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Elementary nets (carpets) over a discrete valuation ring

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Elementary net (carpet) $\sigma = (\sigma_{ij})$ is called closed (admissible) if the elementary net (carpet) group $E(\sigma)$ does not contain a new elementary transvections. The work is related to the question of V. M. Levchuk 15.46 from the Kourovka notebook(closedness (admissibility) of the elementary net (carpet)over a field). Let R be a discrete valuation ring, K be the field of fractions of R, $\sigma = (\sigma_{ij})$ be an elementary net of order n over R, $\omega = (\omega_{ij})$ be a derivative net for σ, and ω_{ij} is ideals of the ring R. It is proved that if K is a field of odd characteristic, then for the closedness (admissibility) of the net σ, the closedness (admissibility) of each pair $(\sigma_{ij}, \sigma_{ji})$ is sufficient for all $i \neq j$.

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Let K be a field, ν a discrete valuation of the field K, R be discrete valuation ring, that is, R be the valuation ring of the ν of the field K (field of fractions of the ring R). We consider an elementary net of order n (elementary carpet) $\sigma = (\sigma_{ij})$ additive subgroups of the ring R, associated with σ derivative net $\omega = (\omega_{ij})$. It is proved that if K is a field of odd characteristic and ω_{ij} are ideals of R, then for the closedness (admissibility) of the net σ, the closedness of each pair $(\sigma_{ij}, \sigma_{ji})$ is sufficient for all $i \neq j$. In the considered case, a positive answer was received to the question of V. M. Levchuk (Kourovskaya notebook [1, question 15.46]) for a special linear group about reduction of the admissibility of an elementary carpet $\mathfrak{A} = \{\mathfrak{A}_r : r \in \Phi\}$ to its admissibility subcarpets $\{\mathfrak{A}_r, \mathfrak{A}_{-r}\}$, $r \in \Phi$ of rank 1. In other words, for the case under consideration it was proved (Theorem 1) that the inclusion elementary transvection of $t_{ij}(\alpha)$ into an elementary group $E(\sigma)$ is equivalent to including $t_{ij}(\alpha)$ in the group $\langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji}) \rangle$ (for any $i \neq j$). In fact, in theorem 1 it is proved (without restrictions on the characteristic of the field K), which of the inclusion $t_{ij}(\alpha) \in E(\sigma)$ follows inclusion $t_{ij}(2\alpha) \in \langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji}) \rangle$. In the final part of the article, we look at an example a symmetric elementary net over the field of rational functions $F(x)$ (over the field of coefficients F) and investigate it (Theorem 2) on closedness for an arbitrary field F other than the field $\mathbb{F}_4$ of four elements. Built examples (see Remark 1) show that the closure of the elementary net is arithmetic character, namely, it essentially depends on the characteristic of the field.

Note that the description of elementary (and complete) nets over locally finite field and the field of fractions of a principal ideal ring are obtained in [2, 3].

In the paper the following standard notations are adopted: R is an arbitrary commutative ring with a unit (in Sections 3 and 4 of R is discrete valuation ring); n is a natural number,

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$n \geqslant 3$, $\sigma = (\sigma_{ij})$ is an elementary net over the ring R of order n. Let e is the identity matrix of order n, e_{ij} is the matrix, its entries at (i; j) are equal to 1, and all other entries are equal to zero; $t_{ij}(\alpha) = e + \alpha e_{ij}$ is an elementary transvection. Let further $t_{ij}(A) = \{t_{ij}(\alpha) : \alpha \in A\}$. For elementary net (carpet) σ we consider the elementary net group $E(\sigma)$ and its subgroup $E_{ij}(\sigma)$, $i \neq j$:

$$E(\sigma) = \langle t_{ij}(\sigma_{ij}) : 1 \leqslant i \neq j \leqslant n \rangle, \quad E_{ij}(\sigma) = \langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji}) \rangle.$$

Let F be a field, then through $F(x) = \left\{ \dfrac{f}{g} : f, g \in F[x], \ g \neq 0 \right\}$ denotes the field of rational functions with coefficients from F.

