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The de Rham Cohomology through Hilbert Space Methods

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We discuss canonical representations of the de Rham cohomology on a compact manifold with boundary. They are obtained by minimising the energy integral in a Hilbert space of differential forms that belong along with the exterior derivative to the domain of the adjoint operator. The corresponding Euler-Lagrange equations reduce to an elliptic boundary value problem on the manifold, which is usually referred to as the Neumann problem after Spencer.

Introduction

When looking for a natural representation of the de Rham cohomology on a compact manifold, one uses diverse homotopy formulas for differential forms. They are of the form $u = Ru + P(du) + d(Pu)$, so that $du = 0$ implies $d(Ru) = 0$ and the cohomology classes of u and Ru coincide. While being a homomorphism of the de Rham complex, R fails to induce any mapping of the de Rham cohomology into closed differential forms unless it vanishes on exact differential forms. If π is a projection onto the subspace of exact forms, then $P \circ (1 - \pi)$ already acts on cohomology classes since

$$u = R(1 - \pi)u + P(du) + d(Pu + Rd^{-1}\pi u) \tag{0.1}$$

for all differential forms u. This is precisely the problem treated by the second author in his bachelor thesis advised by Lev A. Aizenberg, see [12]. In the 1970 s, Aizenberg encouraged his students Sh. Dautov, A. Kytmanov and others to study the Dolbeault cohomology of complex manifolds.

In general there is no canonical projection onto the subspace of exact forms. The classical approach to this topic invokes Hilbert space methods of elliptic theory. The variational formulation of the equation $du = f$ consists in minimising the so-called energy functional

$$F(u) = \frac{1}{2} \left(\|du\|^2 + \|d^*u\|^2 \right) - \Re (f, u)$$

over the subspace of all square integrable forms u such that both u and du belong to the domain of the adjoint operator for d, see [6]. The minimal solution is thought of as canonical. In the case of compact closed manifolds it is given by the classical Hodge theory, see [13, 15].

On compact manifolds with boundary the Euler-Lagrange equations for the energy functional constitute what is usually referred to as the Neumann problem. To the best of our knowledge, this

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boundary value problem was first formulated explicitly in [11] within the more general context of complexes of differential operators.

For the de Rham complex, a coerciveness estimate in the Neumann problem after Spencer was actually proved in various settings in the middle 1950 s. This was already sufficient to conclude on the Fredholm solvability and regularity of the problem. However, no one has given a direct proof of the ellipticity of the Neumann problem within the Boutet de Monvel algebra of boundary value problems, cf. Example 4.1.28 in [13].

The analysis of the Neumann problem after Spencer was undertaken by his PhD student W. Sweeney in a series of papers. In [10] he derived an algebraic condition for coerciveness in the problem. This condition is fulfilled for the de Rham complex, implying the ellipticity of the Neumann problem.

The study of W. Sweeney was well motivated all the more so since the Neumann problem for the de Dolbeault complex had been proved to be subelliptic, see [4,7]. The latter paper gave rise to [5] where subelliptic estimates in the Neumann problem were studied within the framework of general complexes of differential operators.

The purpose of this paper is to give a systematic presentation of Hodge theory for the de Rham complex which is based on the Neumann problem after Spencer. When using Hilbert space methods, we choose the L^2 setting of classical variational calculus. The same technique is known to apply in the setting of Sobolev spaces, see [2]. Although this work leads to new minimal solutions to the inhomogeneous equation $du = f$, no surprising phenomena are found there while the presentation is voluminous.

1.   Representation of the de Rham cohomology

Suppose that $\mathcal{X}$ is a compact C^∞ manifold with boundary of dimension n. Consider the de Rham complex

$$0 \to \Omega^0(\mathcal{X}) \overset{d}{\to} \Omega^1(\mathcal{X}) \overset{d}{\to} \ldots \overset{d}{\to} \Omega^n(\mathcal{X}) \to 0 \tag{1.2}$$

on $\mathcal{X}$, where $\Omega^i(\mathcal{X})$ stands for the space of all differential forms of degree i with C^∞ coefficients on $\mathcal{X}$ and d for the exterior differentiation of forms. We have $d^2 = d \circ d = 0$.

Given any $f \in \Omega^i(\mathcal{X})$, the question of solvability of the inhomogeneous equation $du = f$ is of crucial importance in analysis and geometry. Under what conditions on f does there exist a form $u \in \Omega^{i-1}(\mathcal{X})$ satisfying $du = f$, and how does u depend on f? A necessary condition on f follows immediately from the integration by parts formula.

To formulate it we endow the manifold $\mathcal{X}$ with a Riemannian metric. The induced metric on the bundle of exterior forms of degree i is denoted by $(f, g)_x$, where $x \in \mathcal{X}$. Set

$$(f, g) := \int_{\mathcal{X}} (f, g)_x dx,$$

where dx is a volume form on $\mathcal{X}$.

Lemma 1.1. *In order that equation $du = f$ may be solvable, it is necessary that $(f, h) = 0$ for all $h \in \Omega^i(\mathcal{X})$ satisfying $d^* f = 0$ in $\mathcal{X}$ and $n(h) = 0$ on $\partial \mathcal{X}$.*

Here, d^* stands for the formal adjoint of d and $n(f)$ for the normal part of f on the boundary, see [13, 3.2.2].

