УДК 517.98+530.1

Positive Fixed Points of Cubic Operators on $\mathbb{R}^2$ and Gibbs Measures

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Received 13.03.2019, received in revised form 16.04.2019, accepted 10.07.2019

One model with nearest neighbour interactions of spins with values from the set $[0,1]$ *on the Cayley tree of order three is considered in the paper. Translation-invariant Gibbs measures for the model are studied. Results are proved by using properties of the positive fixed points of a cubic operator in the cone* $\mathbb{R}^2_+$.

Keywords: Cayley tree, Gibbs measure, translation-invariant Gibbs measure, fixed point, cubic operator, Hammerstein's integral operator.
DOI: 10.17516/1997-1397-2019-12-6-663-673.

Introduction

A lot of nonlinear operators are connected with problems in statistical physics, biology, thermodynamics, statistical mechanics and so on. One of the central problem in statistical physics is the existence of phase transitions. Phase transitions are connected with the theory of Gibbs measures [1]. In the theory of Gibbs measures there are a lot of papers devoted to Gibbs measures on a Cayley tree [2]. Splitting Gibbs measure is studied for models on a Cayley that can be divided into three classes: 1) models with a finite set of spin values; 2) models with a countable set of spin values; 3) models with a continuum set of spin values. Let us note that problems of studying Gibbs measures for models with a finite and countable set of spin values on a Cayley tree are reduced to the study of systems of algebraical or functional equations [3–13]. One of the main factors is that studying translation-invariant Gibbs measures for models with a continuum set of spin values is reduced to the study of positive fixed points of non-linear integral operator [14–20].

In the case of continuum set of spin values (i.e., [0,1]) various models with the nearest neighbour interactions on a Cayley tree were considered [14–20]. It was found that the existence of translation-invariant Gibbs measure for the models is equivalent to the existence of a positive fixed point of Hammerstein's nonlinear integral operator [17,20]. It was proved that the existence of translation-invariant Gibbs measures for the models on a Cayley tree of an arbitrary order and uniqueness of translation-invariant Gibbs measures on the Cayley tree of order one was shown (see [16,18]).

It is found that a sufficient condition for the model has the unique translation-invariant splitting Gibbs measure and each constructed model has at least two periodic Gibbs measures (see [15,16]).

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Models on the Cayley tree of order two were considered [20]. The study of translation-invariant Gibbs measures was reduced to the study of positive fixed points of some quadratic operator on $\mathbb{R}^2$. Sufficient conditions were also given such that the model has one, two or three translation-invariant Gibbs measure by using quadratic operators.

In this paper we consider the translation-invariant Gibbs measures for models on the Cayley tree of order three given in [20] .

1. Preliminaries

A Cayley tree $\Gamma^k = (V, L)$ of order $k \geqslant 1$ is an infinite homogeneous tree, i.e., a graph without cycles with exactly $k+1$ edges. Here V is the set of vertices and L is the set of edges.

Let us consider models where the spin takes values from the set $[0,1]$, and it is assigned to the vertices of the tree. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0,1]$. Let us denote $\Omega_A = [0,1]^A$. It is the set of all configurations on A. A configuration σ on V is defined as a function $x \in V \mapsto \sigma(x) \in [0,1]$, and the set of all configurations is $[0,1]^V$. The Hamiltonian of the model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \quad \sigma \in \Omega_V \tag{1}$$

where $J \in R \setminus \{0\}$ and $\beta = \dfrac{1}{T}$, $T > 0$ is temperature, $\xi : (u,v) \in [0,1]^2 \to \xi_{uv} \in \mathbb{R}$ is a given bounded measurable function. As usual, $\langle x, y\rangle$ stands for the nearest neighbour vertices.

We write $x < y$ if the path from x^0 to y goes through x. Vertex y is a direct successor of x if $y > x$ and x, y are nearest neighbours. Let us denote the set of direct successors of x by $S(x)$. Any vertex $x \neq x^0$ has k direct successors, and x^0 has $k+1$ direct successors.

Let $h:\ x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$.

