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On Determinability of a Quotient Divisible Abelian Group of Rank 1 by Its Automorphism Group

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A criterion for determinability of a quotient divisible Abelian group of rank 1 by its automorphism group in the class of quotient divisible Abelian groups of rank 1 is considered in the paper.

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The word “group” will mean an Abelian group. As usual, A is a *torsion group* (a *torsion-free group*) if $\mathrm{t}(A) = A$ ($\mathrm{t}(A) = 0$), where $\mathrm{t}(A)$ is the subgroup of all elements of finite order in A. A nonzero group A is a *torsion-free group of rank* 1 if it is isomorphic to a subgroup of $\mathbb{Q}$; we say that this group is a *rank* 1 *torsion-free group of idempotent type* if A is isomorphic to the additive group of some subring of $\mathbb{Q}$.

Let $A \in \mathbf{X}$, where $\mathbf{X}$ is some class of groups. We say that A is *determined by its automorphism group in the class* $\mathbf{X}$ if $\operatorname{Aut} A \cong \operatorname{Aut} B$ implies $A \cong B$ for every $B \in \mathbf{X}$.

It was proved that a p-group is determined by its automorphism group in the class of all p-groups for $p \geqslant 5$ [1]. Similar result was obtained for $p = 3$ [2]. Rank 1 torsion-free groups are not determined by their automorphism groups, but for direct sums of torsion-free groups of rank 1 (i.e., for *completely decomposable torsion-free groups*) necessary and sufficient conditions for determinability of a group by its automorphism group in some classes were found (see [3, 4]). Among other things, the question of determinability of a group by its automorphism group was considered in the class of completely decomposable torsion-free groups whose direct summands of rank 1 have idempotent types [3]. In particular it was shown that a torsion-free group A of rank 1 and of idempotent type is determined by its automorphism group in the class of all such groups if and only if A is an infinite cyclic group. Let us note that the class of rank 1 torsion-free groups of idempotent type coincides with the class of quotient divisible groups of rank 1 which are torsion-free.

Let us denote the set of all primes by $\mathbb{P}$. Recall (for example, see [5]) that a group A is *p-divisible* if the set $pA = \{pa \mid a \in A\}$ coincides with A; a *divisible group* is a group which is p-divisible for all $p \in \mathbb{P}$.

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Definition. *Suppose that a group A does not contain nonzero divisible torsion subgroups and there is an element $x \in A$ of infinite order such that $A/\langle x\rangle$ is a divisible torsion group. Then A is called the quotient divisible group of rank 1 and x is called the basis element of this group.*

For an arbitrary quotient divisible group A of rank 1 we define a characteristic $\chi = (m_p)_{p\in\mathbb{P}}$ by taking m_p to be the smallest number $m \geqslant 0$ for which $p^m x$ is in $p^n A$ for all $n > 0$ (if there are no such numbers, we set $m_p = \infty$). We say that χ is the *cocharacteristic of the element x*. The cocharacteristic of a basis element of the group A is uniquely determined by A [6] and it is called the *cocharacteristic of the group A*.

The class of all quotient divisible groups of rank 1 is denoted by $\mathcal{QD}_1$. Let us recall the structure of quotient divisible groups of rank 1 (see [6, 7]). Let $\chi = (m_p)_{p\in\mathbb{P}}$ be an arbitrary characteristic and L be the set of all $p \in \mathbb{P}$ for which $0 < m_p < \infty$. We denote the direct product of all rings $\mathbb{Z}/p^{m_p}\mathbb{Z}$ with $p \in L$ by K^χ and we also denote the subring of the field $\mathbb{Q}$ generated by the elements $\dfrac{1}{p}$ such that $m_p < \infty$ by $\mathbb{Q}^\chi$ (if all m_p are infinite then $\mathbb{Q}^\chi = \mathbb{Z}$). If L is a finite set, we define $R^\chi = \mathbb{Q}^\chi \times K^\chi$.

If the set L is infinite then R^χ is the ring of all elements $b = (b_p)_{p\in L} \in K^\chi$ such that for some fraction $\dfrac{u}{v} \in \mathbb{Q}^\chi$ the equality $ue_p = vb_p$ holds for almost all $p \in L$, where e_p is the identity of the ring $\mathbb{Z}/p^{m_p}\mathbb{Z}$). The mapping π that assigns to every such element b the number $\dfrac{u}{v}$ is an epimorphism of R^χ onto $\mathbb{Q}^\chi$. Its kernel $\operatorname{Ker}\pi$ coincides with the direct sum $\mathrm{t}(R^\chi)$ of all $\mathbb{Z}/p^{m_p}\mathbb{Z}$ with $p \in L$.

