$ is zero, where β is defined in (3.9). Hence, the triple (3.5) and the upper horizontal triple of the diagram (3.9) extend to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^3}(a+b-c) & = & \mathcal{O}_{\mathbb{P}^3}(a+b-c) & & & & \\
 & & \downarrow & & \downarrow{\scriptstyle s} & & & & \\
0 & \to & F & \to & E & \xrightarrow{\beta\circ\alpha} & \mathcal{I}_{Z,S}(e-b) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \to & \mathcal{I}_{C_0}(e-a) & \to & \mathcal{I}_{\overline{C}_0}(c-a-b+e) & \xrightarrow{\beta} & \mathcal{I}_{Z,S}(e-b) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The leftmost vertical triple of this diagram twisted by $\mathcal{O}_{\mathbb{P}^3}(c-b-a)$ coincides with (5.3) for $C=C_0$:

$$0\to\mathcal{O}_{\mathbb{P}^3}\to F(c-a-b)\to\mathcal{I}_{C_0}(c-2a-b+e)\to 0. \tag{5.4}$$

Since C_0 is a complete intersection (2.8), it follows that the sheaf $\mathcal{I}_{C_0}(c-2a-b+e)$ has the following locally free $\mathcal{O}_{\mathbb{P}^3}$-resolution:

$$0\to\mathcal{O}_{\mathbb{P}^3}(e-c-a)\to\mathcal{O}_{\mathbb{P}^3}(e-2a)\oplus\mathcal{O}_{\mathbb{P}^3}(e-a-b)\to\mathcal{I}_{C_0}(c-2a-b+e)\to 0. \tag{5.5}$$

Passing to sections in the triple (5.5) and (5.4), we obtain (5.1) and (5.2).

(ii) Note that, since by (5.1) $h^0(F(c-a-b))\le 3$, it clearly follows that the zero-scheme $C=(s)_0$ of any non-zero section $s\in H^0(F(c-a-b))$ has dimension 1. (Indeed, a standard argument in case $\dim C=2$ shows that there exists a positive integer d and nonzero section $s'\in H^0(F(c-a-b-d))$ with $\dim(s')_0=1$, so that $\mathbf{k}s'\otimes H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is a subspace of dimension ≥ 4 of $H^0(F(c-a-b))$ which is a contradiction.) We thus have to treat three cases corresponding to three possible values of $h^0(F(c-a-b))$.

(ii.1) $h^0(F(c-a-b))=1$. Since by (5.4) $C_0=(s)_0$ for some $0\ne s\in H^0(F(c-a-b))$, it follows that, in (5.3), $C=C_0$ which is a complete intersection of the desired form (2.8).

(ii.2) $h^0(F(c-a-b))=2$. In this case $e=a=0$, $b>0$, and for any $0\ne t\in H^0(F(c-b))$, the triple (5.3) becomes:

$$0\to\mathcal{O}_{\mathbb{P}^3}\xrightarrow{t}F(c-b)\to\mathcal{I}_{C_t}(c-b)\to 0,\quad C_t=(t)_0. \tag{5.6}$$

It follows that $h^0(\mathcal{I}_{C_t}(c-b)) = 1$, i. e. there exists a unique surface $S_t \in |\mathcal{O}_{\mathbb{P}^3}(c-b)|$ containing C_t. By construction, the cokernel Q of the evaluation morphism

$$0 \to H^0(F(c-b)) \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\text{ev}} F(c-b)$$

is a sheaf supported on the divisor S_t for any $0 \neq t \in H^0(F(c-b))$. Thus, from the above uniqueness, we have $S_t = S$, where $S \in |\mathcal{O}_{\mathbb{P}^3}(c-b)|$ is a surface containing the curve C_0.

Besides, passing to cohomology in the triples (5.4) and (5.5) twisted by $\mathcal{O}_{\mathbb{P}^3}(b)$, we obtain that $h^0(F(c)) = 1 + 2h^0(\mathcal{O}_{\mathbb{P}^3}(b))$ for $e = a = 0$. This together with the triple (5.6) twisted by $\mathcal{O}_{\mathbb{P}^3}(b)$ implies that $h^0(\mathcal{I}_{C_t}(c)) = 1 + h^0(\mathcal{O}_{\mathbb{P}^3}(b))$. Therefore in view of the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3}(b) \xrightarrow{\cdot S} \mathcal{I}_{C_t}(c) \to \mathcal{O}_S(-C_t)(c) \to 0, \tag{5.7}$$

where S is any surface of the linear series $|\mathcal{O}_{\mathbb{P}^3}(c-b)|$ containing the curve C_t, we obtain $h^0(\mathcal{O}_S(-C_t)(c)) = 1$. From this equality and the fact that the sheaf $\mathcal{O}_S(-C_t)(c)$ has degree 0 with respect to $\mathcal{O}_S(1)$ we deduce that $\mathcal{O}_S(C_t) = \mathcal{O}_S(c)$. Since $S \in |\mathcal{O}_{\mathbb{P}^3}(c-b)|$, it follows that C_t is a complete intersection of the desired form (2.8).

(ii.3) $h^0(F(c-a-b)) = 3$. In this case, $e = a = b = 0$ and, arguing as in the case (ii.2) above, we obtain for any $0 \neq t \in H^0(F(c))$ that $h^0(\mathcal{I}_{C_t}(c)) = 2$. Besides, for any surface $S \in |\mathcal{O}_{\mathbb{P}^3}(c-b)|$ passing through C_t, there is an exact triple (5.7) with $b = 0$. This together with the last equality implies that $h^0(\mathcal{O}_S(-C_t)(c)) = 1$, and, as above, we obtain that C_t is a complete intersection curve of the form (2.8).

Last, note that $C = (s)_0$ is smooth for a general $s \in H^0(F(c-a-b))$, since C_0 is smooth.

(iii) In the case $h^0(F(c-a-b)) \leq 2$, the assertion directly follows from (ii.1-2). Consider the case $h^0(F(c-a-b)) = 3$. Note that, in this case, $a = b = e = 0$. The exact triples

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to F(c) \to \mathcal{I}_{C_0}(c) \to 0 \quad \text{and} \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to 2\mathcal{O}_{\mathbb{P}^3} \to \mathcal{I}_{C_0}(c) \to 0$$

by push-out yield a resolution for $F(c)$ of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to 3\mathcal{O}_{\mathbb{P}^3} \to F(c) \to 0.$$

This resolution shows that, for any 2-dimensional subspace V of $H^0(F(c))$ the cokernel of the evaluation morphism $0 \to V \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\text{ev}} F(c)$ is isomorphic to the sheaf $\mathcal{O}_{S_t}$ for some surface $S_t \in |\mathcal{O}_{\mathbb{P}^3}(c)|$. These surfaces S_t constitute a 2-dimensional linear subseries parametrized by $P(H^0(F(c-a-b))^\vee)$. $\square$

Let $p : \mathbb{P}^3 \times \mathbf{X} \to \mathbf{X}$ be the projection, and set

$$\boldsymbol{\mathcal{W}} := \mathbb{P}((p_*\mathbf{F}(c-a-b))^\vee) \xrightarrow{\boldsymbol{\pi}} \mathbf{X}. \tag{5.8}$$

Note that, by (5.1), (5.2) and the base change, $p_*\mathbf{F}(c-a-b)$ is a locally free sheaf of rank

$$\mathrm{rk}(p_*\mathbf{F}(c-a-b)) = h^0(F(c-a-b)), \tag{5.9}$$

where $h^0(F(c-a-b))$ is given in (5.1). Hence $\boldsymbol{\pi}: \boldsymbol{\mathcal{W}} \to \mathbf{X}$ is a locally trivial projective bundle, and there is a canonical epimorphism of vector bundles on $\boldsymbol{\mathcal{W}}$

$$\epsilon: \ \boldsymbol{\pi}^*((p_*\mathbf{F}(c-a-b))^\vee) \twoheadrightarrow \mathcal{O}_{\boldsymbol{\mathcal{W}}}(1). \tag{5.10}$$

Consider the G-action on $\mathbf{X}$ turning the projection $\Phi: \mathbf{X} \to X$ into a principal G-bundle (see Theorem 2.2(ii)). From the definition of $\boldsymbol{\mathcal{W}}$ and Remark 3.2(iii), it follows that this action lifts to a G-action on $\boldsymbol{\mathcal{W}}$ such that $\boldsymbol{\pi}$ is a G-invariant morphism. We thus obtain a cartesian diagram of principal G-bundles

$$\begin{array}{ccc} \boldsymbol{\mathcal{W}} & \xrightarrow{\tilde{\Phi}} & \mathcal{W} \\ {\scriptstyle \boldsymbol{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{X} & \xrightarrow{\Phi} & X, \end{array} \tag{5.11}$$

where $\mathcal{W} = \boldsymbol{\mathcal{W}}//G$ is a geometric factor, $\tilde{\Phi}: \boldsymbol{\mathcal{W}} \to \mathcal{W}$ is a canonical projection, and $\pi: \mathcal{W} \to X$ is the induced morphism.

Let $\boldsymbol{\mathcal{W}} \xleftarrow{\tilde{p}} \mathbb{P}^3 \times \boldsymbol{\mathcal{W}} \xrightarrow{\tilde{\boldsymbol{\pi}}} \mathbb{P}^3 \times \boldsymbol{\mathcal{W}}$ be the induced projections. The canonical epimorphism ϵ from (5.10) induces a morphism

$$\begin{aligned} \tilde{\mathbf{s}}: \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\boldsymbol{\mathcal{W}}}(-1) &\xrightarrow{\tilde{p}^*(\epsilon^\vee)} \tilde{p}^*\boldsymbol{\pi}^* p_*\mathbf{F}(c-a-b) \\ &= \tilde{\boldsymbol{\pi}}^* p^* p_*\mathbf{F}(c-a-b) \xrightarrow{\tilde{\boldsymbol{\pi}}^* ev} \mathbf{F}_{\boldsymbol{\mathcal{W}}}(c-a-b). \end{aligned} \tag{5.12}$$

(Note that, here, $\mathbf{F}_{\boldsymbol{\mathcal{W}}} = \tilde{\boldsymbol{\pi}}^*\mathbf{F}$, according to our agreement on notation.)