1. Discrete valuation rings

Let K be a field. A discrete valuation on K is a mapping ν of the group $K^* = K \backslash 0$ onto $\mathbb{Z}$ such that [4, ch. 9]

$$\nu(xy) = \nu(x) + \nu(y), \quad \nu(x + y) \geqslant \min(\nu(x), \nu(y))$$

(i.e., ν is a surjective homomorphism). The set R consisting of 0 and all $x \in K^*$ such that $\nu(x) \geqslant 0$ is a ring, called the valuation ring of ν (it is a valuation ring of the field K). An integral domain R is a discrete valuation ring [4, ch. 9] if there is a discrete valuation ν of its field of fractions K, such that R is the valuation ring. Let R be discrete valuation ring, then R is a local ring, and its maximal ideal $\mathfrak{m}$ coincides with the set of those elements $x \in K$, for which $\nu(x) \geqslant 1$: $\mathfrak{m} = R_1 = \{x \in R \ : \nu(x) \geqslant 1\}$, every ideal of R has the form $R_n = \{x \in R \ : \nu(x) \geqslant n\}$, $(n \in \mathbb{N})$. Let $z \in R$ be an element such that $\nu(z) = 1$. Then $R_n = (z^n) = z^n R$, in particular, $\mathfrak{m} = R_1 = zR$. The group of invertible elements R^* of the ring R coincides with the set $\{x \in R : \ \nu(x) = 0\}$ and $1 + \mathfrak{m} \subseteq R^*$.

Consider an example of a discrete valuation ring, which we will be used further in Section 4. Let F be a field, $F(x)$ be a field of rational functions with coefficients from F. We define the function $\nu : F(x) \backslash 0 \longrightarrow \mathbb{Z}$ as follows. For $\dfrac{f}{g} \neq 0$ we set $\nu\left(\dfrac{f}{g}\right) = deg(g) - deg(f)$. So the defined surjective function ν satisfies the properties: $\nu\left(\dfrac{f_1}{g_1} \cdot \dfrac{f_2}{g_2}\right) = \nu\left(\dfrac{f_1}{g_1}\right) + \nu\left(\dfrac{f_2}{g_2}\right)$, $\nu\left(\dfrac{f_1}{g_1} + \dfrac{f_2}{g_2}\right) \geqslant \min\left(\nu\left(\dfrac{f_1}{g_1}\right), \nu\left(\dfrac{f_2}{g_2}\right)\right)$. Therefore, $K = F(x)$ is a discrete field of the valuation ν, $R = R_0 = \left\{\dfrac{f}{g} \in F(x) : deg(g) \geqslant deg(f)\right\}$ is a discrete valuation ring. For non-negative integer $m \geqslant 0$ consider the set

$$R_m = \left\{ \frac{f}{g} \in F(x) : \nu\left(\frac{f}{g}\right) = deg(g) - deg(f) \geqslant m \right\}. \tag{1}$$

Then every ideal of the discrete valuation ring $R = R_0$ has kind R_m and is the main ideal $R_m = \dfrac{1}{x^m} R = (R_1)^m$, next R is a local ring, $\mathfrak{m} = R_1 = \dfrac{1}{x} R$ is the maximal ideal of a local ring R, $R^* = \left\{\dfrac{f}{g} \in R : deg(g) = deg(f)\right\}$, $1 + R_1 \subseteq R^*$.

2. Preliminary results

System $\sigma = (\sigma_{ij})$, $1 \leqslant i, j \leqslant n$, additive subgroups of a ring R are called *net (carpet)* [5, 6] over ring R of order n if $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ for all values of the index i, r, j. A net viewed without a

diagonal is called *elementary net (elementary carpet)* [5–6, 1, question 15.46]. The elementary net $\sigma = (\sigma_{ij}), 1 \leqslant i \neq j \leqslant n$, is called *supplemented* if for some additive subgroups (more precisely, subrings) σ_{ii} of the ring R table (with diagonal) $\sigma = (\sigma_{ij}), 1 \leqslant i, j \leqslant n$, is (full) net. It is well known (see, for example, [5]) that the elementary net $\sigma = (\sigma_{ij})$ is supplemented if and only if $\sigma_{ij}\sigma_{ji}\sigma_{ij} \subseteq \sigma_{ij}$ for any $i \neq j$. Diagonal subgroups σ_{ii} are defined by the formula

$$\sigma_{ii} = \sum_{k \neq i} \sigma_{ki}\sigma_{ik}, \tag{2}$$

where summation is taken over all k other than i.