It can be checked directly that for any characteristic χ the additive group of the ring R^χ is a quotient divisible group of rank 1, while the identity of the ring is the basis element of this group and it has χ as its cocharacteristic. It follows that any quotient divisible group of rank 1 with cocharacteristic χ is isomorphic to the additive group of the ring R^χ [6]. In particular, if $A \in \mathcal{QD}_1$ then the factor group of A with respect to its torsion part $\mathrm{t}(A)$ is a rank 1 group of idempotent type which is p-divisible for every $p \in \mathbb{P}$ such that the p-component A_p is nonzero.

The symbol $\prod$ denotes a restricted direct product of groups. This analogue of the notion of a direct sum is used for multiplicatively written groups.

Proposition 1. *If A is a quotient divisible group of rank 1, then*

$$\operatorname{Aut} A \cong \operatorname{Aut}(A/\mathrm{t}(A)) \times \prod_{p\in L} \operatorname{Aut} A_p, \qquad (*)$$

where L is the set of all $p \in \mathbb{P}$ for which $A_p \neq 0$.

Proof. We may assume that A coincides with one of the rings R^χ. If L is finite then A is the direct sum of the p-components A_p with $p \in L$ and of the group $\mathbb{Q}^\chi \cong A/\mathrm{t}(A)$ which is p-divisible for every $p \in L$. Since each of these direct summands is fully invariant in A, we arrive at the isomorphism $(*)$.

Now let the set L be infinite. It is known that any endomorphism of the additive group of the ring $A = R^\chi$ is a multiplication by some element of A [6, 7]. From this we obtain $\operatorname{Aut} A \cong A^*$, where A^* is the group of invertible elements of the ring A. It is easy to verify that the restriction ρ of the epimorphism $\pi\colon A \to \mathbb{Q}^\chi$ to the group A^* has the group $(\mathbb{Q}^\chi)^*$ as its image. Its kernel $\operatorname{Ker}\rho$ is the set of elements $b = (b_p)_{p\in L} \in A^*$ such that $b_p = e_p$ for almost all $p \in L$.

Let us show that $\operatorname{Ker}\rho$ is a pure subgroup of A^*. Suppose that $b = (b_p)_{p\in L}$ is an element of the group A^* such that $b^n \in \operatorname{Ker}\rho$ (for some natural n). Let us define the element $c = (c_p)_{p\in L} \in A^*$

by setting $c_p = b_p$ if $(b_p)^n \neq e_p$ and $c_p = e_p$ otherwise. Obviously $c^n = b^n$ and $c \in \operatorname{Ker}\rho$ as desired.

$\operatorname{Im}\rho = (\mathbb{Q}^\chi)^*$ is a restricted direct product of cyclic groups. Using Theorem 28.2 [5], we have $A^* \cong (\mathbb{Q}^\chi)^* \times \operatorname{Ker}\rho$. Then

$$\operatorname{Aut} A \cong (\mathbb{Q}^\chi)^* \times \operatorname{Ker}\rho \cong \operatorname{Aut}\mathbb{Q}^\chi \times \prod_{p\in L}(A_p)^* \cong \operatorname{Aut}(A/\mathrm{t}(A)) \times \prod_{p\in L}\operatorname{Aut} A_p.$$

This completes the proof of the proposition. □

Cyclic groups of infinite order and of order n are denoted by Z and $Z(n)$, respectively. It makes no difference whether these groups are written additively or multiplicatively.

Proposition 2. *Suppose all p-components of a quotient divisible group A of rank 1 are nonzero. Then*

$$\operatorname{Aut} A \cong \prod_{\aleph_0} Z \times \prod_{q\in\mathbb{P}}\prod_{k>0}\prod_{\aleph_0} Z(q^k).$$

Proof. It follows from the description of quotient divisible groups of rank 1 that $A/\mathrm{t}(A) \cong \mathbb{Q}$. Using Proposition 1 and decomposing each group $\operatorname{Aut} A_p$ into the product of indecomposable cyclic subgroups, we obtain that

$$\operatorname{Aut} A \cong \prod_{\aleph_0} Z \times \prod_{q\in\mathbb{P}}\prod_{k>0}\prod_{\mathfrak{J}_{q,k}} Z(q^k),$$

where $\mathfrak{J}_{q,k} \leqslant \aleph_0$ for every $q \in \mathbb{P}$ and $k > 0$.