Now, we consider the following equation

$$f(t,x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{tu}) f(u,y)du}{\int_0^1 \exp(J\beta\xi_{0u}) f(u,y)du}. \tag{2}$$

Here and below $f(t,x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0,1]$ and $du = \lambda(du)$ is the Lebesgue measure.

It is known that necessary and sufficient condition of the existence of the splitting Gibbs measure for model (1) is the existence of a solution of equation (2) for any $x \in V \setminus \{x^0\}$. Thus, we know that splitting Gibbs measure μ for model (1) depends on the function $f(t,x)$ and each splitting Gibbs measure corresponds to a solution $f(t,x)$ of equation (2). Let us note that number of the Gibbs measures for model (1) is equal to the number of positive solutions of integral equation (2).

A detailed definition of the splitting Gibbs measure for models with nearest neighbour interactions and continuum set of spin values on the Cayley tree can be found in [14–20]. In what follows the splitting Gibbs measure will be called the Gibbs measure.

Let us note that the analysis of solutions to (2) is not easy. It is difficult to give a full description of the given potential function $\xi_{t,u}$. We study Gibbs measures of model (2) in the case $f(t,x) = f(t)$ for all $x \in S(x)$. Such Gibbs measure is called translation-invariant measure.

We introduce

$$C_+[0,1] = \{f \in C[0,1] : f(x) \geqslant 0\}, \quad C_>[0,1] = C_+[0,1] \setminus \{\theta \equiv 0\}.$$

Let ξ_{tu} be a continuous function. For every $k \in \mathbb{N}$ we consider an integral operator H_k acting in the cone $C_+[0,1]$

$$(H_k f)(t) = \int_0^1 K(t,u) f^k(u)du, \;\; k \in \mathbb{N},$$

where $K(t,u) = \exp(J\beta\xi_{tu})$.

The operator H_k is called the Hammerstein integral operator of order k.

Lemma 1 ([16]). *Let $k \geqslant 2$. Hamiltonian H (1) has a translation-invariant Gibbs measure iff the Hammerstein integral operator H_k has a positive eigenvalue, i.e., the Hammerstein integral equation*

$$H_k f = \lambda f, \;\; f \in C_+[0,1] \tag{3}$$

has a non-zero positive solution for some $\lambda > 0$.

Moreover, if $\lambda_0 > 0$ is an eigenvalue of the operator $H_k, k \geqslant 2$ then an arbitrary positive number is the eigenvalue of the operator H_k. A number of positive eigenfunctions that correspond to positive eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$ of the operator H_k are equal (see [16]). Then we have the following lemma.

Lemma 2. *Let $k \geqslant 2$. A number $N^{tigm}(H)$ of translation-invariant Gibbs measures for model (1) is*

$$N^{tigm}(H) = N_+^{fix}(H_k),$$

where $N_+^{fix}(B)$ is a number of non-trivial positive fixed points of the operator B.

2. Main results

Let $\varphi_1(t),\;\; \varphi_2(t)$ and $\phi_1(t),\;\; \phi_2(t)$ are strictly positive functions that belong to $C_+[0,1]$. We consider Hamiltonian (1) on the Cayley tree Γ^3 with the potential

$$\xi_{t,u} = \frac{ln\,(\phi_1(t)\varphi_1(u) + \phi_2(t)\varphi_2(u))}{J\beta}. \tag{4}$$

We consider the Hammerstein integral operator H_3 on $C_+[0,1]$ in the following form

$$(H_3 f)(t) = \int_0^1 \big(\phi_1(t)\varphi_1(u) + \phi_2(t)\varphi_2(u)\big) f^3(u)du.$$

Let us introduce the following designations

$$\alpha_{11} = \int_0^1 \varphi_1(u)\phi_1^3(u)du, \;\; \alpha_{12} = \int_0^1 \varphi_1(u)\phi_1^2(u)\phi_2(u)du, \;\; \alpha_{21} = \int_0^1 \varphi_1(u)\phi_1(u)\phi_2^2(u)du,$$

$$\alpha_{22} = \int_0^1 \varphi_1(u)\phi_2^3(u)du, \;\; \beta_{11} = \int_0^1 \varphi_2(u)\phi_1^3(u)du, \;\; \beta_{12} = \int_0^1 \varphi_2(u)\phi_1^2(u)\phi_2(u)du,$$

$$\beta_{21} = \int_0^1 \varphi_2(u)\phi_1(u)\phi_2^2(u)du, \qquad \beta_{22} = \int_0^1 \varphi_2(u)\phi_2^3(u)du.$$

It is easy to verify that $\alpha_{ij} > 0$ and $\beta_{ij} > 0$ for all $i,j \in \{1,2\}$.