A full or elementary net $\sigma = (\sigma_{ij})$ is called *irreducible* if all additive subgroups σ_{ij} are different from zero. The elementary net σ is called *closed (admissible)* if the subgroup $E(\sigma)$ does not contain new elemental transvection. Closed are, for example, elementary nets, the diagonal of which can be supplemented with subgroups, getting at the same time (full) net.

Let $\sigma = (\sigma_{ij})$ is an elementary net over the ring R of order $n \geqslant 3$. Consider the set $\omega = (\omega_{ij})$ additive subgroups ω_{ij} of ring R, defined for any $i \neq j$ as follows: $\omega_{ij} = \sum_{k=1}^{n} \sigma_{ik}\sigma_{kj}$, where summation is taken over all k other than i and j. Set $\omega = (\omega_{ij})$ is a supplemented elementary net. We will add elementary net ω to (full) net in a cyclical way, proposed in [7], setting $\omega_{ii} = \sum_{k \neq s} \sigma_{ik}\sigma_{ks}\sigma_{si}$, where summation is done by all $1 \leqslant k \neq s \leqslant n$. We call the constructed net a derivative net (for elementary net σ). Further, for arbitrary $i \neq j$ we set $\Omega_{ij} = \sigma_{ij} + \sigma_{ij}\gamma_{ij}$, where

$$\gamma_{ij} = \Omega_{ij}\Omega_{ji} = \sum_{m=1}^{\infty} (\sigma_{ji}\sigma_{ij})^m, \quad i \neq j.$$

The table $\Omega = (\Omega_{ij})$ is an elementary net, moreover supplemented, then there are fair inclusions $\Omega_{ij}\Omega_{ji}\Omega_{ij} \subseteq \Omega_{ij}$ for any $i \neq j$. By virtue of (2) we add the elementary net Ω to the (full) net, putting $\Omega_{ii} = \sum_{k \neq i} \Omega_{ik}\Omega_{ki}$, where the summation is taken by k, $k \neq i$. The net Ω is called the net associated with the elementary group $E(\sigma)$.

Lemma 1. *For any pairwise distinct i, r, j there are inclusions:* $\Omega_{ir}\Omega_{rj} \subseteq \omega_{ij}$.

Proof. Let i, r, j be pairwise distinct integers. In the beginning, we note that

$$\sigma_{ir}(\sigma_{jr}\sigma_{rj})^s \subseteq \sigma_{ir}, \ \sigma_{rj}(\sigma_{ir}\sigma_{ri})^k \subseteq \sigma_{rj}$$

(the first inclusion is obvious, the second follows from the fact that $\sigma_{rj}(\sigma_{ri}\sigma_{ir}) = \sigma_{ri}(\sigma_{ir}\sigma_{rj}) \subseteq \sigma_{rj}$). Therefore

$$(\sigma_{ir} + \sigma_{ir}(\sigma_{ir}\sigma_{ri})^k)(\sigma_{rj} + \sigma_{rj}(\sigma_{jr}\sigma_{rj})^s) = \sigma_{ir}\sigma_{rj} + \sigma_{ir}\sigma_{rj}(\sigma_{jr}\sigma_{rj})^s +$$

$$(\sigma_{ir}\sigma_{rj})(\sigma_{ir}\sigma_{ri})^k(\sigma_{jr}\sigma_{rj})^s + (\sigma_{ir}\sigma_{rj})(\sigma_{ir}\sigma_{ri})^k = \sigma_{ir}\sigma_{rj} + [\sigma_{ir}(\sigma_{rj}\sigma_{jr})^s]\sigma_{rj} +$$

$$+[\sigma_{ir}(\sigma_{rj}\sigma_{jr})^s][(\sigma_{ri}\sigma_{ir})^k\sigma_{rj}] + \sigma_{ir}[(\sigma_{ri}\sigma_{ir})^k\sigma_{rj}] \subseteq \sigma_{ir}\sigma_{rj} \subseteq \omega_{ij} \subseteq \Omega_{ij}.$$

$\square$

Lemma 1 and Theorem 1 [7] imply the following proposition.

Proposition 1 ([7], Theorem 1)**.** *Elementary net σ induces a derived net ω and the net Ω, associated with the elementary group $E(\sigma)$, with $\omega \subseteq \sigma \subseteq \Omega$, and for any i, r, j relations are fulfilled*

$$\omega_{ir}\Omega_{rj} \subseteq \omega_{ij}, \quad \Omega_{ir}\omega_{rj} \subseteq \omega_{ij}.$$

Further, for any pairwise different i, r, j there are inclusions: $\Omega_{ir}\Omega_{rj} \subseteq \omega_{ij}$.