Proof. Indeed, if $u \in \Omega^{i-1}(\mathcal{X})$ and $h \in \Omega^i(\mathcal{X})$, then we get by Stokes' formula

$$(du, g) = (u, d^*g) + \int_{\partial \mathcal{X}} (t(u), n(g))_x ds, \tag{1.3}$$

where $t(u)$ is the tangential part of u and ds the area form on $\partial\mathcal{X}$. From (1.3) the lemma follows. $\qquad\square$

On choosing $h = d^*w$ with $w \in \Omega^{i+1}(\mathcal{X})$ of compact support in the interior of $\mathcal{X}$ we see that $df = 0$, i.e., f is a closed form. Write $\mathcal{H}^i(\mathcal{X})$ for the subspace of $\Omega^i(\mathcal{X})$ consisting of all h such that $dh = 0$ and $d^*h = 0$ in $\mathcal{X}$ and $n(h) = 0$ on $\partial\mathcal{X}$. The differential forms of $\mathcal{H}^i(\mathcal{X})$ are said to be harmonic. This concept generalise that of harmonic forms on a compact closed manifold, see for instance [13, 15] and elsewhere.

Lemma 1.2. *The natural mapping $\mathcal{H}^i(\mathcal{X}) \hookrightarrow H^i_{\mathrm{dR}}(\mathcal{X})$ is injective.*

Proof. Assume that $f \in \mathcal{H}^i(\mathcal{X})$ and $f = du$ for some form $u \in \Omega^{i-1}(\mathcal{X})$. Then we get

$$\|f\|^2 = (u, d^*f) = 0,$$

i.e., $f = 0$. $\qquad\square$

In connection with the problem of global solvability of overdetermined inhomogeneous systems of differential equations Spencer [11] suggested an approach that in certain cases allowed one to prove the surjectivity of the natural mapping $\mathcal{H}^i(\mathcal{X}) \hookrightarrow H^i_{\mathrm{dR}}(\mathcal{X})$. It may be thought of as an attempt to extend Hodge's theory for elliptic complexes on compact closed manifolds to the case of compact manifolds with boundary. The second order differential operator $\Delta = d^*d + dd^*$ taking forms of degree i into forms of the same degree is called the Laplacian. It is elliptic and formally selfadjoint. For $i = 0$, this is precisely the Laplace-Beltrami operator in differential geometry.

Lemma 1.3. *The space $\mathcal{H}^i(\mathcal{X})$ just amounts to the subspace of $\Omega^i(\mathcal{X})$ which consists of all forms u satisfying $\Delta u = 0$ in $\mathcal{X}$, and $n(u) = 0$ and $n(du) = 0$ on the boundary.*

Proof. We only need to show that if $u \in \Omega^i(\mathcal{X})$ satisfies $\Delta u = 0$ in $\mathcal{X}$, and $n(u) = 0$ and $n(du) = 0$ on $\partial\mathcal{X}$, then $du = 0$ and $d^*u = 0$ in $\mathcal{X}$. To this end, use integration by parts as in (1.3), showing

$$0 = (\Delta u, u) = \|du\|^2 + \|d^*u\|^2,$$

and the lemma follows. $\qquad\square$

By the Neumann problem after Spencer is meant the following boundary value problem in $\mathcal{X}$. Given a form $f \in \Omega^i(\mathcal{X})$, under what conditions does there exist a form $u \in \Omega^i(\mathcal{X})$ satisfying

$$\begin{cases} \Delta u &=& f & \text{in} & \mathcal{X}, \\ n(u) &=& 0 & \text{on} & \partial\mathcal{X}, \\ n(du) &=& 0 & \text{on} & \partial\mathcal{X}, \end{cases} \qquad (1.4)$$

and how does u depend on f?

Lemma 1.4. *For the existence of a solution $u \in \Omega^i(\mathcal{X})$ to problem (1.4) it is necessary that the form f should satisfy $(f, h) = 0$ for all $h \in \mathcal{H}^i(\mathcal{X})$.*

Proof. As is known, for problem (1.4) to be solvable, it is necessary that f should be orthogonal to the space of solutions of the homogeneous boundary value problem which is (formal) adjoint to (1.4) with respect to a Green formula. An easy verification based on formula (1.3) shows that (1.4) is formally selfadjoint. It remains to invoke Lemma 1.3. $\qquad\square$

Suppose that the necessary condition of Lemma 1.4 is also sufficient, i.e., problem (1.4) has a solution $u \in \Omega^i(\mathcal{X})$ for each form $f \in \Omega^i(\mathcal{X})$ orthogonal to $\mathcal{H}^i(\mathcal{X})$. Then, given any $f \in \Omega^i(\mathcal{X})$, the inhomogeneous equation $du = f$ has a solution $u \in \Omega^{i-1}(\mathcal{X})$ if and only if

$$\begin{cases} df & = & 0 \quad\quad \text{and} \\ f & \perp & \mathcal{H}^i(\mathcal{X}). \end{cases} \tag{1.5}$$

To see this, we only need to show that for any $f \in \Omega^i(\mathcal{X})$ satisfying (1.5) there is a $u \in \Omega^{i-1}(\mathcal{X})$ such that $du = f$. By hypothesis, there exists a form $w \in \Omega^i$ such that $\Delta w = f$ in $\mathcal{X}$, and $n(w) = 0$ and $n(dw) = 0$ on the boundary. If we have in addition $df = 0$, then the equality $\Delta w = f$ implies $dd^*dw = 0$ because $d^2 = 0$. Since $n(dw) = 0$, we derive from here that $\|d^*dw\|^2 = (dd^*dw, dw) = 0$ whence $d^*dw = 0$. Using now the equality $n(dw) = 0$ again we obtain $\|dw\|^2 = (w, d^*dw) = 0$, and so $dw = 0$. Hence it follows that $f = dd^*w$, i.e., $u = d^*w$ is a solution of $du = f$, as desired. Note that the solution $u = d^*w$ constructed in this way is minimal in the sense that it is orthogonal to the subspace of $\Omega^{i-1}(\mathcal{X})$ consisting of all closed forms.