We introduce a fourth degree polynomial $P_4(\xi)$

$$P_4(\xi) = \mu_0\xi^4 + \mu_1\xi^3 + 3\mu_2\xi^2 + \mu_3\xi - \mu_4, \tag{5}$$

where

$$\mu_0 = \alpha_{22}, \quad \mu_1 = 3\alpha_{21} - \beta_{22}, \quad \mu_2 = \alpha_{12} - \beta_{21}, \quad \mu_3 = \alpha_{11} - 3\beta_{12}, \quad \mu_4 = \beta_{11}.$$

We use the following designations

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2,$$

$$\theta_k = 2\sqrt{-\frac{p}{3}} cos\left(\frac{\alpha + 2\pi(k-2)}{3}\right), \quad k = \overline{1,2,3},$$

where

$$p = -\frac{3\mu_1^2}{16\mu_0^2} + \frac{3\mu_2}{2\mu_0}, \qquad q = \frac{\mu_1^3}{32\mu_0^3} - \frac{3\mu_1\mu_2}{8\mu_0^2} + \frac{\mu_3}{4\mu_0}$$

and

$$\cos\alpha = -\frac{q}{2}\left(-\frac{3}{p}\right)^{\frac{3}{2}}, \quad \alpha \in [0, \pi].$$

We also introduce

$$\gamma_1 = \theta_3 - \frac{\mu_1}{4\mu_0}, \quad \gamma_2 = \theta_1 - \frac{\mu_1}{4\mu_0}, \quad \gamma_3 = \theta_2 - \frac{\mu_1}{4\mu_0}.$$

Theorem 2.1. *Let $Q \geqslant 0$. Then model (1) on the Cayley tree of order three has the unique translation-invariant Gibbs measure, i.e., $N^{tigm}(H) = 1$.*

Theorem 2.2. *Let $Q < 0$. If one of the following conditions*

(a) $\gamma_2 \leqslant 0,$

(b) $\gamma_2 > 0, \quad P_4(\gamma_2) < 0,$

(c) $\gamma_2 > 0, \quad P_4(\gamma_3) > 0,$

is satisfied then model (1) on the Cayley tree of order three has the unique translation-invariant Gibbs measure, i.e., $N^{tigm}(H) = 1$.

Theorem 2.3. *Let $Q < 0$. If one of the following conditions*

(d) $\gamma_2 > 0, \quad P_4(\gamma_2) = 0,$

(e) $\gamma_2 > 0, \quad P_4(\gamma_3) = 0,$

is satisfied then model (1) on the Cayley tree of order three has two translation-invariant Gibbs measures, i.e., $N^{tigm}(H) = 2$.

Theorem 2.4. *Let $Q < 0$. If the following condition*

(f) $\gamma_2 > 0, \quad P_4(\gamma_2) > 0, \quad P_4(\gamma_3) < 0,$

is satisfied then model (1) on the Cayley tree of order three has three translation-invariant Gibbs measures, i.e., $N^{tigm}(H) = 3$.

3. Positive fixed points of cubic operators on $\mathbb{R}^2$

We introduce

$$\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : \ x \geqslant 0, y \geqslant 0\},$$

$$\mathbb{R}^2_> = \{(x,y) \in \mathbb{R}^2 : \ x > 0, y > 0\}.$$

We consider the following cubic operator (CO) $\mathcal{C}$ on the cone $\mathbb{R}^2_+$

$$\mathcal{C}(x,y) = (a_{11}x^3 + 3a_{12}x^2y + 3a_{21}xy^2 + a_{22}y^3, \ b_{11}x^3 + 3b_{12}x^2y + 3b_{21}xy^2 + b_{22}y^3), \qquad (6)$$

where $a_{ij} > 0$ and $b_{ij} > 0$ for all $i,j \in \{1,2\}$.