We fix q^k and consider the arithmetic progression with initial term $1+q^k$ and difference q^{k+1}. By the Dirichlet theorem (see [8]) this progression contains infinitely many primes. If p is one of such primes and $A_p \cong Z(p^m)$ then $\operatorname{Aut} A_p \cong Z(p-1) \times Z(p^{m-1})$ has a direct factor $Z(q^k)$ since $p-1$ is divisible by q^k but not by q^{k+1}. It follows that each of the cardinals $\mathfrak{J}_{q,k}$ is equal to $\aleph_0$. This proves the proposition. □

Definition. *We shall say that a finite cyclic group A is weakly determined by its automorphism group if for any finite cyclic group $B \ncong A$ the condition $\operatorname{Aut} A \cong \operatorname{Aut} B$ implies that the number of nonzero B_p is greater than the number of nonzero A_p.*

Let us note that if for a group $A \in \mathcal{QD}_1$ and $p \in \mathbb{P}$ we have $A_p = 0$ then group $\mathrm{t}(A)$ is p-divisible, and so the groups A and $A/\mathrm{t}(A)$ are simultaneously p-divisible or not p-divisible.

Theorem 3. *A group $A \in \mathcal{QD}_1$ is determined by its automorphism group in the class $\mathcal{QD}_1$ if and only if $\mathrm{t}(A)$ is a cyclic group (possibly equal to zero) which is weakly determined by its automorphism group and $pA \neq A$ for all $p \in \mathbb{P}$ such that $A_p = 0$.*

Proof. A quotient divisible group of rank 1 with non-zero p-components is not determined by its automorphism group in the class $\mathcal{QD}_1$ because automorphism groups of all such quotient divisible groups are isomorphic to each other as it follows from Proposition 2.

Let us suppose that group $A \in \mathcal{QD}_1$ has infinitely many nonzero p-components and its cocharacteristic $\chi = (m_p)_{p\in\mathbb{P}}$ contains at least one symbol ∞. Let us define $\varphi = (h_p)_{p\in\mathbb{P}} \neq \chi$ by setting

$$h_p = \begin{cases} m_p & \text{if } m_p \neq \infty, \\ 0 & \text{if } m_p = \infty. \end{cases}$$

Then for any group $B \in \mathcal{QD}_1$ with cocharacteristic φ we have $B/\mathrm{t}(B) \cong \mathbb{Q}$ and $B_p \cong A_p$ for every $p \in \mathbb{P}$. Taking into account Proposition 1, it follows from the isomorphisms

$$\mathrm{Aut}(A/\mathrm{t}(A)) \cong Z(2) \times \prod_{\aleph_0} Z \cong \mathrm{Aut}\,\mathbb{Q} \cong \mathrm{Aut}(B/\mathrm{t}(B))$$

that $\mathrm{Aut}\,A \cong \mathrm{Aut}\,B$. Then group A is not determined by its automorphism group in $\mathcal{QD}_1$.

Let us assume that group $B \in \mathcal{QD}_1$ has infinitely many nonzero p-components and its cocharacteristic φ does not contain symbols ∞. Taking into account the case considered in the beginning of the proof, we may assume that φ contains at least one symbol 0. Then φ can be obtained from some characteristic $\chi \neq \varphi$ using the procedure described in the previous paragraph. Therefore for every group $A \in \mathcal{QD}_1$ with cocharacteristic χ we have $A \not\cong B$ and $\mathrm{Aut}\,A \cong \mathrm{Aut}\,B$. Hence B is not determined by its automorphism group in the class $\mathcal{QD}_1$.

It remains to consider the case when the group $A \in \mathcal{QD}_1$ has a finite number of nonzero p-components, i.e., when the group $\mathrm{t}(A)$ is cyclic (then A splits). It follows from Proposition 1 and from the uniqueness (up to isomorphism) of the decomposition of a group into the restricted direct product of indecomposable cyclic groups that isomorphism $\mathrm{Aut}\,A \cong \mathrm{Aut}\,B$ with $B \in \mathcal{QD}_1$ is possible only in the case when $\mathrm{t}(B)$ is a cyclic group, $\mathrm{Aut}(A/\mathrm{t}(A)) \cong \mathrm{Aut}(B/\mathrm{t}(B))$ and $\mathrm{Aut}(\mathrm{t}(A)) \cong \mathrm{Aut}(\mathrm{t}(B))$. We denote the set of all $p \in \mathbb{P}$ for which $A_p \neq 0$ by L and the set of all $p \in \mathbb{P}$ for which the group $A/\mathrm{t}(A)$ is p-divisible by X (then $L \subset X$).