Theorem 1.5. *Assume that problem (1.4) has a solution $u \in \Omega^i(\mathcal{X})$ for each form $f \in \Omega^i(\mathcal{X})$ orthogonal to $\mathcal{H}^i(\mathcal{X})$. If moreover the space $\mathcal{H}^i(\mathcal{X})$ is finite dimensional, then*

$$H^i_{\mathrm{dR}}(\mathcal{X}) \overset{top}{\cong} \mathcal{H}^i(\mathcal{X}).$$

Proof. First we notice that the space $\mathcal{H}^i(\mathcal{X})$ is finite dimensional if and only if it is closed in $L^2(\mathcal{X}, \Lambda^i)$, where Λ^i is the vector bundle of exterior forms of degree i over $\mathcal{X}$. Indeed, as far as the sufficiency is concerned, we may apply the open mapping theorem to conclude that two topologies in $\mathcal{H}^i(\mathcal{X})$ induced from $\Omega^i(\mathcal{X})$ and $L^2(\mathcal{X}, \Lambda^i)$, respectively, coincide. Hence it follows by the Arzela-Ascoli theorem that the identity mapping of $\mathcal{H}^i(\mathcal{X})$ is compact. Fix now any $f \in \Omega^i(\mathcal{X})$. Denote by Hf the orthogonal projection of f into the space $\mathcal{H}^i(\mathcal{X})$. Then the difference $f - Hf$ is orthogonal to the space $\mathcal{H}^i(\mathcal{X})$, so there is a form $w \in \Omega^i(\mathcal{X})$ such that $\Delta w = f - Hf$ in $\mathcal{X}$, and $n(w) = 0$ and $n(dw) = 0$ on the boundary. Therefore, we get

$$f = Hf + d^*dw + dd^*w$$

on $\mathcal{X}$. If $df = 0$, then $dd^*dw = 0$. This implies just as above that $dw = 0$. It follows that $f = Hf + d(d^*w)$, so the natural mapping $\mathcal{H}^i(\mathcal{X}) \hookrightarrow H^i_{\mathrm{dR}}(\mathcal{X})$ is surjective. We now use Lemma 1.2 to see that the natural mapping is an isomorphism of vector spaces without topologies. By the above, the space $H^i(\mathcal{X})$ is separated, and so the natural mapping is actually a homeomorphism by the open mapping theorem. This finishes the proof. $\square$

2.   Variational approach

The solution $u = Gf$ to the Neumann problem after Spencer constructed in Section 1. is minimal in the sence that it is orthogonal to solutions of the corresponding homogeneous problem. It has minimal energy $\|du\|^2 + \|d^*u\|^2 = (f, u)$, as is easy to see.

To recover this solution within the framework of variational calculus, consider the problem of local minima of the functional

$$F(u) = \frac{1}{2} \left(\|du\|^2 + \|d^*u\|^2 \right) - \Re(f, u) \tag{2.6}$$

over the space of all $u \in H^1(\mathcal{X}, \Lambda^i)$ satisfying $n(u) = 0$ on $\partial\mathcal{X}$. Pick any form $v \in H^1(\mathcal{X}, \Lambda^i)$ such that $n(v) = 0$. If F takes on a local minimum at an admissible form u, then the Gâteaux derivative F'_v of F in direction v vanishes at u. It follows that

$$\frac{d}{dt}F(u + tv)\,|_{t=0} = \Re\left((du, dv) + (d^*u, d^*v) - (f, v)\right) = 0.$$

On substituting iv for v we readily conclude that also the imaginary part of the expression on the right-hand side vanishes. Hence

$$(du, dv) + (d^*u, d^*v) = (f, v) \tag{2.7}$$

for all $v \in H^1(\mathcal{X}, \Lambda^i)$ satisfying $n(v) = 0$. This is the weak form of Euler-Lagrange equations for functional (2.6).

Assuming that u is moreover of Sobolev class $H^2(\mathcal{X}, \Lambda^i)$, we can use the integration by parts formula, to get

$$(du, dv) + (d^*u, d^*v) = (\Delta u, v) + \int_{\partial\mathcal{X}} (n(du), t(v))_x ds = (f, u).$$

On choosing v with compact support in the interior of $\mathcal{X}$ we conclude that $\Delta u = f$ in $\mathcal{X}$. Therefore,

$$\int_{\partial\mathcal{X}} (n(du), t(v))_x ds = 0$$

for all differential forms $v \in H^1(\mathcal{X}, \Lambda^i)$ such that $n(v) = 0$, and so $n(du) = 0$ on the boundary. We thus get

Theorem 2.1. *Let* $f \in L^2(\mathcal{X}, \Lambda^i)$. *If* u *is a solution of class* $H^2(\mathcal{X}, \Lambda^i)$ *to the variational problem* $F(u) \mapsto \min$, *then* u *satisfies the Neumann problem after Spencer of (1.4).*

This theorem shows that the Neumann problem after Spencer just amounts to the Euler-Lagrange equations for functional (2.6).