Clearly, an arbitrary non-trivial positive fixed point $(x_0, y_0) \in \mathbb{R}^2_+$ of the CO $\mathcal{C}$ is strictly positive, i.e., $x_0 > 0, \ y_0 > 0$. We denote a number of fixed points of the CO $\mathcal{C}$ that belongs to $\mathbb{R}^2_>$ by $N^{fix}_>(\mathcal{C})$.

Lemma 3. *If $\omega = (x_0, y_0) \in \mathbb{R}^2_>$ is a fixed point of the CO $\mathcal{C}$ then $\omega \in \mathbb{R}^2_>$ and $\xi_0 = \dfrac{y_0}{x_0}$ is a root of the algebraic equation*

$$a_{22}\xi^4 + (3a_{21} - b_{22})\,\xi^3 + 3\,(a_{12} - b_{21})\,\xi^2 + (a_{11} - 3b_{12})\xi - b_{11} = 0. \qquad (7)$$

Proof. Let the point $\omega = (x_0, y_0) \in \mathbb{R}^2_>$ be a fixed point of CO $\mathcal{C}$. Then

$$a_{11}x_0^3 + 3a_{12}x_0^2y_0 + 3a_{21}x_0y_0^2 + a_{22}y_0^3 = x_0,$$

$$b_{11}x_0^3 + 3b_{12}x_0^2y_0 + 3b_{21}x_0y_0^2 + b_{22}y_0^3 = y_0.$$

Taking into account that $\dfrac{y_0}{x_0} = \xi_0$, we obtain

$$a_{11}x_0^3 + 3a_{12}x_0^3\xi_0 + 3a_{21}x_0^3\xi_0^2 + a_{22}x_0^3\xi_0^3 = x_0,$$

$$b_{11}x_0^3 + 3b_{12}x_0^3\xi_0 + 3b_{21}x_0^3\xi_0^2 + b_{22}x_0^3\xi_0^3 = x_0\xi_0.$$

Consequently we have

$$x_0^3\left(a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3\right) = x_0,$$

$$x_0^3\left(b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3\right) = \xi_0 x_0.$$

Hence, we have

$$\frac{1}{\xi_0} = \frac{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}{b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3}.$$

Using the last equality, we obtain

$$a_{22}\xi_0^4 + (3a_{21} - b_{22})\,\xi_0^3 + 3\,(a_{12} - b_{21})\,\xi_0^2 + (a_{11} - 3b_{12})\xi_0 - b_{11} = 0.$$

This completes the proof. $\square$

Lemma 4. *If ξ_0 is a root of algebraic equation (7) then the point $\omega_0 = (x_0, \xi_0 x_0) \in \mathbb{R}^2_>$ is a fixed point of the CO $\mathcal{C}$, where*

$$x_0 = \frac{1}{\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}}. \qquad (8)$$

Proof. Let $\xi_0 > 0$ and ξ_0 is a root of equation (7). We assume that $y_0 = \xi_0 x_0$, where x_0 is given by equality (8) and $\omega_0 = (x_0, \xi_0 x_0)$. From the equality $y_0 = \xi_0 x_0$ we have

$$a_{11}x_0^3 + 3a_{12}x_0^2 y_0 + 3a_{21}x_0 y_0^2 + a_{22}y_0^3 = a_{11}x_0^3 + 3a_{12}x_0^2(\xi_0 x_0) + 3a_{21}x_0(\xi_0 x_0)^2 + a_{22}(\xi_0 x_0)^3 =$$

$$= x_0^3 \cdot \left(a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3\right) = \frac{1}{\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}},$$

i.e.

$$a_{11}x_0^3 + 3a_{12}x_0^2 y_0 + 3a_{21}x_0 y_0^2 + a_{22}y_0^3 = x_0.$$

On the other hand

$$a_{22}\xi_0^4 + (3a_{21} - b_{22})\xi_0^3 + 3(a_{12} - b_{21})\xi_0^2 + (a_{11} - 3b_{12})\xi_0 - b_{11} = 0.$$