Theorem 5.2. *(i) The variety* $\mathcal{W}$ *is described as* $\mathcal{W} = \{(x, C) \mid x = (S, [E]) \in X,$ *and* $C = (s)_0$ *for some* $0 \neq s \in H^0(F(c-a-b))$*, where* F *is determined by the pair* $x = (S, [E])$ *via the reduction step* (3.25)$\}$*. In addition, the morphism* $\pi: \mathcal{W} \to X$ *is given by* $(x, C) \mapsto x$*, and* $\pi^{-1}(x) = P(H^0(F(c-a-b)))$*.*

(ii) The vertical maps $\boldsymbol{\pi}: \boldsymbol{\mathcal{W}} \to \mathbf{X}$ *and* $\pi: \mathcal{W} \to X$ *in* (5.11) *are locally trivial* $\mathbb{P}^m$*-fibrations, where*

$$m = m(e, a, b, c) := h^0(F(c-a-b)) - 1, \tag{5.13}$$

and $h^0(F(c-a-b))$ *is given by* (5.1)*. Therefore,* $\dim \mathcal{W} = \dim X + m(e, a, b, c)$*. In particular, if* $(e, a) \neq (0, 0)$*, then there is an isomorphism* $\pi: \mathcal{W} \xrightarrow{\simeq} X$*.*

(iii) There is an exact $\mathcal{O}_{\mathbb{P}^3 \times \boldsymbol{\mathcal{W}}}$*-triple*

$$0 \to \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\boldsymbol{\mathcal{W}}}(-1) \xrightarrow{\tilde{\mathbf{s}}} \mathbf{F}_{\boldsymbol{\mathcal{W}}}(c-a-b) \to \mathcal{I}_{\tilde{\boldsymbol{\mathcal{C}}}, \mathbb{P}^3 \times \boldsymbol{\mathcal{W}}}(c-2a-b+e) \to 0,$$

where $\tilde{\mathbf{s}}$ *is defined in* (5.12) *and* $\tilde{\boldsymbol{\mathcal{C}}} = (\mathbf{s})_0$ *is a codimension 2 subscheme of* $\mathbb{P}^3 \times \boldsymbol{\mathcal{W}}$*.*

Proof. Statement (i) follows from the base change and the definition of $\boldsymbol{\mathcal{W}}$ and $\mathcal{W}$. In (ii), the local triviality of the fibration $\boldsymbol{\pi}$ is clear, and Theorem 5.1(iii) yields the local triviality of the fibration π. The isomorphism (5.16) is a corollary of (5.1). Statement (iii) follows from the definition of the morphism $\mathbf{s}$ in (5.12). $\square$

Now consider the dense open subset W of $\mathcal{W}$ defined in (4.1):

$$W = \{([E], S, C) \in \mathcal{W} \mid C \text{ is smooth}\} \xrightarrow{\text{dense open}} \mathcal{W}, \tag{5.14}$$

and set

$$\mathbf{W} := \boldsymbol{\mathcal{W}} \times_{\mathcal{W}} W \xrightarrow{\boldsymbol{\pi}} \mathbf{X}, \qquad \mathbf{F_W} := \mathbf{F}_{\boldsymbol{\mathcal{W}}}|_{\mathbb{P}^3 \times \mathbf{W}}, \quad \boldsymbol{\mathcal{C}} := \tilde{\boldsymbol{\mathcal{C}}} \times_{\mathcal{W}} W.$$

In view of Theorem 5.1(ii), the morphisms $\mathbf{W} \xrightarrow{\pi} \mathbf{X}$ and $W \xrightarrow{\pi} X$ are surjective. Thus from Theorem 5.2 we obtain

Corollary 5.3. *(i) $\mathbf{W} \xrightarrow{\pi} \mathbf{X}$ and $W \xrightarrow{\pi} X$ are open subfibrations of locally trivial $\mathbb{P}^m$-fibrations, where $m = m(e, a, b, c)$ is defined in (5.13), and*

$$\dim \mathcal{W} = \dim X + m(e, a, b, c). \tag{5.15}$$

In particular, if $(e, a) \neq (0, 0)$, then there is an isomorphism

$$\pi : W \xrightarrow{\simeq} X. \tag{5.16}$$

(ii) There is an exact $\mathcal{O}_{\mathbb{P}^3 \times \mathbf{W}}$-triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{W}}(-1) \xrightarrow{\mathbf{s}} \mathbf{F_W}(c - a - b) \to \mathcal{I}_{\boldsymbol{\mathcal{C}}, \mathbb{P}^3 \times \mathbf{W}}(c - 2a - b + e) \to 0, \tag{5.17}$$

where $\mathbf{s} = \tilde{\mathbf{s}}|_{\mathbb{P}^3 \times \mathbf{W}}$. This triple being restricted onto $\mathbb{P}^3 \times \{\mathbf{w}\}$, for an arbitrary point $\mathbf{w} \in \mathbf{W}$, coincides with the triple (5.3), in which we set $s = \mathbf{s} \otimes \mathbf{k}(\mathbf{w})$, $F = \mathbf{F_W}|_{\mathbb{P}^3 \times \{\mathbf{w}\}}$, and $C = \boldsymbol{\mathcal{C}} \cap \mathbb{P}^3 \times \{\mathbf{w}\}$.

Remark 5.4. *According to Theorem 2.2(ii) and the above Corollary,*

$$\mathbf{W} \xrightarrow{\boldsymbol{\pi}} \mathbf{X} \xrightarrow{\boldsymbol{\theta}} Y \xrightarrow{\varphi} N$$

is a composition of two open subfibrations of projective fibrations and of a principal bundle. Hence, since N is a reduced scheme by [9], it follows that $\mathbf{W}$ is a reduced scheme.
(ii) Applying the functor $\tilde{\boldsymbol{\pi}}^*$ to the epimorphism $\boldsymbol{\psi}$ in (3.31) we obtain an epimorphism $\boldsymbol{\psi}_{\mathbf{W}} : \ \mathbf{F_W} \twoheadrightarrow (\mathbf{L}_{\boldsymbol{\Gamma}}^{\vee}(b, 1))_{\mathbf{W}}$, hence also an epimorphism

$$\boldsymbol{\psi}_{\mathbf{W}} : \ \mathbf{F_W} \twoheadrightarrow (\mathbf{L}_{\boldsymbol{\Gamma}}^{\vee}(b, 1))_{\mathbf{W}}. \tag{5.18}$$

6 A new family $\underline{\mathbf{F}}$ of reflexive sheaves

In this section, we construct a new family $\underline{\mathbf{F}}$ of reflexive sheaves with Chern classes (3.26) and with the same properties as that of the sheaves of the family $\mathbf{F}$ – see Theorem 6.3. As above, we fix the numbers e, a, b, c which determine an Ein component $N(e,a,b,c)$ of $M(e,n)$, where $n = c^2 - a^2 - b^2 - e(c-a-b)$. Take an arbitrary point $(S,C) \in \mathrm{R}$ and compute the number $h^0(\mathcal{O}_C(c+a-e))$. Since C is a complete intersection curve $C = S \cap S'$ (see Remark 4.4(i)), we obtain the equality

$$\det N_{C/\mathbb{P}^3} = \mathcal{O}_C(2c-a-b) \tag{6.1}$$

and the exact triples

$$0 \to \mathcal{I}_C(c+a-e) \to \mathcal{O}_{\mathbb{P}^3}(c+a-e) \to \mathcal{O}_C(c+a-e) \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(2a+b-c-e) \to \mathcal{O}_{\mathbb{P}^3}(a+b-e) \oplus \mathcal{O}_{\mathbb{P}^3}(2a-e) \to \mathcal{I}_C(c+a-e) \to 0.$$

These triples yield

$$\begin{aligned} &h^0(\mathcal{O}_C(c+a-e)) \\ &= \binom{c+a-e+3}{3} - \binom{a+b-e+3}{3} - \binom{2a-e+3}{3} + \delta(e,a,b,c), \end{aligned} \tag{6.2}$$

where

$$\delta(e,a,b,c) = \begin{cases} \binom{2a+b-c-e+3}{3}, & \text{if } c \le 2a+b-e, \\ 0, & \text{if } c > 2a+b-e. \end{cases} \tag{6.3}$$

For an arbitrary point $x = (S,C) \in \mathrm{R}$, consider the groups

$$\mathrm{Ext}^i(x) := \mathrm{Ext}^i(\mathcal{I}_C(c-2a-b+e), \mathcal{O}_{\mathbb{P}^3}), \quad i = 0, 1.$$

From (6.1) it follows that

$$\begin{aligned} &\mathcal{E}xt^1(\mathcal{I}_C(c-2a-b+e), \mathcal{O}_{\mathbb{P}^3}) = \mathcal{E}xt^2(\mathcal{O}_C(c-2a-b+e), \mathcal{O}_{\mathbb{P}^3}) = \\ &\det N_{C/\mathbb{P}^3}(2a+b-c-e) \simeq \mathcal{O}_C(c+a-e). \end{aligned} \tag{6.4}$$

Since

$$h^i(\mathcal{H}om(\mathcal{I}_C(c-2a-b+e), \mathcal{O}_{\mathbb{P}^3})) = h^i(\mathcal{O}_{\mathbb{P}^3}(2a+b-c-e)) = 0, \ i = 0,1,2,$$

from (6.2), (6.4), and the spectral sequence of local-to-global Ext's, we obtain

$$\dim \mathrm{Ext}^0(x) = h^0(\mathcal{O}_{\mathbb{P}^3}(2a+b-c-e)), \tag{6.5}$$

$$\begin{aligned} &\dim \mathrm{Ext}^1(x) = h^0(\mathcal{O}_C(c+a-e)) = \\ &\binom{c+a-e+3}{3} - \binom{a+b-e+3}{3} - \binom{2a-e+3}{3} + \delta(e,a,b,c). \end{aligned} \tag{6.6}$$

Remark 6.1. Consider the incidence subvariety Σ of $\mathbb{P}^3 \times \mathrm{R}$ defined in (4.10). In view of (6.5)–(6.6), the dimensions of the groups $\mathrm{Ext}^1(x)$ do not depend on the point $x = (S, C) \in \mathrm{R}$, so that the sheaves

$$\mathcal{E}_i := \mathcal{E}xt^i_{p_2}(\mathcal{I}_{\Sigma,\mathbb{P}^3\times\mathrm{R}}(c - 2a - b + e), \mathcal{O}_{\mathbb{P}^3\times\mathrm{R}}), \quad i = 0, 1,$$

by [3] commute with the base change in the sense of [20, Remark 1.5]. In particular, the sheaf $\mathcal{E}_1$ is a locally free $\mathcal{O}_\mathrm{R}$-sheaf of rank

$$\mathrm{rk}\,\mathcal{E}_1 = h^0(\mathcal{O}_C(c + a - e))$$

and for any $x = (S, C) \in \mathrm{R}$ one has the base change isomorphism $\mathcal{E}_1 \otimes \mathbf{k}(x) \overset{\simeq}{\to} \mathrm{Ext}^1(x)$.