At extreme steps the Neumann problem after Spencer is quite classical. At step 0 it coincides with the usual Neumann problem for the inhomogeneous Laplace-Beltrami operator. The Neumann problem at step n amounts to the classical Dirichlet problem. Simple calculations show that the Neumann problem at an arbitrary step i reduces locally to the Dirichlet and Neumann problems for coefficients of a differential form u. Thus the Neumann problem is a regular elliptic boundary value problem and it may be investigated by standard techniques. By ellipticity is meant the ellipticity in the Boutet de Monvel algebra of boundary value problems on $\mathcal{X}$, see [1]. Along with the ellipticity of the Laplace operator in the interior of $\mathcal{X}$ this requires the invertibility of a boundary symbol. The latter property admits an equivalent algebraical description which is usually referred to as Shapiro-Lopatinskij condition. Analytically it implies a coercive estimate for the functional F which is a far reaching generalisation of A. Korn's (1908) inequality. Such an estimate for differential forms on compact manifolds with boundary was first proved in [3].

Lemma 2.2. *There are positive constants c and k with the property that*

$$\|du\|^2 + \|d^*u\|^2 \geqslant c \|u\|^2_{H^1(\mathcal{X}, \Lambda^i)} - k \|u\|^2 \tag{2.8}$$

for all $u \in H^1(\mathcal{X}, \Lambda^i)$ satisfying $n(u) = 0$ on the boundary of $\mathcal{X}$.

Proof. See Theorem 4.2 in [6] which is a simple consequence of Gaffney's inequality [3]. $\square$

Since the Neumann problem after Spencer is elliptic in Boutet de Monvel's agebra on $\mathcal{X}$, it possesses a parametrix in the algebra. When focused on the left upper corners of (2×2)-matrices in the algebra, this means that there is a pseudodifferential operator G' of order -2 on $\mathcal{X}$, which maps $L^2(\mathcal{X}, \Lambda^i)$ into the subspace of $H^2(\mathcal{X}, \Lambda^i)$ consisting of all forms u with $n(u) = 0$ and $n(du) = 0$ on $\partial\mathcal{X}$, and satisfies $G'\Delta = I$ and $\Delta G' = I$ up to operators of order -1 on $\mathcal{X}$. But we will not develop this point here.

3.   Weak form of the Neumann problem

By the above, (2.7) is a weak formulation of the Neummann problem. If the functional F takes on a local minimum at a form $u \in H^1(\mathcal{X}, \Lambda^i)$, then equality (2.7) is fulfilled, and so f is orthogonal to all $h \in H^1(\mathcal{X}, \Lambda^i)$ satisfying $dh = 0$ and $d^*h = 0$ in $\mathcal{X}$, and $n(h) = 0$ on $\partial \mathcal{X}$.

To handle problem (2.7) we introduce as usual the so-called Dirichlet norm on $\Omega^i(\mathcal{X})$ by

$$D(u) = \left(\|du\|^2 + \|d^*u\|^2 + \|u\|^2 \right)^{1/2}.$$

Denote by $\mathcal{D}^i$ the completion of the space of all $u \in \Omega^i$, such that $n(u) = 0$ on $\partial \mathcal{X}$, under the Dirichlet norm. Since $D(u) \geqslant \|u\|$, we can identify $\mathcal{D}^i$ with a subspace of $L^2(\mathcal{X}, \Lambda^i)$. Lemma 2.2 shows readily that $\mathcal{D}^i$ coincides with the domain of the energy functional F.

Let $f \in L^2(\mathcal{X}, \Lambda^i)$. We look for a differential form $u \in \mathcal{D}^i$ which satisfies the equality $(du, dv) + (d^*u, d^*v) = (f, v)$ for all $v \in \mathcal{D}^i$. As mentioned above, if $u \in \mathcal{D}^i$ is a solution to (2.7), then $\Delta u = f$ weakly in the interior of $\mathcal{X}$. If $w \in H^1(\mathcal{X}, \Lambda^{i+1})$ and $(d^*w, v) = (w, dv)$ for all $v \in \mathcal{D}^i$, then $n(w) = 0$ on the boundary, and conversely. However, this is no longer true for differential forms of the form $w = du$, where $u \in \mathcal{D}^i$, for $n(du)$ need not be defined on the boundary. (The normal part $n(du)$ can be still defined for those $u \in \mathcal{D}^i$ which satisfy $d^*du = 0$ weakly in the interior of $\mathcal{X}$.)

Lemma 3.1. *If $u \in \mathcal{D}^i \cap H^2(\mathcal{X}, \Lambda^i)$ is a solution to equation (2.7), then $n(du) = 0$ on $\partial \mathcal{X}$.*

Proof. Indeed, using (2.7) yields

$$
\begin{aligned}
(d^* du, v) &= (\Delta u, v) - (d d^* u, v) = \\
&= (f, v) - (d^* u, d^* v) = \\
&= (du, dv)
\end{aligned}
$$

for all $v \in \mathcal{D}^i$, and the lemma follows.   □

Thus, if $f \in \Omega^i(\mathcal{X})$ and for a solution $u \in \mathcal{D}^i$ of (2.7) we could prove the regularity up to the boundary, i.e., $u \in \Omega^i(\mathcal{X})$, then u would be a solution of problem (1.4).