Then we obtain

$$b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3 = a_{11}\xi_0 + 3a_{12}\xi_0^2 + 3a_{21}\xi_0^3 + a_{22}\xi_0^4 = \xi_0\left(a_{11} + 3a_{12}\xi + 3a_{21}\xi^2 + a_{22}\xi^3\right).$$

From the last equality we have

$$\frac{\xi_0}{\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3}} = \frac{b_{11} + 3b_{12}\xi_0 + 3b_{21}\xi_0^2 + b_{22}\xi_0^3}{(\sqrt{a_{11} + 3a_{12}\xi_0 + 3a_{21}\xi_0^2 + a_{22}\xi_0^3})^3} =$$

$$= b_{11}x_0^3 + 3b_{12}x_0^2 y_0 + 3b_{21}x_0 y_0^2 + b_{22}y_0^3 = y_0.$$

This completes the proof. □

We introduce

$$\mu_0 = a_{22}, \quad \mu_1 = 3a_{21} - b_{22}, \quad \mu_2 = a_{12} - b_{21}, \quad \mu_3 = a_{11} - 3b_{12}, \quad \mu_4 = b_{11}$$

and define the fourth degree polynomial $P_4(\xi)$

$$P_4(\xi) = \mu_0\xi^4 + \mu_1\xi^3 + 3\mu_2\xi^2 + \mu_3\xi - \mu_4. \tag{9}$$

Lemma 5. *The CO $\mathcal{C}$ has at least one positive fixed point in $\mathbb{R}^2_>$, i.e., $N^{fix}_>(\mathcal{C}) \geqslant 1$.*

Proof. It is clear that $P_4(0) = -b_{11}$ and $P_4(+\infty) = +\infty$. It means that there exists $c > 0$ such that $P_4(c) = 0$. According to lemma 4, (x_0, cx_0) is a fixed point of CO $\mathcal{C}$ and

$$x_0 = \frac{1}{\sqrt{a_{11} + 3a_{12}c + 3a_{21}c^2 + a_{22}c^3}}.$$

□

Lemma 6. *A number of strictly positive fixed points of the CO $\mathcal{C}$ less than or equal to three, i.e., $1 \leqslant N^{fix}_>(\mathcal{C}) \leqslant 3$.*

Proof. We have the following table for the number of sign changes of the coefficients of the polynomial $P_4(\xi)$ (Tab. 1).

Using this table and the Descartes rule, we can conclude that a number of positive solutions of the polynomial $P_4(\xi)$ is not more than three (see [22, pp. 27–29]), i.e., $1 \leqslant N^{fix}_>(\mathcal{C}) \leqslant 3$. □

Let us introduce

$$Q = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2,$$

Table 1.

$P_4(\xi)$	μ_0	μ_1	μ_2	μ_3	μ_4	the number of sign changes
1.	+	+	+	+	−	1
2.	+	+	+	−	−	1
3.	+	+	−	−	−	1
4.	+	−	−	−	−	1
5.	+	−	−	+	−	3
6.	+	−	+	+	−	3
7.	+	+	−	+	−	3
8.	+	−	+	−	−	3

$$\theta_k = 2\sqrt{-\frac{p}{3}}cos\left(\frac{\alpha+2\pi(k-2)}{3}\right), \quad k=\overline{1,2,3},$$

where

$$p = -\frac{3\mu_1^2}{16\mu_0^2}+\frac{3\mu_2}{2\mu_0}, \qquad q = \frac{\mu_1^3}{32\mu_0^3}-\frac{3\mu_1\mu_2}{8\mu_0^2}+\frac{\mu_3}{4\mu_0}$$

and

$$\cos\alpha = -\frac{q}{2}\left(-\frac{3}{p}\right)^{\frac{3}{2}}, \quad \alpha\in[0,\pi].$$

We also introduce

$$\gamma_1 = \theta_3-\frac{\mu_1}{4\mu_0}, \quad \gamma_2 = \theta_1-\frac{\mu_1}{4\mu_0}, \quad \gamma_3 = \theta_2-\frac{\mu_1}{4\mu_0}.$$