Consider the rational variety

$$\mathbf{T} := \mathbb{P}(\mathcal{E}_1^\vee) \tag{6.7}$$

with its structure morphism $\boldsymbol{\mu} : \mathbf{T} \to \mathrm{R}$ which is a locally trivial projective fibration with fibre of dimension $h^0(\mathcal{O}_C(c+a-e)) - 1$. We thus obtain from (4.5) and (6.6) the formula for the dimension of $\mathbf{T}$:

$$\begin{aligned} \dim \mathbf{T} = &\binom{c-b+3}{3} + \binom{c-a+3}{3} - \binom{b-a+3}{3} + \binom{c+a-e+3}{3} \\ &- \binom{a+b-e+3}{3} - \binom{2a-e+3}{3} + \delta(e, a, b, c) - 3. \end{aligned} \tag{6.8}$$

By construction, $\mathbf{T}$ has a set-theoretical description (4.7), and the structure morphism $\boldsymbol{\mu}$ coincides with (4.8). In addition, each point $t = (S, C, \xi) \in \mathbf{T}$ defines a non-trivial (class of proportionality of an) extension of $\mathcal{O}_{\mathbb{P}^3}$-sheaves

$$\xi : \qquad 0 \to \mathcal{O}_{\mathbb{P}^3}(a + b - c) \to F_t \to \mathcal{I}_C(e - a) \to 0. \tag{6.9}$$

Remark 6.2. This is the well-known Serre construction (cf. [12], [13], [22]). In particular, F_t is a reflexive sheaf with Chern classes given by (3.26).

Globalizing over $\mathbf{T}$ the triple (6.9), we arrive at the following result.

Theorem 6.3. *On $\mathbb{P}^3 \times \mathbf{T}$, there is a sheaf $\underline{\mathbf{F}}$ defined as the universal extension sheaf (4.9). The sheaf $\underline{\mathbf{F}}$ is a family of reflexive sheaves (6.9) on $\mathbb{P}^3$ with the base $\mathbf{T}$.*

In the remaining part of this section, we study the question of producing a generalized null correlation bundle E from an arbitrary reflexive sheaf F of the family $\underline{\mathbf{F}}$. A hint for this is given by the triple (3.32). In this triple, a generalized null correlation bundle E is obtained from F by an analogue of the "inverse reduction step" (cf. Remark 3.2(i)) as a kernel of an epimorphism $F \twoheadrightarrow \mathcal{O}_S(b)$. In fact, the following theorem which will be used in the next section is true.

Theorem 6.4. *Consider a subset T of $\mathbf{T}$ consisting of those points $t = (S, C, \xi) \in \mathbf{T}$ for which there exists an epimorphism $\psi : F_t \twoheadrightarrow \mathcal{O}_S(b)$, with F_t given by an extension (6.9), such that $E = \ker \psi$ is locally free. Then T is nonempty and E is a generalized null correlation bundle, $[E] \in N_{\mathrm{nc}}$.*

Proof. Clearly, T is nonempty: it is enough to take a point $\mathbf{x} = (S, y) \in \mathbf{X}$ and set $[E] = \varphi(y)$, so that the data (F, C, ξ) are determined by the pair $\big(S, [E]\big)$ as in Theorem 5.1; in particular, ξ is defined as the extension class of the triple (5.3). Then for the point $t = (S, C, \xi)$ by (5.3) the sheaf $F_t = F$ coincides with the sheaf $F_{\mathbf{x}}$ in the triple (3.32), and this triple shows that $t \in T$.

Now take $t = (S, C, \xi) \in T$ and consider the triple (3.32) twisted by $\mathcal{O}_{\mathbb{P}^3}(c - b + m)$:

$$0 \to E(m) \to F(c - b + m) \xrightarrow{\psi} \mathcal{O}_S(c + m) \to 0, \quad m \in \mathbb{Z}. \tag{6.10}$$

Respectively, the triple (6.9) twisted by $\mathcal{O}_{\mathbb{P}^3}(c - b + m)$ yields

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a + m) \xrightarrow{i} F(c - b + m) \xrightarrow{\theta} \mathcal{I}_C(m + c - a - b + e) \to 0. \tag{6.11}$$

Besides we have a standard exact triple

$$\begin{gathered} 0 \to \mathcal{O}_{\mathbb{P}^3}(-c + e + m) \to \mathcal{O}_{\mathbb{P}^3}(-b + e + m) \oplus \mathcal{O}_{\mathbb{P}^3}(-a + e + m) \to \\ \mathcal{I}_C(m + c - a - b + e) \to 0. \end{gathered} \tag{6.12}$$

Substituting $m \le b - c$ into (6.11) and (6.12) and using the inequalities $c > a + b,\ e \le 0$, we obtain $h^0(F(m)) \le 0,\ m \le 0$. Besides, since $Z \ne \emptyset$ and $e - b \le 0$, $h^0(\mathcal{I}_{Z,S}(e - b + m)) = 0,\ m \le 0$. Hence the triple (3.25) twisted by $\mathcal{O}_{\mathbb{P}^3}(m)$ yields

$$h^0(E(m)) = 0, \quad m \le 0. \tag{6.13}$$

In particular, $h^0(E) = 0$, i. e. E is stable.

Now consider the triples (6.10) and (6.11) and the morphisms ψ and i therein. If the composition $\psi \circ i$ vanishes, then i is a section of $E(-a)$ which contradicts (6.13) since $a \ge 0$. Hence, the composition $\psi \circ i$ factors as

$$\psi \circ i : \ \mathcal{O}_{\mathbb{P}^3}(a + m) \overset{\psi'}{\twoheadrightarrow} \mathcal{O}_S(a + m) \xrightarrow{i'} \mathcal{O}_S(c + m).$$

Denote $U = \mathbb{P}^3 \setminus \mathrm{Sing}F$. Since by (6.11) the morphism $i|_U : \mathcal{O}_U(a + m) \to F(c - b + m)|_U$ is a section of the locally free sheaf $F(c - b + m)|_U$ vanishing at the curve $C \cap U$, it follows that $i' : \mathcal{O}_S(a + m) \to \mathcal{O}_S(c + m)$ is a multiplication by the equation of the divisor C in S. Hence $\mathrm{coker}\, i' = \mathcal{O}_C(c + m)$, and we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^3}(a+b-c+m) & \xrightarrow{\cdot S} & \mathcal{O}_{\mathbb{P}^3}(a+m) & \xrightarrow{\psi'} & \mathcal{O}_S(a+m) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & & \\
0 & \to & E(m) & \to & F(c-b+m) & \xrightarrow{\psi} & \mathcal{O}_S(c+m) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\
0 & \to & \mathcal{I}_{\tilde{C}}(c-a-b+e+m) & \to & \mathcal{I}_C(c-a-b+e+m) & \xrightarrow{\psi''} & \mathcal{O}_C(c+m) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

in which ψ'' is induced by the morphisms ψ and ψ', and $\tilde{C}$ is a certain double scheme structure on the curve C. Consider the bottom horizontal and left vertical triples in this diagram:

$$0 \to \mathcal{I}_{\tilde{C}}(c-a-b+e+m) \to \mathcal{I}_C(c-a-b+e+m) \xrightarrow{\psi''} \mathcal{O}_C(c+m) \to 0, \quad (6.14)$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(a+b-c+m) \to E(m) \to \mathcal{I}_{\tilde{C}}(m+c-a-b+e) \to 0. \quad (6.15)$$

By [23, Example 3.3], from the triple (6.14) one obtains that the cohomology $H^0_*(\mathcal{O}_{\mathbb{P}^3})$-module $H^1_*(\mathcal{I}_{\tilde{C}})$ (as a graded module over the graded ring $H^0_*(\mathcal{O}_{\mathbb{P}^3}) \simeq \mathbf{k}[x_0, x_1, x_2, x_3]$) has one generator. Hence, the triple (6.15) yields that the cohomology $H^0_*(\mathcal{O}_{\mathbb{P}^3})$-module $H^1_*(E)$ also has one generator. This together with [9, Prop. 1.3] shows that E is a generalized null correlation bundle. □

7 Family of generalized null correlation bundles $\underline{\mathbf{E}}$ associated to $\underline{\mathbf{F}}$

In this section, starting with the family $\underline{\mathbf{F}}$ with rational base $\mathbf{T}$, we produce a family $\underline{\mathbf{E}}$ of generalized null correlation bundles with certain rational base V. For this, we first prove Theorem 7.1 in which we state certain properties of the restriction of a reflexive sheaf F_t of the family $\underline{\mathbf{F}}$ onto a surface S, where $t = (S, C, \xi) \in \mathbf{T}$. From these properties, it follows that the set T from Theorem 6.4 is a dense open subset T of $\mathbf{T}$. Theorem 7.1 then also leads to a construction of a desired rational family V as of a dense open subset of a locally trivial projective fibration over T.

Theorem 7.1. *In conditions and notation of Theorem 6.4, let* $t = (S, C, \xi) \in T$ *and* $F = F_t$. *Then, the following statements hold*

(i) $\dim \mathrm{Hom}(F, \mathcal{O}_S(b)) = \binom{b+c-e+3}{3} - \binom{2b-e+3}{3} + 1$.

(ii) the set $P(\mathrm{Hom}(F, \mathcal{O}_S(b)))^* = \{\mathbf{k}\psi \in P(\mathrm{Hom}(F, \mathcal{O}_S(b))) \mid \psi : F \to \mathcal{O}_S(b)$ is surjective and $\ker \psi$ is locally free$\}$ *is nonempty, hence dense open in* $P(\mathrm{Hom}(F, \mathcal{O}_S(b)))$.