Proposition 2 ([8], Proposition 2)**.** *Let be σ is an elementary net of order n over R, Ω – a net associated with the elementary group $E(\sigma)$. If $a = (\delta_{ij} + a_{ij}) \in E(\sigma)$, then $a_{ij} \in \Omega_{ij}$.*

Using the nets $\omega = (\omega_{ij})$ and $\Omega = (\Omega_{ij})$, which are defined for the elementary net σ, we will build a new net τ follows. In the elementary net Ω to the position $(1, 2)$ instead of Ω_{12} we put ω_{12}, and the position $(2, 1)$ instead of Ω_{21} we set ω_{21}. According to proposition 1 table thus obtained will be an elementary net, and it is a supplemented elementary net. We will add its up to the (full) net as follows: we set $\tau_{ii} = \Omega_{ii}$, $i = 3, \ldots, n$,

$$\tau_{11} = \omega_{11} + \Omega_{13}\Omega_{31} + \ldots + \Omega_{1n}\Omega_{n1}, \quad \tau_{22} = \omega_{22} + \Omega_{23}\Omega_{32} + \ldots + \Omega_{2n}\Omega_{n2}.$$

According to proposition 1, the table τ is a net and has the form:

$$\tau = \begin{pmatrix} \tau_{11} & \omega_{12} & \Omega_{13} & \ldots & \Omega_{1n} \\ \omega_{21} & \tau_{22} & \Omega_{23} & \ldots & \Omega_{2n} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \ldots & \Omega_{3n} \\ . & . & . & \ldots & . \\ \Omega_{n1} & \Omega_{n2} & \Omega_{n3} & \ldots & \Omega_{nn} \end{pmatrix}.$$

Proposition 3 ([9], Theorem)**.** *Let $n \geqslant 3$, $t_{21}(\alpha) \in E(\sigma)$. Then $t_{21}(\alpha) = ah$, $a \in E_{12}(\sigma)$, $h \in G(\tau)$. If*

$$a = diag\left(\begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix}, e_{n-2} \right), \quad h = diag\left(\begin{pmatrix} 1 + h_{11} & h_{12} \\ h_{21} & 1 + h_{22} \end{pmatrix}, e_{n-2} \right),$$

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix} \begin{pmatrix} 1 + h_{11} & h_{12} \\ h_{21} & 1 + h_{22} \end{pmatrix}, \tag{3}$$

then $a_{ii}, h_{ii} \in \tau_{11} \cap \tau_{22} \cap \gamma_{12}$. $i = 1, 2$,

$$a_{11} = h_{22}, \quad \alpha, a_{21} \in \Omega_{21}, \quad \alpha - a_{21} \in \omega_{21}, \quad a_{12}, h_{12} \in \omega_{12}, \quad h_{21} \in \omega_{21},$$

$$a_{22} + a_{11}, \ a_{22} - h_{11}, \ a_{22} + h_{22}, \ h_{11} + h_{22}, \ h_{11} + a_{11} \in \omega_{11} \cap \omega_{22}.$$

Proposition 4. *Let the conditions of proposition 3 be satisfied. Then*
    (1) $a_{11}a_{21} \in \omega_{21}$;
    (2) if $t_{21}(2a_{21}) \in E_{12}(\sigma)$, then $t_{21}(2\alpha) \in E_{12}(\sigma)$.

Proof. (1) By the condition of proposition 3 $a_{21} \in \Omega_{21}$, $a_{11} = h_{22} \in \tau_{22}$, but then according to Lemma 1 [9] we have $\tau_{22}\Omega_{21} \subseteq \omega_{21}$, whence $a_{11}a_{21} \in \omega_{21}$. (2) According to proposition 3, we have

$$\alpha - a_{21} \in \omega_{21} \subseteq \sigma_{21} \Longrightarrow t_{21}(2(\alpha - a_{21})) \in E_{12}(\sigma) \Longrightarrow t_{21}(2\alpha) \in E_{12}(\sigma).$$

<div align="right">□</div>

3. The main result

Let K be a field, ν a discrete valuation of the field K. In this section R is a discrete valuation ring, that is $R = \{x \in K : \ \nu(x) \geqslant 0\}$ – valuation ring of the ν of the field K (fields of fractions of the ring R).