Lemma 3.2. *The space of solutions of the homogeneous equation, corresponding to (2.7), is finite-dimensional.*

Proof. Denote the space in question by H^i, i.e., H^i consists of all forms $u \in \mathcal{D}^i$ satisfying (2.7) with $f = 0$. This is a closed subspace of $\mathcal{D}^i$ because

$$|(du, dv) + (d^*u, d^*v)| \leqslant D(u)\, D(v)$$

for all $u, v \in \mathcal{D}^i$. Hence, when endowed with the Dirichlet norm, H^i is a Banach space. If $u \in H^i$, then choosing $v = u$ in (2.7) yields $\|du\|^2 + \|d^*u\|^2 = 0$, i.e., $du = 0$ and $d^*u = 0$ in $\mathcal{X}$. It follows that the Dirichlet norm on H^i coincides with the L^2-norm. By Lemma 2.2,

$$\|u\|^2_{H^1(\mathcal{X}, \Lambda^i)} \leqslant \frac{k}{c} \|u\|^2$$

for all $u \in H^i$. Thus, the topology on H^i induced from $H^1(\mathcal{X}, \Lambda^i)$ coincides with that induced from $L^2(\mathcal{X}, \Lambda^i)$. Since the embedding $H^1(\mathcal{X}, \Lambda^i) \hookrightarrow L^2(\mathcal{X}, \Lambda^i)$ is compact, which is due to Rellich's theorem, we conclude that the identity mapping of H^i is compact.   □

In fact problem (2.7) is hypoelliptic in the sense that. for each right-hand side $f \in \Omega^i(\mathcal{X})$, all solutions $u \in \mathcal{D}^i$ of the problem are infinitely differentiable up to the boundary in $\mathcal{X}$. More

precisely, if $f \in H^s(\mathcal{X}, \Lambda^i)$ with a nonnegative integer s, then every solution $u \in \mathcal{D}^i$ to problem (2.7) is of Sobolev class $H^{s+1}(\mathcal{X})$. Since the scale of Sobolev spaces shrinks to $C^\infty(\mathcal{X})$, the above assertion follows. As usual, the proof of regularity in Sobolev spaces is very technical. We dwell on regularity in Section 5. Having disposed of this step, we can now proceed with constructing a canonical solution to (2.7).

Corollary 3.3. *The space of solutions of the homogeneous equation corresponding to weak problem (2.7) coincides with $\mathcal{H}^i(\mathcal{X})$.*

Proof. By hypoellipticity, if $u \in \mathcal{D}^i$ is a solution to (2.7), then $v \in \Omega^i(\mathcal{X})$. On applying Lemma 3.1 we see that $n(du) = 0$ on $\partial\mathcal{X}$. To complete the proof it suffices to apply Lemma 1.3. $\qquad\square$

By Lemma 3.2, the space $\mathcal{H}^i(\mathcal{X})$ is finite-dimensional. Hence, it is a closed subspace of $L^2(\mathcal{X}, \Lambda^i)$. The orthogonal projection H of $L^2(\mathcal{X}, \Lambda^i)$ onto $\mathcal{H}^i(\mathcal{X})$ is given by

$$Hu = \sum_{j=1}^{\beta_i} (u, h_j^i)\, h_j^i$$

for $u \in L^2(\mathcal{X}, \Lambda^i)$, where $\{h_j^i\}_{j=1,\ldots,\beta_i}$ is an orthonormal basis in $\mathcal{H}^i(\mathcal{X})$. This is a smoothing operator.

Theorem 3.4. *Given any form $f \in L^2(\mathcal{X}, \Lambda^i)$ orthogonal to $\mathcal{H}^i(\mathcal{X})$, the variational problem $F(u) \to \min$ has a uniques solution $u \in \mathcal{D}^i$ orthogonal to $\mathcal{H}^i(\mathcal{X})$.*

As the Euler-Lagrange equations show, the condition $f \perp \mathcal{H}^i(\mathcal{X})$ is also necessary for the variational problem to possess a solution $u \in \mathcal{D}^i$.

Proof. We first observe that if $u \in \mathcal{D}^i$ is a solution to the variational problem and $h \in \mathcal{H}^i(\mathcal{X})$ an arbitrary form then $u + h$ is also a solution, for

$$
\begin{aligned}
F(u + h) &= \frac{1}{2}\left(\|d(u+h)\|^2 + \|d^*(u+h)\|^2\right) - \Re\,(f, u+h) = \\
&= F(u),
\end{aligned}
$$

as desired. On the other hand, if F takes on a local minimum at two forms $u_1, u_2 \in \mathcal{D}^i$, then the Euler-Lagrange equations imply $(du, dv) + (d^*u, d^*v) = 0$ for all $v \in \mathcal{D}^i$, where $u := u_2 - u_1$. Taking $v = u$ we deduce that $du = 0$ and $d^*u = 0$, i.e., $u \in \mathcal{H}^i(\mathcal{X})$. So, $u_2 = u_1 + h$, where $h \in \mathcal{H}^i(\mathcal{X})$. It follows that the problem $F(u) \to \min$ has at most one solution orthogonal to $\mathcal{H}^i(\mathcal{X})$. We now proceed as follows.