(iii) For any point $\mathbf{k}\psi \in P(\mathrm{Hom}(F, \mathcal{O}_S(b)))^*$, *the sheaf*

$$E_\psi := (\ker(F \twoheadrightarrow \mathcal{O}_S(b)))(c-b)$$

is a generalized null correlation bundle, $[E_\psi] \in N_{\mathrm{nc}}$.

Proof. (i) Note that the natural epimorphism $\rho : \mathcal{I}_C(e-a) \twoheadrightarrow \mathcal{I}_{C,S}(e-a) \simeq \mathcal{O}_S(-C)(e-a) \simeq \mathcal{O}_S(e-c)$ composed with the epimorphism $\theta : F \twoheadrightarrow \mathcal{I}_C(e-a)$ from the triple (6.11) for $m = b-c$ gives an epimorphism

$$\rho \circ \theta : F \twoheadrightarrow \mathcal{O}_S(e-c).$$

Restricting it onto S yields an exact triple: $0 \to \mathcal{I}_{Z,S}(b) \to F|_S \to \mathcal{O}_S(e-c) \to 0$. This triple together with the triple $0 \to \mathcal{I}_{Z,S}(b) \to \mathcal{O}_S(b) \to \mathcal{O}_Z \to 0$ by push-out yield the following exact triples:

$$0 \to \mathcal{O}_S(b) \xrightarrow{u} (F|_S)^{\vee\vee} \to \mathcal{O}_S(e-c) \to 0, \tag{7.1}$$

$$0 \to F|_S \to (F|_S)^{\vee\vee} \to \mathcal{O}_Z \to 0, \tag{7.2}$$

where $(F|_S)^{\vee\vee} = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{H}om_{\mathcal{O}_S}(F|_S, \mathcal{O}_S), \mathcal{O}_S)$. On the other hand, restricting onto S the epimorphism $\psi : F \twoheadrightarrow \mathcal{O}_S(b)$ from the triple (6.10) with $m = b-c$, we obtain an exact triple $0 \to \mathcal{I}_{Z,S}(e-c) \to F|_S \to \mathcal{O}_S(b) \to 0$. As above, by push-out this triple yields the exact triple

$$0 \to \mathcal{O}_S(e-c) \to (F|_S)^{\vee\vee} \xrightarrow{v} \mathcal{O}_S(b) \to 0. \tag{7.3}$$

Now, consider the morphisms u and v in the triples (7.1) and (7.3). If their composition $v \circ u : \mathcal{O}_S(b) \to \mathcal{O}_S(b)$ iz zero, this implies that there exists a nonzero morphism $\mathcal{O}_S(b) \to \mathcal{O}_S(e-c)$, contrary to the condition that $e-c-b < 0$. Hence $v \circ u : \mathcal{O}_S(b) \to \mathcal{O}_S(b)$ is an isomorphism. This means that both triples (7.1) and (7.3) split. Thus

$$(F|_S)^{\vee\vee} \simeq \mathcal{O}_S(b) \oplus \mathcal{O}_S(e-c). \tag{7.4}$$

Remark that, since $\dim Z = 0$, it follows that

$$\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_S(b)) = \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_S(b)) = 0,$$

the triple (7.2) yields the isomorphisms

$$\mathrm{Hom}(F, \mathcal{O}_S(b)) \simeq \mathrm{Hom}(F|_S, \mathcal{O}_S(b)) \simeq \mathrm{Hom}((F|_S)^{\vee\vee}, \mathcal{O}_S(b)).$$

This together with (7.4) shows that

$$\mathrm{Hom}(F,\mathcal{O}_S(b))=H^0(\mathcal{O}_S)\oplus H^0(\mathcal{O}_S(b+c-e)).$$

Whence, (i) follows.

Statements (ii) and (iii) are immediate consequences of Theorem 6.4. □

Now, return to the family $\underline{\mathbf{F}}$ of reflexive sheaves on $\mathbb{P}^3\times\mathbf{T}$, and recall that $\mathbf{T}$ is a rational variety (see (6.7)) with the projection $r\circ\mu:\mathbf{T}\to\mathbf{P}$. Let $\underline{\Gamma}:=\Gamma\times_{\mathbf{P}}\mathbf{T}\subset\mathbb{P}^3\times\mathbf{T}$ be the family of surfaces in $\mathbb{P}^3$ with base $\mathbf{T}$, together with the natural projection $\underline{\Gamma}\to\mathbf{T}$, the fibre of which over an arbitrary point $t=(S,C,\xi)\in\mathbf{T}$ is a surface S. Consider an $\mathcal{O}_{\mathbf{T}}$-sheaf

$$\mathcal{A}=\mathcal{E}xt^0_{\mathrm{pr}_2}(\underline{\mathbf{F}},\mathcal{O}_{\underline{\Gamma}}(b)),$$

where $\mathrm{pr}_2:\mathbb{P}^3\times\mathbf{T}\to\mathbf{T}$ is the projection. The base change and Theorem 7.1(i) show that

$$\mathcal{A}\otimes\mathbf{k}(t)=\mathrm{Hom}(F_t,\mathcal{O}_S(b)),\quad F_t=\underline{\mathbf{F}}|_{\mathbb{P}^3\times\{t\}},\tag{7.5}$$

and $\mathcal{A}$ is a locally free $\mathcal{O}_{\mathbf{T}}$-sheaf of rank

$$\mathrm{rk}\,\mathcal{A}=\binom{b+c-e+3}{3}-\binom{2b-e+3}{3}+1.\tag{7.6}$$

Since $\mathbf{T}$ is a rational variety, the scheme

$$\mathbf{U}:=\mathbb{P}(\mathcal{A}^\vee)\xrightarrow{\boldsymbol{\lambda}}\mathbf{T}\tag{7.7}$$

is a rational variety and its structure morphism $\boldsymbol{\lambda}:\mathbf{U}\to\mathbf{T}$ is a locally trivial projective fibration with fibre of dimension $\mathrm{rk}\,\mathcal{A}-1$. Thus by (6.8) and (7.6):

$$\begin{aligned}\dim\mathbf{U}=&\binom{c-b+3}{3}+\binom{c-a+3}{3}-\binom{b-a+3}{3}\\&+\binom{c+a-e+3}{3}-\binom{a+b-e+3}{3}-\binom{2a-e+3}{3}\\&+\binom{b+c-e+3}{3}-\binom{2b-e+3}{3}+\delta(e,a,b,c)-3.\end{aligned}\tag{7.8}$$

In view of (7.5), we have the set-theoretic description of $\mathbf{U}$ as:

$$\mathbf{U}=\{(t,\mathbf{k}\psi)\mid t=(S,C,\xi)\in\mathbf{T},\ \mathbf{k}\psi\in P(\mathrm{Hom}(F_t,\mathcal{O}_S(b)))\}.\tag{7.9}$$

On $\mathbf{U}$ there is a tautological subbundle morphism

$$j:\ \mathcal{O}_{\mathbf{U}}\to\mathbf{A}\otimes\mathcal{O}_{\mathbf{U}}(1),$$

where $\mathcal{O}_{\mathbf{U}}(1)$ is the Grothendieck sheaf and $\mathbf{A} := \boldsymbol{\lambda}^*\mathcal{A}$. From Theorem 7.1(ii), it follows that

$$U = \{(t, \mathbf{k}\psi) \in \mathbf{U} \mid \mathbf{k}\psi \in P(\mathrm{Hom}(F_t, \mathcal{O}_S(b)))^*\} \tag{7.10}$$

is a nonempty open (hence dense) subset of $\mathbf{U}$. Since $\boldsymbol{\lambda} : \mathbf{U} \to \mathbf{T}$ is a projective fibration, it is flat. Hence by the openness of flat morphisms (see, e. g., [11, Ch. III, Exc. 9.1]) the set $T = \lambda(U)$ is a nonempty open (hence dense) subset of $\mathbf{T}$. We now set

$$\begin{gathered} \Gamma_{\mathbf{U}} := \underline{\Gamma} \times_{\mathbf{T}} \mathbf{U}, \quad \Gamma_U := \Gamma_{\mathbf{U}} \times_{\mathbf{U}} U = \underline{\Gamma} \times_T U, \\ A := \mathbf{A}_U, \quad \mathcal{O}_B(1) := (\mathcal{O}_{\mathbf{U}}(1))_U, \quad \lambda := \boldsymbol{\lambda}|_U, \end{gathered} \tag{7.11}$$

and let

$$\mathrm{can} : \underline{\mathbf{F}}_{\mathbf{U}} \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{A} \to \mathcal{O}_{\Gamma_{\mathbf{U}}}(b)$$

be the canonical evaluation morphism. Consider the universal morphism

$$\boldsymbol{\Psi} : \ \underline{\mathbf{F}}_{\mathbf{U}} \to \mathcal{O}_{\Gamma_{\mathbf{U}}}(b) \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{U}}(1) \tag{7.12}$$

defined as the composition

$$\boldsymbol{\Psi} : \ \underline{\mathbf{F}}_{\mathbf{U}} \xrightarrow{\mathrm{id} \otimes j} \underline{\mathbf{F}}_{\mathbf{U}} \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes (A \otimes \mathcal{O}_{\mathbf{U}}(1)) \xrightarrow{\mathrm{can} \otimes \mathrm{id}} \mathcal{O}_{\Gamma_{\mathbf{U}}}(b) \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{U}}(1).$$

By Theorem 7.1(iii),

$$\begin{aligned} U = \{\mathbf{u} = (t, \mathbf{k}\psi) \in \mathbf{U} \mid t = (S, C, \xi) \in T, \ \ \boldsymbol{\Psi}|_{\mathbb{P}^3 \times \{\mathbf{u}\}} : \ \underline{\mathbf{F}} \otimes \mathbf{k}(t) \to \mathcal{O}_S(b) \\ \text{is surjective and } [\ker \boldsymbol{\Psi}(c-b)|_{\mathbb{P}^3 \times \{\mathbf{u}\}}] \in N_{\mathrm{nc}}\} \end{aligned} \tag{7.13}$$

is a dense open subset of $\mathbf{U}$, and we obtain a well-defined morphism

$$q : \ U \to N_{\mathrm{nc}}, \quad \mathbf{u} \mapsto [\ker \boldsymbol{\Psi}(c-b)|_{\mathbb{P}^3 \times \{\mathbf{u}\}}]. \tag{7.14}$$

Set

$$V := q^{-1}(N), \quad \underline{\mathbf{E}} := (\ker \boldsymbol{\Psi})(c-b)|\mathbb{P}^3 \times V. \tag{7.15}$$

Remark 7.2. (i) Note that V is nonempty. Indeed, for any $([E], S, C) \in W$, the point $f([E], S, C)$ defined in (4.14) belongs to V.
(ii) Since N is a dense open subset of N_{nc}, it follows that V is a dense open subset of U, hence also of $\mathbf{U}$, i. e. V is a rational variety of dimension given by formula (7.8). In addition, comparing (7.8) with (2.1), we obtain

$$\dim V = \dim N + \delta(e, a, b, c) + t(e, a, b), \tag{7.16}$$

and from (2.42) and (7.16) it follows that

$$\dim X - \dim V = \tau - \delta(e, a, b, c) - t(e, a, b). \tag{7.17}$$

(iii) Clearly, from (7.15) follows the exact triple (4.12).