Theorem 3.5. *Let $Q \geqslant 0$ then the CO $\mathcal{C}$ has the unique fixed point in $\mathbb{R}^2_>$, i.e., $N^{fix}_>(\mathcal{C}) = 1$.*

Proof.

a) Let $Q > 0$. Then the equation $P_4'(\xi) = 0$ has one real root. This root is stationary point of the function $P_4(\xi)$. Furthermore we have $P_4(0) = -b_{11} < 0$, $P_4(\pm\infty) = +\infty$. Consequently, there exists the unique number $\xi_0 > 0$ such that $P_4(\xi_0) = 0$.

b) Let $Q = 0$. Then the equation $P_4'(\xi) = 0$ has a multiple root and all of its roots are real. A simple root (of multiplicity 1) is stationary point of the function $P_4(\xi)$. Also $P_4(0) = -b_{11} < 0$, $P_4(\pm\infty) = +\infty$. It indicates that there exist the unique number $\xi_0 > 0$ such that $P_4(\xi_0) = 0$. □

Theorem 3.6. *Let $Q < 0$. If the CO $\mathcal{C}$ satisfies one of the following conditions*

(*a*) $\gamma_2 \leqslant 0$,

(*b*) $\gamma_2 > 0$, $P_4(\gamma_2) < 0$,

(*c*) $\gamma_2 > 0$, $P_4(\gamma_3) > 0$,

then the CO $\mathcal{C}$ has the unique fixed point in $\mathbb{R}^2_>$, i.e., $N^{fix}_>(\mathcal{C}) = 1$.

Proof. Let $Q < 0$. We have

$$P_4'(\xi) = 4\mu_0\xi^3+3\mu_1\xi^2+6\mu_2\xi+\mu_3. \tag{10}$$

One can find roots of the equation $P_4'(\xi) = 0$ by the Vieta method (see [23]). From $Q < 0$ it turns out that numbers $\theta_1,\ \theta_2,\ \theta_3$ are real and $\theta_3 < \theta_1 < \theta_2$. By the Vieta method numbers $\gamma_1, \gamma_2, \gamma_3$ are roots of the polynomial $P_4'(\xi)$. Then the polynomial $P_4'(\xi)$ (10) has the following form

$$P_4'(\xi) = 4\mu_0(\xi - \gamma_1)(\xi - \gamma_2)(\xi - \gamma_3).$$

It follows that function $P_4(\xi)$ is an increasing (decreasing) function on the set $(\gamma_1, \gamma_2) \cup (\gamma_3, +\infty)$ $((-\infty, \gamma_1) \cup (\gamma_2, \gamma_3))$. The function $P_4(\xi)$ has a local maximum value at the point γ_2 and local minimum values at the points γ_1 and γ_3.

(a) Let $\gamma_2 < 0$. It is clear that $\min_{\xi\in(\gamma_2,+\infty)} P_4(\xi) = P_4(\gamma_3)$ and function $P_4(\xi)$ is an increasing function on the interval $(\gamma_3, +\infty)$. On the other hand, we have $P_4(0) < 0$. Consequently, we obtain $P_4(\gamma_3) < 0$. It means that polynomial $P_4(\xi)$ has the unique positive root.

(b) Let $\gamma_2 > 0$ and $P_4(\gamma_2) < 0$. Then $\max_{\xi\in(\gamma_1,\gamma_3)} P_4(\xi) = P_4(\gamma_2) < 0$. Function $P_4(\xi)$ is an increasing function on the interval $(\gamma_3, +\infty)$. Then polynomial $P_4(\xi)$ has the unique positive root ξ_1 and $\xi_1 \in (\gamma_3, +\infty)$.