Theorem 1. *Let $\sigma = (\sigma_{ij})$ – elementary net over a discrete valuated ring R, $\omega = (\omega_{ij})$ is a derivative net, and ω_{ij} is the ideal of R for all $i \neq j$. If $t_{ij}(\alpha) \in E(\sigma)$, then $t_{ij}(2\alpha) \in E_{ij}(\sigma) = = \langle t_{ij}(\sigma_{ij}), t_{ji}(\sigma_{ji}) \rangle$. In particular, if K is a field of odd characteristic, then $t_{ij}(\alpha) \in E_{ij}(\sigma)$.*

Proof. Without loss of generality, we set $i = 2$, $j = 1$. Let $\mathfrak{m}$ be the maximal ideal of R. If $\omega_{21} = R$ (and then $\sigma_{21} = R$), then the conclusion of the theorem obviously, so we will assume that $\omega_{21} \subseteq \mathfrak{m}$. Let $t_{21}(\alpha) \in E(\sigma)$. Then, according to proposition 3, we have $t_{21}(\alpha) = ah$, $a \in E_{12}(\sigma)$, $h \in G(\tau)$,

$$a = diag\left(\begin{pmatrix} 1 + a_{11} & a_{12} \\ a_{21} & 1 + a_{22} \end{pmatrix}, e_{n-2}\right), \quad h = diag\left(\begin{pmatrix} 1 + h_{11} & h_{12} \\ h_{21} & 1 + h_{22} \end{pmatrix}, e_{n-2}\right),$$

and equality (3) holds. To prove the theorem according to Proposition 4 (2), it suffices to show that $t_{21}(2a_{21}) \in E_{12}(\sigma) = \langle t_{12}(\sigma_{12}), t_{21}(\sigma_{21}) \rangle$. From (3), according to Proposition 3, we have $(a_{12}h_{21} \in \omega_{21} \cdot \omega_{12})$

$$(1 + a_{11})(1 + h_{11}) \in 1 + \omega_{21} \cdot \omega_{12} \subseteq 1 + \mathfrak{m} \subseteq R^* \Longrightarrow (1 + a_{11})(1 + h_{11}) \in R^*.$$

Therefore

$$0 = \nu[(1 + a_{11})(1 + h_{11})] = \nu(1 + a_{11}) + \nu(1 + h_{11}) \Longrightarrow \nu(1 + a_{11}) = 0.$$

Consequently, $1 + a_{11}$ is an invertible element of the ring R. Further, because $a_{12} \in \omega_{12}$ и ω_{12} is an ideal of ring R, then $\dfrac{-a_{12}}{1 + a_{11}} \in \omega_{12} \subseteq \sigma_{12}$. Therefore, the inclusion $a \in E_{12}(\sigma)$ implies what

$$at_{12}\left(\frac{-a_{12}}{1 + a_{11}}\right) = \begin{pmatrix} 1 + a_{11} & 0 \\ a_{21} & (1 + a_{11})^{-1} \end{pmatrix} \in E_{12}(\sigma). \tag{4}$$

Further, since (( )^T is the transposition of the matrix and (12) is matrix-permutation)

$$(12)\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}^T (12) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad (12)\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^T (12) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$

then the matrix

$$(12)\begin{pmatrix} 1 + a_{11} & 0 \\ a_{21} & (1 + a_{11})^{-1} \end{pmatrix}^T (12) = \begin{pmatrix} (1 + a_{11})^{-1} & 0 \\ a_{21} & (1 + a_{11}) \end{pmatrix}$$

also is contained in the group $E_{12}(\sigma)$. Here it should be noted that the matrix (4) is represented as a product of elementary transvections from $t_{21}(\sigma_{21})$ and $t_{12}(\sigma_{12})$. It follows that the matrix

$$\begin{pmatrix} 1 & 0 \\ 2a_{21}(1 + a_{11}) & 1 \end{pmatrix} = \begin{pmatrix} (1 + a_{11})^{-1} & 0 \\ a_{21} & (1 + a_{11}) \end{pmatrix}\begin{pmatrix} 1 + a_{11} & 0 \\ a_{21} & (1 + a_{11})^{-1} \end{pmatrix}$$

is contained in $E_{12}(\sigma)$. According to proposition 4 (1) $a_{21}a_{11} \in \omega_{21} \subseteq \sigma_{21}$, hence $t_{21}(a_{21}a_{11}) \in E_{12}(\sigma)$, and therefore $t_{21}(2a_{21}) \in E_{12}(\sigma)$. Then, according to proposition 4 (2) $t_{21}(2\alpha) \in E_{12}(\sigma)$. $\square$