8 Relation between $\underline{\mathbf{E}}$ and E. Proof of the main theorem

We are now ready to prove the main result of the paper, Theorem 8.1, which follows from the relation between the families $\underline{\mathbf{E}}$ and $\mathbf{E}$. (The exact form of this relation is the isomorphism (8.4).)

Theorem 8.1. *(i) There is an isomorphism of varieties*

$$f: W \overset{\simeq}{\to} V. \tag{8.1}$$

(ii) For e, a, b, c with $e \in \{-1, 0\}$ and $b \geq a \geq 0,\ c > a + b$, the variety N, hence also the variety $N(e, a, b, c)$ is at least stably rational. Furthermore, on $\mathbb{P}^3 \times W$ there exists a family of generalized null correlation bundles $\underline{\mathbf{E}}_W$ for which the corresponding modular morphism $W \to N$ coincides with $\theta \circ \pi$ in the diagram (4.2).

(iii) Assume $(e, a) \neq (0, 0),\ c > 2a+b-e$, and $b > a$. Then $\tau = 0$, $N(e, a, b, c)$ is a rational variety, and its open dense subset $N \simeq X \simeq W$ is a fine moduli space, i.e. the $\mathcal{O}_{\mathbb{P}^3 \times N}$-sheaf $\underline{\mathbf{E}}_W$ is a universal family of generalized null correlation bundles over N.

Proof. (i) The desired map $f : W \to V$ was set-theoretically defined in (4.14). We have to show that this is the underlying map of a certain morphism. We first construct a $PGL(N_m)$-invariant morphism

$$\mathbf{f}_V: \ \mathbf{W} \to V. \tag{8.2}$$

For this, consider the triple (5.17) and remark that the subscheme $\boldsymbol{\mathcal{C}}$ in this triple is a family with base $\mathbf{W}$ of complete intersection curves from R (see (4.3)). Thus, by the universality of the Hilbert scheme, there exists a morphism $\mathbf{f}_0 : \ \mathbf{W} \to \mathrm{R}$ such that $\boldsymbol{\mathcal{C}} = \Sigma \times_{\mathrm{R}} \mathbf{W}$. Hence,

$$\mathcal{I}_{\boldsymbol{\mathcal{C}}, \mathbb{P}^3 \times \mathbf{W}}(c - 2a - b + e) \simeq (\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f}_0)^* \mathcal{I}_{\Sigma, \mathbb{P}^3 \times \mathrm{R}}(c - 2a - b + e).$$

Now, consider the triples (5.17) and (4.9) as families of extensions of $\mathcal{O}_{\mathbb{P}^3}$-sheaves with bases $\mathbf{W}$ and $\mathbf{T}$, respectively. Use Remark 6.1 and the fact that $\mathbf{W}$ is reduced (see Remark 5.4) to apply the universal property of the scheme $\mathbf{T}$ (see [20, Cor. 4.4]). By this universal property there is a uniquely defined morphism $\mathbf{f}_1 : \ \mathbf{W} \to \mathbf{T}$ such that $\mathbf{f}_0 = \boldsymbol{\mu} \circ \mathbf{f}_1$ and such that the triple (5.17) is obtained by applying the functor $(\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f}_1)^*$ to the triple (4.9). In particular,

$$\mathbf{F}_{\mathbf{W}} \simeq \underline{\mathbf{F}}_{\mathbf{W}}. \tag{8.3}$$

By (8.3) and the universal property of the scheme $\mathbf{U}$ over $\mathbf{T}$, there is a unique morphism $\mathbf{f}_V : \ \mathbf{W} \to \mathbf{V}$ such that $\mathbf{f}_1 = \boldsymbol{\lambda} \circ \mathbf{f}_V$ and such that the epimorhism

$\boldsymbol{\psi}_{\mathbf{W}}: \mathbf{F}_{\mathbf{W}} \twoheadrightarrow (\mathbf{L}_{\rho}^{\vee}(b,1))_{\mathbf{W}}$ in (5.18) is obtained from the universal morphism $\boldsymbol{\Psi}$ in (7.12) by aplying the functor $(\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f}_V)^*$. As $\boldsymbol{\psi}_{\mathbf{W}}$ is surjective, from the description (7.15) of V it follows that

$$\mathbf{f}_V(\mathbf{W}) \subset V$$

and $\mathbf{E}_{\mathbf{W}} = \ker \boldsymbol{\psi}_{\mathbf{W}}$ is a family of locally free sheaves on $\mathbb{P}^3$. Moreover, (3.31), (7.15), and (8.3) yield

$$\mathbf{E}_{\mathbf{W}} \simeq \underline{\mathbf{E}}_{\mathbf{W}}. \tag{8.4}$$

Furthermore, as the $PGL(N_m)$-principal bundle $\tilde{\Phi}: \mathbf{W} \to W$ is a categorical factor, and the morphism $\mathbf{f}_V : \mathbf{W} \to V$ by construction is $PGL(N_m)$-invariant, it follows that there exists a morphism

$$f: W \to V$$

such that $\mathbf{f}_V = f \circ \tilde{\Phi}$. Clearly, f is pointwise just the map $([E], S, C) \mapsto (S, C, \xi, \mathbf{k}\psi)$ given in (4.14).

We have to show that f is an isomorphism. For this, remark that the sheaf $\mathbf{D} = \mathrm{pr}_{2*}\, \underline{\mathbf{E}}(m)$, where $\mathrm{pr}_2 : \mathbb{P}^3 \times V \to V$ is the projection, is a locally free $\mathcal{O}_V$-sheaf of rank N_m, and the evaluation morphism $ev\colon \mathrm{pr}_2^* \mathbf{D} \to \underline{\mathbf{E}}(m)$ is surjective (see Section 2). Now consider a locally free $\mathcal{O}_V$-sheaf $\mathcal{K} = \mathcal{H}om(\mathbf{k}^{N_m} \otimes \mathcal{O}_V, \mathbf{D})$ and the corresponding scheme $\mathbb{V}(\mathcal{K}^\vee) = \mathrm{Spec}(\mathrm{Sym}^{\cdot} \mathcal{K}^\vee)$. There is an open dense subset $\mathbf{Y} = \mathrm{Isom}(\mathbf{k}^{N_m} \otimes \mathcal{O}_V, \mathbf{D})$ of $\mathbb{V}(\mathcal{K}^\vee)$ consisting of (fibrewise) invertible homomorphisms from $\mathbf{k}^{N_m} \otimes \mathcal{O}_V$ to $\mathbf{D}$, together with the projection $v: \mathbf{Y} \to V$ and the canonical isomorphism $\mathrm{can}: \mathbf{k}^{N_m} \otimes \mathcal{O}_{\mathbb{P}^3 \times \mathbf{Y}} \xrightarrow{\simeq} (\mathrm{id}_{\mathbb{P}^3} \times v)^* \mathbf{D}$. This isomorphism, being twisted by $\mathcal{O}_{\mathbb{P}^3}(-m) \boxtimes \mathcal{O}_{\mathbf{Y}}$, together with the above epimorphism ev yields an epimorphism

$$\boldsymbol{\mathcal{B}} \boxtimes \mathcal{O}_{\mathbf{Y}} \xrightarrow{\mathrm{can}} (\mathrm{id}_{\mathbb{P}^3} \times v)^* \mathbf{D}(-m) \overset{ev}{\twoheadrightarrow} \underline{\mathbf{E}}_{\mathbf{Y}},$$

where $\boldsymbol{\mathcal{B}} = \mathbf{k}^{N_m} \otimes \mathcal{O}_{\mathbb{P}^3}(-m)$ (see Section 2). Thus, by the universal property of the open subset Y of the Quot-scheme $Q = \mathrm{Quot}_{\mathbb{P}^3}(\boldsymbol{\mathcal{B}}, P)$ introduced in Theorem 2.2(ii), there exists a uniquely defined morphism $\tilde{\mathbf{q}}: \mathbf{Y} \to Q$ such that

$$\underline{\mathbf{E}}_{\mathbf{Y}} \simeq \mathbb{E}_{\mathbf{Y}}, \tag{8.5}$$

where $\mathbb{E}$ is the universal quotient sheaf on $\mathbb{P}^3 \times Q$. Note that, by (7.14),

$$\varphi \circ \tilde{\mathbf{q}} = q \circ v, \tag{8.6}$$

where $\varphi : Y \to N$ is a principal $PGL(N_m)$-bundle (2.13). In particular,

$$\tilde{\mathbf{q}}(\mathbf{V}) \subset Y.$$

Next, the group $\mathbf{k}^*$ naturally acts on $\mathbf{V}$ by homotheties, so that

$$\mathbf{V} := \mathbf{Y} // \mathbf{k}^* \tag{8.7}$$

is a categorical quotient. Therefore, v as a principal $GL(N_m)$-bundle decomposes as $v = \mathbf{\Phi} \circ \nu$, where $\nu : \mathbf{Y} \to \mathbf{V}$ is a principal $\mathbf{k}^*$-bundle and

$$\mathbf{\Phi} : \ \mathbf{V} \to V \tag{8.8}$$

is a principal $PGL(N_m)$-bundle. Since the morphism $\tilde{\mathbf{q}}$ is $\mathbf{k}^*$-invariant, it decomposes as

$$\tilde{\mathbf{q}} = \mathbf{q} \circ \nu,$$

where

$$\mathbf{q} : \mathbf{V} \to Y \tag{8.9}$$

is a $PGL(N_m)$-equivarint morphism. Thus, as the principal $PGL(N_m)$-bundles $\mathbf{\Phi} : \ \mathbf{V} \to V$ and $\varphi : \ Y \to N$ are categorical quotients, there exists a morphism $q : \ V \to N$ making the diagram

$$\begin{CD} Y @<{\mathbf{q}}<< \mathbf{V} \\ @V{\varphi}VV @VV{\mathbf{\Phi}}V \\ N @<{q}<< V \end{CD} \tag{8.10}$$

cartesian. In addition, similar to (8.5) we see that the sheaf $\underline{\mathbf{E}}_{\mathbf{V}}$ satisfies the relation

$$\underline{\mathbf{E}}_{\mathbf{V}} \simeq \mathbb{E}_{\mathbf{V}}. \tag{8.11}$$

Note that since V is irreducible, so is $\mathbf{V}$.