(c) Let $\gamma_2 > 0$ and $P_4(\gamma_3) > 0$. Then $\max_{\xi\in(\gamma_1,\gamma_3)} P_4(\xi) = P_4(\gamma_2) > 0$ and $\min_{\xi\in(\gamma_2,+\infty)} P_4(\xi) = P_4(\gamma_3) > 0$. Using inequality $P_4(0) < 0$, we obtain that polynomial $P_4(\xi)$ has the unique positive root ξ_1 and $\xi_1 \in (0, \gamma_2)$. □

Theorem 3.7. *Let $Q < 0$. If the CO $\mathcal{C}$ satisfies one of the following conditions*

(d) $\gamma_2 > 0,\quad P_4(\gamma_2) = 0,$

(e) $\gamma_2 > 0,\quad P_4(\gamma_3) = 0,$

then the CO $\mathcal{C}$ has two fixed points in $\mathbb{R}^2_>$, i.e., $N^{fix}_>(\mathcal{C}) = 2$.

Proof. (d) Let $\gamma_2 > 0$ and $P_4(\gamma_2) = 0$. Then $\max_{\xi\in(\gamma_1,\gamma_3)} P_4(\xi) = P_4(\gamma_2) = 0$ and $\xi_1 = \gamma_2$ is the root of the polynomial $P_4(\xi)$. Since $P_4(\xi)$ is an increasing function on the interval (γ_3, ∞), the polynomial $P_4(\xi)$ has a root $\xi_2 \in (\gamma_3, \infty)$ for $\gamma_3 > 0$ and $P_4(\gamma_3) < 0$. It is clear that polynomial $P_4(\xi)$ does not have any other roots in the (γ_3, ∞).

(e) Let $\gamma_2 > 0,\ \ P_4(\gamma_3) = 0$. Function $P_4(\xi)$ is an increasing function on the $(-\infty, \gamma_2)$. Then polynomial $P_4(\xi)$ has a positive root $\xi_1 \in (0, \gamma_2)$. We have $\min_{\xi\in(\gamma_2,\infty)} P_4(\xi) = P_4(\gamma_3) = 0$. Then $\xi_2 = \gamma_3$ is the second positive root of the polynomial $P_4(\xi)$. The polynomial $P_4(\xi)$ does not have another root. □

Theorem 3.8. *Let $Q < 0$. If the CO $\mathcal{C}$ satisfies the following condition*

(f) $\gamma_2 > 0,\quad P_4(\gamma_2) > 0,\quad P_4(\gamma_3) < 0,$

then the CO $\mathcal{C}$ has three fixed points in $\mathbb{R}^2_>$, i.e., $N^{fix}_>(\mathcal{C}) = 3$.

Proof. Let $\gamma_2 > 0,\ \ P_4(\gamma_2) > 0,\ \ P_4(\gamma_3) < 0$. Function $P_4(\xi)$ is an increasing function on the set $(\gamma_1, \gamma_2) \cup (\gamma_3, +\infty)$ and a decreasing function on the interval (γ_2, γ_3). Then polynomial $P_4(\xi)$ has three positive roots $\xi_1 \in (0, \gamma_2),\ \ \xi_2 \in (\gamma_2, \gamma_3)$ and $\xi_3 \in (\gamma_3, \infty)$, as $P_4(0) = -b_{11} < 0$, $P_4(\gamma_2) > 0,\ P_4(\gamma_3) < 0,\ \ P_4(+\infty) = +\infty$. □

4. Proofs of the main results

Let $\varphi_1(t),\ \varphi_2(t)$ and $\phi_1(t),\ \phi_2(t)$ are strictly positive functions that belong to $C_+[0,1]$. We consider the Hammerstein integral operator H_3 on $C_+[0,1]$ in the following form

$$(H_3 f)(t) = \int_0^1 (\phi_1(t)\varphi_1(u) + \phi_2(t)\varphi_2(u))\, f^3(u)du$$

and cubic operator $\mathcal{C}$ on $\mathbb{R}^2$ has the form

$$\mathcal{C}(x,y) = (\alpha_{11}x^3 + 3\alpha_{12}x^2y + 3\alpha_{21}xy^2 + \alpha_{22}y^3, \quad \beta_{11}x^3 + 3\beta_{12}x^2y + 3\beta_{21}xy^2 + \beta_{22}y^3).$$