Write $\mathcal{R}^i$ for the orthogonal complement of $\mathcal{H}^i(\mathcal{X})$ in $\mathcal{D}^i$. We give $\mathcal{R}^i$ the Hilbert space structure induced from $\mathcal{D}^i$, i.e., that determined by the Dirichlet scalar product $D(u, v) = (du, dv) + (d^*u, d^*v) + (u, v)$. Since $\mathcal{H}^i$ is closed in the norm $D(u)$, we get $\mathcal{D}^i = \mathcal{H}^i(\mathcal{X}) \oplus \mathcal{R}^i$.

The restriction of the energy functional F to $\mathcal{R}^i$ is continuous. Our next objective is to show that F is bounded from below on $\mathcal{R}^i$. To this end, we invoke the estimate

$$\|u\|^2 \leqslant C\left(\|du\|^2 + \|d^*u\|^2\right)$$

for all $u \in \mathcal{R}^i$, with C a constant independent of u. This estimate is a consequence of Gaffney's inequality (2.8) and the closed graph theorem, cf. Lemma 4.2.15 in [13]. Therefore,

$$
\begin{aligned}
F(u) &\geqslant \frac{1}{2C}\,\|u\|^2 - \|f\|\,\|u\| = \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{C}}\|u\| - \sqrt{C}\,\|f\|\right)^2 - \frac{C}{2}\,\|f\|^2
\end{aligned}
\tag{3.9}
$$

for all $u \in \mathcal{R}^i$.

By (3.9), F is bounded from below on $\mathcal{R}^i$ by $-(C/2)\|f\|^2$. Set

$$m = \inf_{u \in \mathcal{R}^i} F(u)$$

and choose any minimising sequence $\{u_\nu\}$ in $\mathcal{R}^i$, i.e., $F(u_\nu) \to m$ as $\nu \to \infty$. The sequence $\{F(u_\nu)\}$ is bounded. Using (3.9) once again we see that $\{u_\nu\}$ is bounded in $L^2(\mathcal{X}, \Lambda^i)$, for

$$\|u\| \leqslant C\|f\| + \sqrt{(C\|f\|)^2 + 2C\,F(u)}$$

whenever $u \in \mathcal{R}^i$. Substituting this estimate into Gaffney's inequality (2.8) yields readily $F(u_\nu) \geqslant c\|u_\nu\|_{H^1(\mathcal{X}, \Lambda^i)}^2 - Q$ for all $\nu = 1, 2, \ldots$, where $Q > 0$ is a constant independent of ν. Thus, the sequence $\{u_\nu\}$ is actually bounded in $H^1(\mathcal{X}, \Lambda^i)$, and the rest of the proof runs within the direct methods of variational calculus, see [8]. $\qquad\square$

Since (2.7) constitute the Euler-Lagrange equations of the variational problem $F(y) \mapsto \min$, Theorem 3.4 concerns the solvability of the weak Neumann problem after Spencer, too.

Corollary 3.5. *There is a bounded linear operator G in $L^2(\mathcal{X}, \Lambda^i)$ with the property that*
1) G maps $L^2(\mathcal{X}, \Lambda^i)$ continuously into $\mathcal{D}^i$, preserves the property of being C^∞ up to the boundary, and satisfies $HG = 0$ and $GH = 0$.
2) Each form $f \in L^2(\mathcal{X}, \Lambda^i)$ splits weakly as the sum of three pairwise orthogonal terms

$$f = Hf + d^*dGf + dd^*Gf. \qquad (3.10)$$

In classical papers the operator G was referred to as the Green operator. In [4] and subsequent papers G is called the Neumann operator (denoted by N). When acting on $f \in L^2(\mathcal{X}, \Lambda^i)$, the operator G need not fulfill $n(dGf) = 0$ on the boundary. This is precisely what is meant by 'weakly' in *2)*. The decomposition of (3.10) just amounts to saying that Gf satisfies (2.7) with $f - Hf$ in place of f on the right-hand side. If $f \in H^1(\mathcal{X}, \Lambda^i)$, then Gf is of Sobolev class $H^2(\mathcal{X})$, and so $n(dGf) = 0$ by Lemma 3.1. In this case all of three summands in (3.10) are pairwise orthogonal.

Proof. Pick any $f \in L^2(\mathcal{X}, \Lambda^i)$. The difference $f - Hf$ is orthogonal to $\mathcal{H}^i(\mathcal{X})$. Therefore, there is a form $u \in \mathcal{D}^i$ satisfying equation (2.7) with $f - Hf$ in place of f. Set $Gf = u - Hu$. This operator is well defined because, if u_1 and u_2 are two solutions of (2.7) with $f - Hf$ in place of f, then the difference $u = u_2 - u_1$ is in $\mathcal{H}^i(\mathcal{X})$ whence

$$(u_2 - Hu_2) - (u_1 - Hu_1) = u - Hu = 0.$$

Clearly, $Gf \in \mathcal{D}^i$ satisfies equation (2.7) with $f - Hf$ in place of f. Moreover, we get both $HG = 0$ and $GH = 0$ by the very construction.

Consider now two Hilbert spaces $U = L^2(\mathcal{X}, \Lambda^i) \ominus \mathcal{H}^i(\mathcal{X})$, endowed with the norm $\|\cdot\|$, and $V = \mathcal{D}^i$, endowed with the norm $D(\cdot)$, and the mapping $M : U \to V$ defined by $M(f) = u$, where $u \in \mathcal{D}^i$ is the solution of (2.7) orthogonal to $\mathcal{H}^i(\mathcal{X})$. The graph of M is easily verified to be closed in $U \times V$. By the closed graph theorem, M is bounded, i.e., there is a constant $c > 0$ such that $D(Mf) \leqslant c\|f\|$ for all $f \in \mathcal{R}^i$. On applying this estimate to the form $f - Hf$ we conclude readily that

$$D(Gf) \leqslant c\|f - Hf\| \leqslant c\|f\|,$$

i.e., G is a bounded operator from $L^2(\mathcal{X}, \Lambda^i)$ to $\mathcal{D}^i$.