4. Examples of not closed symmetric nets over a field of rational functions

Let F be a field, $F(x)$ be a field of rational functions with coefficients from F. Consider the discrete valuation ν of the field $F(x)$ (see section 1): $\nu\left(\frac{f}{g}\right) = deg(g) - deg(f)$. Then $R = R_0 =$

$$= \left\{ \frac{f}{g} \in F(x) : deg(g) \geqslant deg(f) \right\} \text{ is the valuation ring of } \nu. \text{ The ideals of the ring } R \text{ have the}$$

form R_m, $m \geqslant 0$ (see (1)). Further, $\mathfrak{m} = R_1 = \dfrac{1}{x} R$ is the maximal ideal of a local ring R.

Put $B = \dfrac{F}{x} + R_4$. Consider the table $\tau = (\tau_{ij})$ of order $n \geqslant 2$, for which $\tau_{12} = \tau_{21} = B$, $\tau_{ij} = R_4$ for the remaining $i \neq j$:

$$\tau = \begin{pmatrix} * & B & R_4 & ... & R_4 \\ B & * & R_4 & ... & R_4 \\ R_4 & R_4 & R_4 & ... & R_4 \\ . & . & . & ... & . \\ R_4 & R_4 & R_4 & ... & * \end{pmatrix}.$$

The table τ is an elementary net since $R_4 R_4 \subseteq R_4 \subseteq B$, $R_4 B \subseteq R_4$, and τ is not supplemented elementary net as $\dfrac{1}{x^3} \in B^3 \backslash B$.

Theorem 2. *Let* $B = \dfrac{F}{x} + R_4$. *If* $|F| \geqslant 5$, *then elementary net* $\sigma = \begin{pmatrix} * & B \\ B & * \end{pmatrix}$ *is not closed (in particular, τ is not closed).*

Proof. Put $z = \dfrac{1}{x} \in B$, $B = Fz + R_4$. Then for any $\xi \in F$, $\xi \neq 0$ an element $\xi z^3 = \dfrac{1}{\xi x^3}$ is not contained in the subgroup B. For the proofs of the theorem are sufficient to show that $t_{12}(\xi z^3)$ is contained in $E(\sigma) = \langle t_{12}(B), t_{21}(B) \rangle$ for some $\xi \in F$, $\xi \neq 0$. Now the proof of the theorem follows from the following Lemma 2. $\square$

Lemma 2. *Let* $|F| \geqslant 5$. *Put* $q \in F$, $q \neq 0$, $q \neq 1$, $q^2 - q + 1 \neq 0$. *Set*

$$b = q^2 - q + 1 \in F, \quad z_1 = \frac{-q^2 z^6 - q z^8}{b} \in R_6, \quad 1 + z_1 \in 1 + R_6 \in R^*.$$

Then

$$[t_{21}(z), t_{12}(z)] \cdot t_{21}(qz) t_{12}\Big(\frac{-z}{q-1}\Big) t_{21}\big(z(q-1)^2\big) t_{12}\Big(\frac{z}{(q-1)b}\Big) \times$$

$$\times t_{21}\Big(\frac{-qz^5 + q(q-1)z^7}{1 + z_1}\Big) t_{12}\Big(\frac{q^2 z^5 + q z^7}{b^2}\Big) t_{21}(-bz) = t_{12}\Big(\frac{-qz + q^2 z^3}{b} + \xi\Big) \in E(\sigma),$$

where $\xi \in R_5 \subseteq B$, $\dfrac{-qz}{b} \in B$. *In particular, since* $\dfrac{-qz}{b} + \xi \in B$, *then* $t_{12}\big(\frac{q^2 z^3}{b}\big) \in E(\sigma)$.