1) Suppose that equality $A_p = 0$ does not imply $pA \neq A$, i.e., that set X strictly contains L. Let us choose a set $Y \subset \mathbb{P}$ so that $L \subset Y$, $Y \neq X$ and $|Y| = |X|$. If R is the subring of the field $\mathbb{Q}$ generated by elements $\frac{1}{q}$ such that $q \in Y$ then for the group $B = R \oplus \mathrm{t}(A) \in \mathcal{QD}_1$ we have $A \not\cong B$ (because $A/\mathrm{t}(A) \not\cong R$) and

$$\mathrm{Aut}(B/\mathrm{t}(B)) \cong \mathrm{Aut}\,R \cong Z(2) \times \prod_{|X|} Z \cong \mathrm{Aut}(A/\mathrm{t}(A)),$$

whence we obtain $\mathrm{Aut}\,B \cong \mathrm{Aut}\,A$. This means that the group A is not determined by its automorphism group in the class $\mathcal{QD}_1$.

2) Now assume that $X = L$ but the group $\mathrm{t}(A)$ is not weakly determined by its automorphism group. Then there is a finite cyclic group $G \not\cong \mathrm{t}(A)$ such that $\mathrm{Aut}\,G \cong \mathrm{Aut}(\mathrm{t}(A))$, and the number of non-zero primary components of G does not exceed $|L|$. Let us choose a set $Y \subset \mathbb{P}$ so that it contains all $q \in \mathbb{P}$ for which $G_q \neq 0$ and so that $|Y| = |L|$. If R is the subring of the field $\mathbb{Q}$ generated by elements $\frac{1}{q}$ with $q \in Y$ then for the group $B = R \oplus G \in \mathcal{QD}_1$ we have $B \not\cong A$ and $\mathrm{Aut}(B/\mathrm{t}(B)) \cong Z(2) \times Z^{|L|} \cong \mathrm{Aut}(A/\mathrm{t}(A))$, whence we obtain $\mathrm{Aut}\,B \cong \mathrm{Aut}\,A$. Consequently, in this case the group A is again not determined by its automorphism group in $\mathcal{QD}_1$.

3) Finally, suppose that $X = L$ and the group $\mathrm{t}(A)$ is weakly determined by its automorphism group. Let $\mathrm{Aut}\,A \cong \mathrm{Aut}\,B$ with $B \in \mathcal{QD}_1$. Then it follows from the isomorphism $\mathrm{Aut}(A/\mathrm{t}(A)) \cong \mathrm{Aut}(B/\mathrm{t}(B))$ that the set of all $q \in \mathbb{P}$ such that $B/\mathrm{t}(B)$ is a q-divisible group has cardinality $|L|$. This means that B has at most $|L|$ non-zero primary components. Then $\mathrm{Aut}(\mathrm{t}(A)) \cong \mathrm{Aut}(\mathrm{t}(B))$ implies the isomorphism $\mathrm{t}(A) \cong \mathrm{t}(B)$. Thus for every $p \in L$ we have $B_p \neq 0$, and hence $B/\mathrm{t}(B)$ is a p-divisible group. Therefore $A/\mathrm{t}(A) \cong B/\mathrm{t}(B)$ and so $A \cong B$. We obtain that group A is determined by its automorphism group in the class $\mathcal{QD}_1$. The theorem is proved. □

The theorem just proved implies that a quotient divisible group of rank 1 with cocharacteristic

$$\chi = (m_2, m_3, m_5, \dots)$$

is determined by its automorphism group in the class $\mathcal{QD}_1$ if and only if χ does not contain symbols 0 and $m_p = \infty$ for almost all $p \in \mathbb{P}$. The finite m_p defines a cyclic group which is weakly determined by its automorphism group.

It can be verified directly that if $n \leqslant 50$ then $Z(n)$ is weakly determined by its automorphism group for $n = 1$, 5, 8, 11, 13, 16, 17, 23, 24, 25, 29, 31, 32, 37, 41, 47. From this we immediately obtain examples of quotient divisible groups of rank 1 which are determined by their automorphism groups in the class $\mathcal{QD}_1$. For instance, at $n = 5$ and $n = 24$ we have quotient divisible groups of rank 1 with cocharacteristics $(\infty, \infty, 1, \infty, \dots, \infty, \dots)$ and $(3, 1, \infty, \infty, \dots, \infty, \dots)$, respectively.

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Об определяемости факторно делимой абелевой группы ранга 1 своей группой автоморфизмов

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Получен критерий определяемости факторно делимой абелевой группы ранга 1 ее группой автоморфизмов в классе факторно делимых абелевых групп ранга 1.

Ключевые слова: определяемость, факторно делимая группа, группа автоморфизмов.