Here

$$\alpha_{11} = \int_0^1 \varphi_1(u)\phi_1^3(u)du, \quad \alpha_{12} = \int_0^1 \varphi_1(u)\phi_1^2(u)\phi_2(u)du, \quad \alpha_{21} = \int_0^1 \varphi_1(u)\phi_1(u)\phi_2^2(u)du,$$

$$\alpha_{22} = \int_0^1 \varphi_1(u)\phi_2^3(u)du, \quad \beta_{11} = \int_0^1 \varphi_2(u)\phi_1^3(u)du, \quad \beta_{12} = \int_0^1 \varphi_2(u)\phi_1^2(u)\phi_2(u)du,$$

$$\beta_{21} = \int_0^1 \varphi_2(u)\phi_1(u)\phi_2^2(u)du, \qquad \beta_{22} = \int_0^1 \varphi_2(u)\phi_2^3(u)du.$$

It is clear that $\alpha_{ij} > 0$ and $\beta_{ij} > 0$ for all $i, j \in \{1, 2\}$.

Lemma 7. *The Hammerstein integral operator H_3 has a non-trivial positive fixed point iff the CO $\mathcal{C}$ has a non-trivial positive fixed point and $N_+^{fix}(H_3) = N_>^{fix}(\mathcal{C})$.*

Proof. Let the Hammerstein integral operator H_3 has a nontrivial positive fixed point $f(t) \in C_+[0,1]$. Let us introduce

$$c_1 = \int_0^1 \varphi_1(u)f^3(u)du \tag{11}$$

and

$$c_2 = \int_0^1 \varphi_2(u)f^3(u)du. \tag{12}$$

It is clear that $c_1 > 0$, $c_2 > 0$, i.e. $(c_1, c_2) \in \mathbb{R}^2_>$. Then function $f(t)$ satisfies the equality

$$f(t) = \phi_1(t)c_1 + \phi_2(t)c_2 \tag{13}$$

and $f(t) \in C_>[0,1]$.

Consequently, from (11) and (12) we have the following two identities for parameters c_1, c_2

$$c_1 = \alpha_{11}c_1^3 + 3\alpha_{12}c_1^2c_2 + 3\alpha_{21}c_1c_2^2 + \alpha_{22}c_2^3,$$

$$c_2 = \beta_{11}c_1^3 + 3\beta_{12}c_1^2c_2 + 3\beta_{21}c_1c_2^2 + \beta_{22}c_2^3.$$

Therefore, the point (c_1, c_2) is the fixed point of the CO $\mathcal{C}$.

(b) Let us assume that point (x_0, y_0) is a non-trivial positive fixed point of the CO $\mathcal{C}$ and x_0, y_0 satisfy the following equalities

$$\alpha_{11}x_0^3 + 3\alpha_{12}x_0^2y_0 + 3\alpha_{21}x_0y_0^2 + \alpha_{22}y_0^3 = x_0,$$

$$\beta_{11}x_0^3 + 3\beta_{12}x_0^2y_0 + 3\beta_{21}x_0y_0^2 + \beta_{22}y_0^3 = y_0.$$

It is easy to verify that function $f_0(t) = \phi_1(t)x_0 + \phi_2(t)y_0$ is the fixed point of the Hammerstein integral operator H_3 and $f_0(t) \in C_>[0,1]$ for $(x_0, y_0) \in \mathbb{R}^2_+$. This completes the proof. □

Taking into account potential (4), Lemma 2 and Lemma 7, the following equality holds for model (1) on the Γ^3

$$N^{tigm}(H) = N_+^{fix}(H_3) = N_>^{fix}(\mathcal{C}).$$

Using the last equality and Theorems 3.5–3.8, we obtain Theorems 2.1–2.4, respectively.

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Положительные неподвижные точки кубических операторов на $\mathbb{R}^2$ и меры Гиббса

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В этой статье мы рассматриваем модель с взаимодействиями ближайших соседей и с множеством $[0, 1]$ *значений спина на дереве Кэли третьего порядка. Трансляционно-инвариантные меры Гиббса для модели исследованы свойствами положительных неподвижных точек кубического оператора в конусе* $\mathbb{R}^2_+$.

Ключевые слова: дерево Кэли, мера Гиббса, трансляционно-инвариантные меры Гиббса, неподвижная точка, кубический оператор, интегральный оператор Гаммерштейна.