By construction, Gf is the unique solution in $\mathcal{D}^i \cap (\mathcal{H}^i(\mathcal{X}))^\perp$ to the equation $(du, dv) + (d^*u, d^*v) = (f - Hf, v)$ for all $v \in \mathcal{D}^i$. If $f \in \Omega^i(\mathcal{X})$, then the difference $f - Hf$ is in $\Omega^i(\mathcal{X})$. Since problem (2.7) is hypoelliptic, it follows that Gf is also in $\Omega^i(\mathcal{X})$. We now use Lemma 3.1 to see that Gf satisfies not only $n(Gf) = 0$ but also $n(dGf) = 0$ on $\partial\mathcal{X}$. $\qquad\square$

We put off studying further properties of the Green operator G to the next section. Meanwhile, we notice that decomposition (3.10) allows one to get a continuous homotopy between the identity and H endomorphisms of the de Rham complex. Such a homotopy is given by the operator $P = d^*G$ because $df = 0$ implies $dGf = 0$ for all $f \in \Omega^i(\mathcal{X})$.

4.   The Green operator

While the variational approach leads to the weak formulation of the Neumann problem after Spencer, the direct study of problem (1.4) by Hilbert space methods suggests to consider the closure of the Laplace operator under stronger boundary conditions $n(u) = 0$ and $n(du) = 0$. To the best of our knowledge the direct study first appeared in [3].

Set $\mathcal{L}^i = L^2(\mathcal{X}, \Lambda^i)$. Denote by T the maximal operator from $\mathcal{L}^i$ into $\mathcal{L}^{i+1}$ generated by the exterior derivative acting from $\Omega^i(\mathcal{X})$ into $\Omega^{i+1}(\mathcal{X})$. This is an unbounded closed densely defined operator, and so its adjoint $T^* : \mathcal{L}^{i+1} \to \mathcal{L}^i$ is well defined. Write $\mathcal{D}_{T^i}$ and $\mathcal{D}_{T^{i*}}$ for the domains of these operators acting on $\mathcal{L}^i$ and $\mathcal{L}^{i+1}$, respectively. We introduce an operator L on $\mathcal{L}^i$ with a domain $\mathcal{D}_{L^i}$ which has the property that $Lu = \Delta u$ for all $u \in \mathcal{D}_{L^i} \cap \Omega^i(\mathcal{X})$. Namely, let $\mathcal{D}_{\mathcal{L}^i}$ be the space of all $u \in \mathcal{D}_{T^i} \cap \mathcal{D}_{T^{i-1*}}$ such that $Tu \in \mathcal{D}_{T^{i*}}$ and $T^*u \in \mathcal{D}_{T^{i-1}}$. Then L is defined by $Lu = T^*Tu + TT^*u$. The Neumann problem after Spencer reads as follows. Given a form $f \in \mathcal{L}^i$, is there $u \in \mathcal{D}_{L^i}$ such that $Lu = f$, and how does u depend on f? Gaffney's inequality applies to show that $\mathcal{D}_{L^i}$ actually consists of all $u \in H^2(\mathcal{X}, \Lambda^i)$ satisfying $n(u) = 0$ and $n(du) = 0$ on the boundary. Moreover, the range $L\mathcal{D}_{L^i}$ is closed for all $i = 0, 1, \ldots, n$. This is typical for elliptic boundary value problems.

Let $f \in \mathcal{L}^i$. Then we get $f = Hf + Lu$, where $u \in \mathcal{D}_{L^i}$. The Green operator $G : \mathcal{L}^i \to \mathcal{D}_{L^i}$ is defined by $Gf = u - Hu$. Obviously, G is well defined. The properties of the Green operator generalise those of the Green operator from Hodge theory, see for instance [15].

Theorem 4.1. *As defined above, the Green operator G possesses the following properties:*

1) G is bounded, selfadjoint, $HG = 0$ and $GH = 0$, and each $f \in \mathcal{L}^i$ admits the orthogonal decomposition $f = Hf + T^*TGf + TT^*Gf$.

2) If $f \in \mathcal{D}_{T^i}$ and $Tf = 0$, then $TGf = 0$. Moreover, G and T commute on the domain of T^i.

3) If $f \in \mathcal{D}_{T^{i-1*}}$ and $T^*f = 0$, then $T^*Gf = 0$. Moreover, G and T^* commute on the domain of T^{i-1*}.

Proof. 1) Both equalities $HG = 0$ and $GH = 0$ and the orthogonal decomposition follow immediately from the definition of the Green operator. Further, by the closed graph theorem there exists a constant $c > 0$ with the property that $\|Lu\| \geqslant c\|u\|$ for all $u \in \mathcal{D}_{L^i}$ which are orthogonal to $\mathcal{H}^i(\mathcal{X})$. Applying this estimate to Gf, we obtain

$$\|Gf\| \leqslant \frac{1}{c}\|LGf\| = \frac{1}{c}\|f - Hf\| \leqslant \frac{1}{c}\|f\|$$

for all $f \in \mathcal{L}^i$. Hence G is bounded. Finally, the selfadjointness of G follows immediately from that of L because

$$(Gf, g) = (Gf, Hg + LGg) = (Gf, LGg) = (LGf, Gg) = (f, Gg)$$

whenever $g \in \mathcal{L}^i$.