Proof. We set

$$S = [t_{21}(z), t_{12}(z)] \cdot t_{21}(qz) t_{12}\Big(\frac{-z}{q-1}\Big) t_{21}\big(z(q-1)^2\big) t_{12}\Big(\frac{z}{(q-1)b}\Big)$$

It is easy to check the formula

$$S = \begin{pmatrix} [1 - qz^2 + q^2 z^4 - q(q-1)z^6] & \Big[\dfrac{-qz + q^2 z^3 - q(q-1)z^5 - qz^7}{b}\Big] \\ [bz + qz^5 - q(q-1)z^7] & \Big[1 - \dfrac{q^2 z^6 + qz^8}{b}\Big] \end{pmatrix}.$$

Note that $(S)_{22} = 1 + z_1 \in R^*$. Recall that $z_1 = \dfrac{-q^2 z^6 - q z^8}{b}$. Therefore

$$S \cdot t_{21}\Big(\frac{-qz^5 + q(q-1)z^7}{1 + z_1}\Big) = \begin{pmatrix} [1 - qz^2 + q^2 z^4 + \xi_1] & \Big[\dfrac{-qz + q^2 z^3 - q(q-1)z^5 - qz^7}{b}\Big] \\ bz & [1 + z_1] \end{pmatrix},$$

where $\xi_1 \in R_6$. Therefore

$$S \cdot t_{21}\left(\frac{-qz^5 + q(q-1)z^7}{1 + z_1}\right)t_{12}\left(\frac{q^2z^5 + qz^7}{b^2}\right) =$$

$$\left(\begin{matrix} [1 - qz^2 + q^2z^4 + \xi_1] & \left[\dfrac{-qz + q^2z^3}{b} + \xi\right] \\ bz & 1 \end{matrix}\right),$$

where $\xi \in R_5 \subseteq B$. Consequently,

$$S \cdot t_{21}\left(\frac{-qz^5 + q(q-1)z^7}{1 + z_1}\right)t_{12}\left(\frac{q^2z^5 + qz^7}{b^2}\right)t_{21}(-bz) = t_{12}\left(\frac{-qz + q^2z^3}{b} + \xi\right).$$

$\square$

Remark 1. *If $F = \mathbb{F}_2$, then the elementary net τ is closed (see [10]). Theorem 2 is also valid for the case of a $F = \mathbb{F}_3$ field of three elements (we do not provide evidence because of its bulkiness). Thus, for a full study on the closure of the elementary net τ remains to consider the case of a field $F = \mathbb{F}_4$ of four elements.*

Remark 2. *Example of the elementary net τ allows you to build similar examples over arbitrary a field in which there is a transcendent element over the simple subfield of K.*

Remark 3. *An example of the elementary net τ allows you to build similar examples over arbitrary a discrete valuation ring R (valuation ring of the discrete valuation field of ν (see section 1)), which in addition is a F-algebra over the field F: set $B = Fz + z^4 R$, where*

$$\nu(z) = 1, \ \ \mathfrak{m} = (z) = zR, \ \ \ R = \{x \in K : \nu(x) \geqslant 0\}, \ \ \ Fz^3 \cap B = 0.$$

For example, if φ is an irreducible polynomial of the ring $F[x]$ and the valuation of ν in the field of rational functions $F(x)$ is determined according to the degree of a polynomial φ in a rational fraction $\dfrac{f}{g}$, $(f, g) = 1$. Then $R = F(x)_{(\varphi)}$ coincides with the localization of the polynomial ring $F[x]$ with respect to a prime ideal (φ), further, $\mathfrak{m} = (\varphi) = \varphi \cdot R$, $B = F \cdot \varphi + \varphi^4 \cdot R$.

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Элементарные сети (ковры) над дискретно нормированным кольцом

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Элементарная сеть (ковер) $\sigma = (\sigma_{ij})$ называется замкнутой(допустимой), если элементарная сетевая (ковровая) группа $E(\sigma)$ не содержит новых элементарных трансвекций. Работа связана с вопросом В. М. Левчука 15.46 из Коуровской тетради о замкнутости (допустимости) элементарной сети (ковра) над полем. Пусть R — дискретно нормированное кольцо, K — поле частных кольца R, $\sigma = (\sigma_{ij})$ — элементарная сеть (ковер) порядка n над R, $\omega = (\omega_{ij})$ – производная сеть для σ, причем ω_{ij} — идеалы кольца R. Доказано, что если K — поле нечетной характеристики, то для замкнутости (допустимости) сети σ достаточна замкнутость (допустимость) каждой пары $(\sigma_{ij}, \sigma_{ji})$ для всех $i \neq j$.

Ключевые слова: сети, ковры, элементарная сеть, замкнутая сеть, производная сеть, элементарная сетевая группа, трансвекция, дискретно нормированное кольцо.