We now construct the morphism

$$\mathbf{f} : \ \mathbf{W} \to \mathbf{V} \tag{8.12}$$

making the square in the diagram

$$\begin{CD} \mathbf{W} @>{\mathbf{f}}>> \mathbf{V} \\ @V{\tilde{\Phi}}VV {\searrow}{\scriptstyle \mathbf{f}_B} @VV{\mathbf{\Phi}}V \\ W @>{f}>> V \end{CD} \tag{8.13}$$

cartesian. For this, note that by the universal property of the Quot-scheme Q the family of generalized null correlation bundles $\mathbf{E}_{\mathbf{W}}$ on $\mathbb{P}^3 \times \mathbf{W}$ defines a morphism

$$\eta : \ \mathbf{W} \to Q \tag{8.14}$$

such that, by definition,

$$\eta(\mathbf{W}) \subset Y,$$

and the diagram

$$\begin{CD} \mathbf{W} @>{\eta}>> Y \\ @V{\tilde{\Phi}}VV @VV{\varphi}V \\ W @>{q \circ f}>> N \end{CD} \tag{8.15}$$

is cartesian. From the cartesian diagrams (8.10) and (8.15) by transitivity of fibred products follows the existence of the desired morphism $\mathbf{f}$ satisfying (8.13).

Now consider the composition $V \xrightarrow{\lambda} T \xrightarrow{\mu} \mathrm{R} \xrightarrow{r} \mathrm{P}$ of natural morphisms in diagram (4.13), and the induced graph of incidence Γ_B (see (7.11)). Let $\gamma: \ \Gamma_V := \Gamma_U \times_U V \to V$ be the projection and the set

$$\mathbf{B} := (R^2\gamma_*(\underline{\mathbf{E}}(c-e-4)|_{\Gamma_V}))^{\vee}.$$

A standard base change and the Serre duality (cf. (2.24)) imply that $\mathbf{B}$ is a line bundle on V with a fibre over an arbitrary point $\mathbf{u} = (S, C, \xi, \mathbf{k}\psi) \in V$ (we use the notaion from (7.13)) given by

$$\mathbf{B} \otimes \mathbf{k}(\mathbf{u}) = H^0(E(-b)|_S),$$

where $E = \underline{\mathbf{E}}|_{\mathbb{P}^3 \times \{\mathbf{u}\}}$. Comparing this with (2.36) and (2.39) and using (8.11), we obtain an epimorphism $\mathbf{q}^*\mathbb{U} \twoheadrightarrow \mathbf{B}$. Now, by the universal property of $\boldsymbol{\mathcal{X}} = \mathbb{P}(\mathbb{U}) \xrightarrow{\boldsymbol{\theta}} \mathcal{Y}$ defined in (2.38) (see, e. g., [11, Ch. II, Prop. 7.12]), there is a morphism

$$\mathbf{g}: \ \mathbf{Y} \to \boldsymbol{\mathcal{X}}$$

such that $\mathbf{q} = \boldsymbol{\theta} \circ \mathbf{g}$ and $\mathbf{B} \simeq \mathbf{g}^*\mathcal{O}_{\mathbb{P}(\mathbb{U})}(1)$. Therefore, in view of (8.11), we have

$$\underline{\mathbf{E}}_{\mathbf{V}} \simeq (\mathrm{id}_{\mathbb{P}^3} \times \mathbf{g})^*\mathbf{E} = \mathbf{E}_{\mathbf{V}}. \tag{8.16}$$

In addition, since $\mathbf{q}(\mathbf{V}) \subset Y$ by (8.9), it follows from diagram (2.43) that

$$\mathbf{g}(\mathbf{Y}) \subset \mathbf{X}.$$

Futhermore, the morphism $\mathbf{g}: \ \mathbf{Y} \to \mathbf{X}$ is an equivariant morphism of principal $PGL(N_m)$-bundles $\boldsymbol{\Phi}: \ \mathbf{Y} \to V$ and $\Phi: \ \mathbf{X} \to X$. Hence there exists a morphism

$$g: \ V \to X$$

making the diagram

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow{\ \mathbf{g}\ } & \mathbf{V} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\Phi}} \\ X & \xleftarrow{\ g\ } & V \end{array} \tag{8.17}$$

cartesian.

We now proceed to constructing the morphism

$$h: \ V \to W, \tag{8.18}$$

inverse to f. For this, we first construct the morphism

$$\mathbf{h}: \ \mathbf{V} \to \mathbf{W}$$

such that

$$\boldsymbol{\pi} \circ \mathbf{h} = \mathbf{g}, \quad \pi \circ h = g, \quad \text{and} \quad \tilde{\boldsymbol{\Phi}} \circ \mathbf{h} = h \circ \boldsymbol{\Phi}, \tag{8.19}$$

where $\tilde{\boldsymbol{\Phi}} : \mathbf{W} \to W$ is a principal $PGL(N_m)$-bundle in the diagram (5.11), and $\boldsymbol{\pi}$, respectively, π are the projections given in that diagram. Remark that, since the sheaf $\mathbf{F_V}$ (respectively, the sheaf $\underline{\mathbf{F}}_{\mathbf{V}}$) is determined by the sheaf $\mathbf{E_V}$ (respectively, by the sheaf $\underline{\mathbf{E}}_{\mathbf{V}}$) uniquely up to an isomorphism (see Remark 3.2(iii)), the isomorphism (8.16) implies an isomorphism

$$\underline{\mathbf{F}}_{\mathbf{V}} \simeq \mathbf{F_V}.$$

Using this isomorphism, rewrite the left morphism in the exact triple (4.9) twisted by $\mathcal{O}_{\mathbb{P}^3}(c-a-b) \boxtimes \mathcal{O}_{\mathbf{T}}$ and lifted onto $\mathbb{P}^3 \times \mathbf{V}$ as

$$i: \ (\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbf{T}}(1))_{\mathbf{V}} \to \underline{\mathbf{F}}_{\mathbf{V}}(c-a-b) \simeq \mathbf{F_V}(c-a-b).$$

Consider the diagram of natural projections

$$\begin{array}{ccc} \mathbb{P}^3 \times \mathbf{X} & \xleftarrow{\mathrm{id}_{\mathbb{P}^3} \times \mathbf{g}} & \mathbb{P}^3 \times \mathbf{V} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathbf{p}} \\ \mathbf{X} & \xleftarrow{\ \ \mathbf{g}\ \ } & \mathbf{V} \end{array}$$

and apply the functor $\mathbf{p}_*$ to the monomorphism i. We obtain a subbundle morphism

$$\iota: \ \Theta^*\mathcal{O}_{\mathbf{T}}(1) \to \mathbf{p}_*\mathbf{F_V}(c-a-b), \qquad \Theta := \lambda \circ \boldsymbol{\Phi}.$$

Note that $\mathbf{p}_*\mathbf{F_V}(c-a-b)$ is a locally free sheaf (cf. (5.9)) for which the base change yields an isomorphism

$$\mathbf{p}_*\mathbf{F_Y}(c-a-b) \simeq \mathbf{g}^* p_* \mathbf{F}(c-a-b),$$

hence an epimorphism of locally free sheaves

$$\mathbf{g}^*(p_*\mathbf{F}(c-a-b))^\vee \twoheadrightarrow \Theta^*\mathcal{O}_{\mathbf{T}}(-1)$$

defined as the composition

$$\epsilon_{\mathbf{V}}: \mathbf{g}^*(p_*\mathbf{F}(c-a-b))^\vee \simeq (\mathbf{g}^* p_*\mathbf{F}(c-a-b))^\vee \simeq (\mathbf{p}_*\mathbf{F_V}(c-a-b))^\vee \xrightarrow{\iota^\vee} \Theta^*\mathcal{O}_{\mathbf{T}}(-1).$$

Comparing $\epsilon_{\mathbf{Y}}$ with the canonical epimorphism ϵ in (5.10), we obtain by the universal property of the projective bundle $\boldsymbol{\pi} : \mathbf{W} \to \mathbf{X}$ in (5.8) that there exists a morphism $\mathbf{h}: \ \mathbf{V} \to \mathbf{W}$ satisfying the first relation (8.19) and such that $\mathbf{h}^*\epsilon = \epsilon_{\mathbf{V}}$, $\mathbf{h}^*\mathcal{O}_{\mathbf{W}}(1) \simeq \Theta^*\mathcal{O}_{\mathbf{T}}(-1)$. By construction, the morphism $\mathbf{h}$ is $PGL(N_m)$-equivariant, so that it descends to the morphism $h: V \to W$ satisfying the last two relations in (8.19).