2) Suppose $f \in \mathcal{D}_{T^i}$ satisfies $Tf = 0$. From $T\mathcal{D}_{T^i} \subset \mathcal{D}_{T^{i+1}}$ and the orthogonal decomposition we get $T^*TGf \in \mathcal{D}_{T^i}$, and so $Tf = 0$ implies $TT^*TGf = 0$. Hence it easily follows that $TGf = 0$. Moreover, for any form $f \in \mathcal{D}_{T^i}$, we obtain $Tf = TT^*TGf$ on the one hand, and $Tf = TT^*GTf$ on the other hand. Therefore, we get $L(TGf - GTf) = 0$, and since $TGf - GTf$ is orthogonal to $\mathcal{H}^{i+1}(\mathcal{X})$ then $TGf - GTf = 0$, as desired.

3) Part 3) is proved by analogy with part 2). $\qquad\square$

The spectral invariance of Boutet de Monvel's algebra shows that G is an operator of order -2 in the algebra, see [9]. Since the operators G and T commute, the orthogonal decomposition of *1)* becomes $f = Hf + (T^*G)Tf + T(T^*G)f$ for all $f \in \mathcal{D}_{T^i}$. Thus, on sufficiently smooth forms $P = T^*G$ is a very special parametrix of the de Rham complex.

5.   Spectral problems

Let U be a complex Hilbert space of functions on $\mathcal{X}$ and A a mapping of U with domain $\mathcal{D}_A$. Given any $f \in U$, we look for a solution $u \in \mathcal{D}_A$ to the equation $Au = f$ under an additional condition $Bu = 0$. This latter can be e.g. a boundary condition. From the viewpoint of variational calculus the condition $Bu = 0$ should determine a convex subset of $\mathcal{D}_A$. If allowing convex domains we can include the additional condition into the definition of $\mathcal{D}_A$. Then one can pass on to a variational formulation of the problem and invoke the Euler-Lagrange equations for its study. Consider the eigenvalue problem $Au = f + \lambda u$ under the same additional conditions $u \in \mathcal{D}_A$, where λ is a complex parameter. This is a perturbation of the original equation, and if it is small or large is determined by the order of A relative to some scale of function spaces with compact embedding. For instance, the order of a pseudodifferential operator on $\mathcal{X}$ is evaluated relative to the scale of Sobolev spaces H^s on $\mathcal{X}$, where $s \geqslant s_0$ is an integer. The order is defined to be the least real number m with the property that the mapping $\mathcal{D}_A \cap H^s \to U \cap H^{s-m}$ generated by A is continuous for all s satisfying $s \geqslant s_0 + m$. Then the perturbation $A - \lambda I$ of A is small, provided $m > 0$, and large, provided $m < 0$. The perturbation is said to regular or singular, respectively. Any singular perturbation results in changing the order. Conversely, $A = (A - \lambda I) + \lambda I$ is a perturbation of $A - \lambda I$. The inverse $R_A(\lambda) = (A - \lambda I)^{-1}$, if exists, is called the resolvent of the operator A. It is thought of as a function of λ with values in operators acting from U to $\mathcal{D}_A$. The most intricate case is that of operators of order $m = 0$, where spectral theory requires additional scales of function spaces. For an operator A of positive order, $A - \lambda I$ is a perturbation of A by a compact operator. Such perturbations preserve the class of Fredholm operators. Assume that $A - \lambda I$ is Fredholm for some particular value $\lambda = \lambda_0$. Then there is a parametrix P for A under the additional condition $u \in \mathcal{D}_A$. This means that P maps U continuously into the domain of A and the equalities $P(A - \lambda_0 I) = I$ and $(A - \lambda I)P = I$ are fulfilled up to compact operators on $\mathcal{D}_A$ and U, respectively. From the parametrix construction for an elliptic pseudodifferential operator one sees that P has order $-m$. If P is nonlinear, $PA = P((A - \lambda_0 I) + \lambda_0 I)$ need not be equal to $P(A - \lambda_0 I)$ modulo compact operators. Hence, P fails to be a left parameterix of A which is responsible for uniqueness and regularity of solutions. On the other hand, the second equality implies readily that $AP = I + \lambda_0 P$ reduces to I up to a compact operator, i.e., P is still a right parametrix for A which is responsible for existence. This suggests to distinguish those nonlinear operators A which possess linear parametrices P. Another way is to mean by parametrices alone linear mappings. Then, if $A - \lambda I$ has a parametrix P for a particular value $\lambda = \lambda_0$, then P is a parametrix of $A - \lambda I$ for all $\lambda \in \mathbb{C}$. This is also true for existence and regularity theorems.

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Когомологии де Рама посредством методов гильбертовых пространств

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Германия

Обсуждаются канонические представления когомологий де Храма на компактном многообразии с краем. Они получены путем минимизации интеграла энергии в гильбертовом пространстве дифференциальных форм, которые наряду с внешней производной принадлежат области присоединенного оператора. Соответствующие уравнения Эйлера-Лагранжа сводятся к эллиптической краевой задаче на многообразии, которую обычно называют проблемой Неймана после Спенсера.

Ключевые слова: комплекс де Рхама, когомология, теория Ходжа, проблема Неймана.