Remark that, by (8.13), $\mathbf{f}_V = \mathbf{\Phi} \circ \mathbf{f}$. Therefore, from (8.4) we obtain $\mathbf{E_W} \simeq (\underline{\mathbf{E}}_\mathbf{V})_\mathbf{W} = (\mathrm{id}_{\mathbb{P}^3} \times \mathbf{f})^* \underline{\mathbf{E}}_\mathbf{V}$. This together with (8.16) yields:

$$\mathbf{E_W} \simeq (\mathrm{id}_{\mathbb{P}^3} \times (\mathbf{h} \circ \mathbf{f}))^* \mathbf{E_W}. \tag{8.20}$$

Now a standard argument shows that

$$\mathbf{h} \circ \mathbf{f} = \mathrm{id}_\mathbf{W}. \tag{8.21}$$

Indeed, consider the Quot-scheme

$$Q_\mathbf{W} := \mathrm{Quot}_{\mathbb{P}^3 \times \mathbf{W}/\mathbf{W}}(\boldsymbol{\mathcal{B}} \boxtimes \mathcal{O}_\mathbf{W}, P) \simeq \mathbf{W} \times Q \tag{8.22}$$

and the embedding

$$\Delta = (\mathrm{id}, \eta) : \ \mathbf{W} \to Q_\mathbf{W}, \ \mathbf{w} \mapsto (\mathbf{w}, \eta(\mathbf{w})),$$

where the morphism η is defined in (8.14). Then in view of the universal property of $Q_\mathbf{W}$ the relation (8.20) shows that the composition $\mathbf{W} \xrightarrow{\mathbf{h} \circ \mathbf{f}} \mathbf{W} \xrightarrow{\Delta} Q_\mathbf{W}$ coincides with Δ. Hence, since Δ is an embedding, (8.21) follows.

Similar to (8.21) one shows that

$$\mathbf{f} \circ \mathbf{h} = \mathrm{id}_\mathbf{V}. \tag{8.23}$$

(For this, use (8.4) to obtain, similar to (8.20), an isomorphism $\mathbf{E_V} \simeq (\mathrm{id}_{\mathbb{P}^3} \times (\mathbf{f} \circ \mathbf{h}))^* \mathbf{E_V}$, and then argue as in (8.22), with $Q_\mathbf{W}$ substituted by $Q_\mathbf{V}$, to achieve (8.23).) From (8.21) and (8.23) it follows that $\mathbf{h} = \mathbf{f}^{-1}$. In particular, $\mathbf{h}$ is a $PGL(N_m)$-equivariant isomorphism, and we obtain a cartesian diagram of principal $PGL(N_m)$-bundles

$$\begin{array}{ccc} \mathbf{W} & \xleftarrow[\simeq]{\mathbf{h}} & \mathbf{Y} \\ {\scriptstyle \mathbf{\Phi}} \downarrow & & \downarrow {\scriptstyle \mathbf{\Phi}} \\ W & \xleftarrow{h} & B. \end{array} \tag{8.24}$$

Whence, since $\mathbf{h}$ is an inverse to $\mathbf{f}$, the morphism h is an isomorphism inverse to f. Note that h is pointwise just the map $(S, C, \xi, \mathbf{k}\psi) \mapsto ([E], S, C)$ given in (4.15).

(ii) Since $W \simeq V$, the stable rationality of N now outcomes from the rationality of V (see Remark 7.2(ii)) and the local triviality of the $\mathbb{P}^m$-fibration $\pi : W \to X$ (Theorem 5.2(ii)) and of the $\mathbb{P}^\tau$-fibration $\theta : X \to N$ (Theorem 2.2(i)). In addition, the isomorphism $f : W \xrightarrow{\simeq} V$ yields the desired family $\underline{\mathbf{E}}_W = (\mathrm{id}_{\mathbb{P}^3} \times f)^* \underline{\mathbf{E}}_V$ of generalized null correlation bundles on $\mathbb{P}^3 \times W$ for which in view of the relation (8.4) the corresponding modular morphism $W \to N$ is just the composition of locally trivial projective bundles $\pi : W \to X$ and $\theta : X \to N$.

(iii) From statement (i) and formulas (5.15) and (7.17) it follows that

$$\tau = \delta(e, a, b, c) + t(e, a, b) - m(e, a, b, c). \tag{8.25}$$

This together with (2.2), (5.13), (5.1) and (6.3) shows that, under the conditions $(e, a) \neq (0, 0)$, $c > 2a + b - e$ and $b > a$, one has

$$\tau = 0.$$

Therefore, by Theorem 5.2(ii) (see (5.16)) and Theorem 2.2(i),

$$W \xrightarrow[\simeq]{\pi} X \xrightarrow{\theta} N(e, a, b, c)$$

is a $\mathbb{P}^0$-fibration, hence an isomorphism. Therefore, by the rationality of $V \simeq W$, $N(e, a, b, c) = \overline{V}$ is rational.

In addition, $\underline{\mathbf{E}}_W \simeq \mathbf{E}$ is a universal family of generalized null correlation bundles over N. This yields that the scheme N together with the universal family $\mathbf{E}$ over it is a fine moduli space in the sense that it represents the functor F : $(\text{Schemes})^0 \to \text{Sets}$ defined in the following usual way. For a given scheme X, $F(X)$ is the set of equivalence classes of flat families with base X of generalized null correlation bundles on $\mathbb{P}^3$ belonging to N. Recall that, by definition, the two families $\mathcal{E}$ and $\mathcal{E}'$ over X are equivalent if they are isomorphic up to a twist by a pullback of a line bundle from X. Thus, to the equivalence class $\{\mathcal{E}\} \in F(X)$ of a family $\mathcal{E}$ there corresponds a morhism $X \to N$ such that $\{\mathcal{E}\} = \{\mathbf{E}_X\}$. □

From Theorem 8.1 and the result of L. Ein [9] now immediately follows

Corollary 8.2. *For both* $e = 0$ *and* $e = -1$*, the union of the spaces* $M(e, n)$ *over all* $n \geq 1$ *contains an infinite series of rational components.*

The following remarks are in order.

Remark 8.3. Fine moduli for n even. There is a well-known sufficient condition for the (given component of the) Gieseker-Maruyama moduli space to be fine – see [17, Cor. 4.6.6]. In case of $M(0, n)$ with n even this condition fails, and there were no known examples of components of $M(0, n)$ when these moduli components were fine moduli spaces. (On the contrary, there are known certain components of $M(0, n)$ for n even, e. g., the instanton components which are not fine – see [16].) Theorem 8.1(ii) provides a series of fine (open dense subsets of) moduli components $N(e, a, b, c)$ for $c > 2a + b - e,\ b > a,\ (e, a) \neq (0, 0)$, and $n = c^2 - a^2 - b^2$ even, this series clearly being infinite – see [19].

Remark 8.4. In 1984 V. K. Vedernikov [28] constructed, for $1 \leq l \leq k$, a family $V_1(k, l) \subset M(0, 2kl + 2l - l^2)$; for $1 \leq 2l \leq k$, a family $V_2(k, l) \subset M(0, k^2 + 2k + 1 - l^2)$; for $1 \leq 2l \leq k + 2$, a family $V_3(k, l)k^2 + 3k + 2 + 2l - 2l^2)$. Later in 1987 (see [29]), he constructed one more family, $V_4(k) \subset M(0, (k + 1)^2)$ for $k \geq 1$. From the results of L. Ein, 1988, see [9], it follows that Ein components $N(e, a, b, c)$ with approriate a, b, c contain these Vedernikov's families $V_1(k, l)$ and $V_4(k)$, respectively, $V_2(k, l)$ and $V_3(k, l)$, as their open dense subsets in special cases when $e = a = 0$, respectively, $a = b$. More precisely, the closures $\overline{V_i(k,l)}$ of the families $V_i(k,l)$ in $M(e,n)$ are:

$$\begin{array}{lll} \overline{V_1(k,l)} = N(0,a,b,c) & \text{for} & a=0, \quad b=k+1-l, \quad c=k+1, \\ \overline{V_2(k,l)} = N(0,a,b,c) & \text{for} & a=b=l, \quad c=k+1, \\ \overline{V_3(k,l)} = N(-1,a,b,c) & \text{for} & a=b=l-1, \quad c=k+1, \\ \overline{V_4(k)} = N(0,a,b,c) & \text{for} & a=b=0, \quad c=k+1. \end{array} \tag{8.26}$$

In [28], it is asserted that $V_1(k,l)$ is rational. However, the construction of rationality of $V_1(k,l)$ presented in [28, Section 3] coincides with ours and thus, by Theorem 8.1, yields only stable rationality of $V_1(k,l)$. Indeed, in this case, $\tau = 0$ by (8.25), but $m = m(0,0,k+1-l,k+1) = 1$ by (5.13) and (5.1), so that, $\pi : B_\tau \to V_1(k,l)$ is a locally trivial $\mathbb{P}^1$-bundle with B_τ rational. So the problem of rationality of $V_1(k,l)$ remains open.

The construction of rationality of $V_2(k,l)$ provided in [28, Sections 5-6] differs from ours. According to Theorem 8.1, the rationality of $V_2(k,l)$ is covered by our result in the range $k \geq 3l \geq 3$ and, respectively, not covered in the range $2 \leq 2l \leq k \leq 3l-1$.

In [28, Section 7], the rationality of $V_3(k,l)$ is asserted without proof. On the other hand, in this case the rationality (respectively, stable rationality) of $V_3(k,l)$ follows from Theorem 8.1 for $k \geq 3l-2$ (respectively, for $2l-2 \leq k \leq 3l-3$).

Last, the rationality of $V_4(k)$ is proved in [29]. It is not covered by Theorem 8.1. Indeed, in this case we obtain from (5.13) and (5.1) that $m = 2$, and Theorem 8.1 yields stable rationality of $V_4(k)$.

Summarizing the above and using (8.26), we conclude that the result of Theorem 8.1 covers Vedernikov's (proven) results in case $e = 0,\ a = b > 0,\ c > 3a$ and improves them in case $e = a = 0,\ b > 0$.

Remark 8.5. As it is known (see [23, Prop. 3.1], [9]), the cohomology module $H^1_*(E)$ of a generalized null correlation bundle $[E] \in N_{\mathrm{nc}}$ has one generator as a graded module over $\mathbf{k}[x_0,x_1,x_2,x_3]$. Using this, A. P. Rao in [23, Prop. 3.1 and Remark 3.2] constructed big enough rational families B of generalized null correlation bundles from N_{nc} with a given cohomology module $H^1_*(E)$. It follows that N_{nc} can be filled by unirational varieties $\Phi(B)$ of dimension big enough, where $\Phi :\ B \to N_{\mathrm{nc}}$ is the modular morphism. This shows that N_{nc} is at least rationally connected (which also follows from their stable rationality), and it possibly might give an alternative approach to the problem of rationality of Ein components.

9 Components of $M(e,n)$ for small n

In this section, we enumerate the known components (including the Ein components) of the Gieseker-Maruyama moduli space $M(e,n)$ for small values of n, namely, for $n \leq 20$ in both cases (i) $e = 0$ and (ii) $e = -1$. We specify those of these components which are rational, respectively, stably rational. Their dimensions are also given.

(i) $e = 0$. The complete description of all the components of $M(0,n)$ is currently known only for $n \leq 5$.

(i.1) $M(0,1)$ is irreducible: $M(0,1) \simeq \mathbb{P}^5 \setminus G$, where G is the Grassmannian $Gr(2,4)$ embedded in $\mathbb{P}^5$ by Plücker – see, e.g., [12] or [22]. Here $M(0,1)$ is an Ein component with $a = b = 0,\ c = 1$.

(i.2) $M(0,2)$ is an irreducible 13-dimensional rational variety, and any sheaf in $M(0,2)$ is an instanton bundle – see [12, Section 9]. Note that $M(0,2)$ is not an Ein component.

(i.3) $M(0,3)$ consists of two rational irreducible 21-dimensional components: the instanton component I_3 any sheaf of which is an instanton bundle, and the Ein component $N(0,0,1,2)$ any sheaf of which is a generalized null correlation bundle, i. e. $N(0,0,1,2) = N(0,0,1,2)^{\mathrm{nc}}$ – see [10].

(i.4) $M(0,4)$ consists of two irreducible 29-dimensional components: the instanton component I_4 any sheaf of which is a mathematical instanton bundle with spectrum $(0,0,0,0)$, and the Ein component $N(0,0,0,2)$ – see [4], [5], [8], [14]. The rationality of $N(0,0,0,2)$ is proved in [8] and reproved in [29] by another method. It is also shown in [8] that $N(0,0,0,2) \setminus N_{\mathrm{nc}} \neq \emptyset$.

(i.5) $M(0,5)$ has three irreducible components, according to a recent result of C. Almeida, M. Jardim, A. Tikhomirov and S. Tikhomirov [1]. The first one is the 37-dimensional rational instanton component I_5 [7], [24], [18], a general sheaf of which is a mathematical instanton bundle. The next one is the 40-dimensional Ein component $N(0,0,2,3)$ – see [9], [10, Theorem 4.7], [14], and it coincides with the component Q_2 of $M(0,5)$ introduced by Ellingsrud and Strømme (we use the notation from Section 1). This component is stably rational by Theorem 8.1. (A weaker statement about unirationality of $N(0,0,2,3) = Q_2$ was mentioned in Section 1.) The third one is a 37-dimensional component M_b described as the closure in $M(0,5)$ of the set $\{[E] \in M(0,5) \mid E \text{ is a cohomology bundle of a monad of the type } 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$.

(i.6) $M(0,6)$ contains the instanton component I_6 of dimension 45 (see [25]) and at least one more component of dimension ≥ 45 which contains a (possibly open) locally closed subset $M_6 = \{[E] \in M(0,6) \mid E \text{ is the cohomology bundle of a monad } 0 \to 2\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to 8\mathcal{O}_{\mathbb{P}^3} \to 2\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$ – see [14, Table 5.3, $c_2 = 6$, (2,i)], where $\dim M_6 = 45$ by Barth's formula [4, p. 216]. However, $M(0,6)$ does not contain Ein components, since there are no integer solutions for a, b, c satisfying the conditions $b \geq a \geq 0,\ c > a + b,\ c^2 - a^2 - b^2 = 6$ – see [19, Section 2].

(i.7) $M(0,7)$ contains at least four irreducible components. They are: the instanton component I_7 of dimension 53 [24], the two Ein components $N(0,0,3,4)$ and $N(0,1,1,3)$ of dimensions 65 and 55, respectively, and a component of dimension ≥ 53 which contains a locally closed subset $M_7 = \{[E] \in M(0,7) \mid E$ is the cohomology bundle of a monad $0 \to 3\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to 10\mathcal{O}_{\mathbb{P}^3} \to 3\mathcal{O}_{\mathbb{P}^3}(1) \oplus$

$\mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$ – see [14, Table 5.3, case $c_2 = 7$, (2,i)], where $\dim M_7 = 52$ by Barth's formula [loc. cit.]. Here the Ein components $N(0,0,3,4)$ and $N(0,1,1,3)$ are stably rational by Theorem 8.1, and there are no other Ein components in $M(0,7)$ by [19, Section 2].

(i.8) $M(0,8)$ contains at least three irreducible components. They are: the instanton component I_8 of dimension 61 [25], the Ein component $N(0,0,1,3)$ of dimension 62, and a component of dimension ≥ 61 which contains a (possibly open) locally closed subset $M_8 = \{[E] \in M(0,8) \mid E$ is the cohomology bundle of a monad $0 \to 4\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to 12\mathcal{O}_{\mathbb{P}^3} \to 4\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$ – see [14, Table 5.3, case $c_2 = 8$, (2,i)], where $\dim M_8 = 61$ by Barth's formula. Here the Ein component $N(0,0,1,3)$ is stably rational by Theorem 8.1, and there are no other Ein components in $M(0,8)$ by [19, Section 2].

We complete, using [1, Main Theorem 1], [19, Section 2] and [26, Theorem 3], the list of all currently known irreducible components of $M(0,n)$ for $9 \leq n \leq 20$. For these values of n, besides the Ein components and the instanton components I_n of dimension $8n-3$, $9 \leq n \leq 20$, (the rationality or stable rationality of these I_n's is unknown), the known irreducible components are 6 more components. They are: 1) component of dimension 69 of $M(0,9)$, 2) component of dimension 77 of $M(0,10)$, 3) component of dimension 85 of $M(0,11)$, 4) component of dimension 93 of $M(0,12)$, 5) component of dimension 135 of $M(0,17)$, 6) component of dimension 141 of $M(0,18)$.

Below we list the Ein components of $M(0,n)$ for $9 \leq n \leq 20$. Their rationality or stable rationality follows from Theorem 8.1 and Remark 8.4, and their dimensions are given by (2.1).

$n = 9$: $N(0,0,0,3)$ rational of dimension 69, $N(0,0,4,5)$ stably rational of dimension 96;

$n = 10$: no Ein components;

$n = 11$: $N(0,0,5,6)$ stably rational of dimension 133, $N(0,1,2,4)$ stably rational of dimension 98;

$n = 12$: $N(0,0,2,4)$ stably rational of dimension 104;

$n = 13$: $N(0,0,6,7)$ stably rational of dimension 176;

$n = 14$: $N(0,1,1,4)$ rational of dimension 117;

$n = 15$: $N(0,0,1,4)$ stably rational of dimension 123, $N(0,0,7,8)$ stably rational of dimension 225, $N(0,1,3,5)$ stably rational of dimension 152;

$n = 16$: $N(0,0,0,4)$ rational of dimension 129, $N(0,0,3,5)$ stably rational of dimension 158;

$n = 17$: $N(0,0,8,9)$ stably rational of dimension 280, $N(0,2,2,5)$ stably rational of dimension 170;

$n = 18$: no Ein components;

$n = 19$: $N(0,0,9,10)$ stably rational of dimension 341, $N(0,1,4,6)$ stably rational of dimension 218;

$n = 20$: $N(0,0,4,6)$ stably rational of dimension 224, $N(0,1,2,5)$ rational of dimension 187.

(ii) $e = -1$. The scheme $M(-1, n)$ is known to be nonempty only for $n = 2m,\ m \geq 1$ [13]. Moreover, Hartshorne in [13] produced a family H_m of bundles with minimal spectrum from $M(-1, 2m)$, using the Serre construction similar to that of 'tHooft instanton bundles from I_m. (For the notion of spectrum see [13, Section 7].) Hartshorne showed that, for each m, the family H_m is contained in a unique irreducible $(16m - 5)$-dimensional component of $M(-1, 2m)$ which is smooth along H_m. Denote this component by Y_{2m}.

Now observe the spaces $M(-1, 2m)$ for $m = 1, 2, 3$.

(ii.1) $M(-1, 2) = Y_2$ is an irreducible rational variety of dimension 11 [15].

(ii.2) $M(-1, 4)$ has two irreducible components: the rational component Y_4 of dimension 27, and the rational component M of dimension 28 which consists of bundles with maximal spectrum [2].

(ii.3) $M(-1, 6)$ has at least three irreducible components: the component Y_3 of the expected dimension 43; the Ein component $N(-1, 0, 0, 2)$ which, by Theorem 8.1, is a rational variety of the expected dimension 43; the Ein component $N(-1, 0, 2, 3)$ which, by Theorem 8.1, is a stably rational variety of dimension 50. Note that these two Ein components differ by the spectra of bundles therein (see [27]). Besides, as it follows from [19], there are no other Ein components in $M(-1, 6)$.

We complete the list of all known irreducible components of $M(-1, n)$ for $8 \leq n \leq 20$, n even. Besides the components Y_n of dimension $8n - 5$, the rationality or stable rationality of which is unknown, these are Ein components of $M(-1, n)$. (As above, here [19, Section 2], Theorem 8.1, Remark 8.4, and (2.1) are used.)

$n = 8 :$ $N(-1, 0, 3, 4)$ stably rational of dimension 78, $N(-1, 1, 1, 3)$ stably rational of dimension 67;

$n = 10 :$ $N(-1, 0, 1, 3)$ rational of dimension 80, $N(-1, 0, 4, 5)$ stably rational of dimension 112;

$n = 12 :$ $N(-1, 0, 0, 3)$ rational of dimension 93, $N(-1, 0, 5, 6)$ stably rational of dimension 152, $N(-1, 1, 2, 4)$ stably rational of dimension 116;

$n = 14 :$ $N(-1, 0, 2, 4)$ rational of dimension 128, $N(-1, 0, 6, 7)$ stably rational of dimension 198;

$n = 16 :$ $N(-1, 0, 7, 8)$ stably rational of dimension 250, $N(-1, 1, 1, 4)$ stably rational of dimension 143, $N(-1, 1, 3, 5)$ stably rational of dimension 176;

$n = 18 :$ $N(-1, 0, 1, 4)$ rational of dimension 154, $N(-1, 0, 3, 5)$ rational of dimension 188, $N(-1, 0, 8, 9)$ stably rational of dimension 308, $N(-1, 2, 2, 5)$ stably rational of dimension 197;

$n = 20 :$ $N(-1, 0, 0, 4)$ rational of dimension 165, $N(-1, 0, 9, 10)$ stably rational of dimension 372, $N(-1, 1, 4, 6)$ stably rational of dimension 248.

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