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Adiabatic Limit in Yang–Mills Equations in $\mathbb{R}^4$

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Our goal is to present an approach to the proof of the harmonic spheres conjecture based on the adiabatic limit construction. This construction allows to associate with an arbitrary Yang–Mills G-field on the Euclidean 4-dimensional space a harmonic map of the Riemann sphere to the loop space of the group G.

Keywords: Yang-Mills fields, loop spaces, adiabatic limit, harmonic maps.
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1. Harmonic spheres conjecture

Let G be a compact Lie group and $\Omega G = C^\infty(S^1, G)/G$ denotes its based loop space. In paper [1] it was proved the following theorem relating G-instantons on $\mathbb{R}^4$ with holomorphic maps of the Riemann sphere to ΩG.

Theorem 1 (Atiyah–Donaldson theorem)**.** *There exists a bijective correspondence between*

$$\left\{\begin{matrix}\textit{moduli space of G-}\\ \textit{instantons on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{space of based holomorphic maps } S^2 =\\ \mathbb{CP}^1 \to \Omega G\end{matrix}\right\}.$$

Under a based map $S^1 \to \Omega G$ we understand here a map sending the north pole of the Riemann sphere S^2 to the origin $[G]$ of the homogeneous space $\Omega G = C^\infty(S^1, G)/G$. This correspondence may be also considered as a correspondence between

$$\left\{\begin{matrix}\text{local minima of Yang–}\\ \text{Mills action on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{local minima of the energy}\\ \text{functional on } S^2 \to \Omega G\end{matrix}\right\}.$$

Switching from the local minima to arbitrary critical points of these functionals, we obtain the formulation of the harmonic spheres conjecture:

$$\left\{\begin{matrix}\text{critical points of Yang–Mills ac-}\\ \text{tion on } \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{critical points of the energy func-}\\ \text{tional on } S^2 \to \Omega G\end{matrix}\right\}.$$

In other words, the conjecture asserts that it should exist a bijective correspondence between

$$\left\{\begin{matrix}\text{moduli space of Yang–Mills } G\text{-fields on}\\ \mathbb{R}^4\end{matrix}\right\} \longleftrightarrow \left\{\text{space of based harmonic maps } S^2 \to \Omega G\right\}.$$

This conjecture may be also considered as a "realification" of Atiyah–Donaldson theorem in which instantons are replaced by arbitrary Yang–Mills fields and holomorphic maps $S^2 \to \Omega G$ are replaced by arbitrary harmjnic maps.

Unfortunately, the proof of Atiyah–Donaldson theorem essentially uses the theory of holomorphic vector bundles over the space $\mathbb{CP}^2$ and by this reason does not extend to arbitrary Yang–Mills fields and harmonic maps.

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The first idea of the proof of conjecture (cf. [12]) was to use the twistor approach. In other words, switch from the formulated conjecture to its twistor version by replacing both sides of the correspondence, proclaimed by the conjecture, to their twistor analogues. To the proof of so obtained twistor conjecture one can already apply purely holomorphic methods. The twistor interpetation of the space of harmonic maps $S^2 \to \Omega G$ was constructed in a joint paper of the author with I. V. Beloshapka [2]. On the other hand, in papers by Isenberg–Green–Yasskin [4], Witten [15] and Manin [6] it was given the twistor realization of Yang–Mills fields. However, we are still not able to establish a correspondence between these two twistor spaces (the arising difficulties are discussed in the paper [11]). By this reason we propose to use for the proof of the harmonic spheres conjecture another approach presented in this paper.

This approach is based on the adiabatic limit construction for the Yang–Mills equations on $\mathbb{R}^4$ proposed by A. D. Popov in [8]. The adiabatic limit was already successfully used earlier for the study of Ginzburg–Landau and Seiberg–Witten equations (cf. [7, 10, 13, 14]). The adiabatic limit construction for the Yang–Mills equations on $\mathbb{R}^4$ uses an interesting parameterization of the sphere S^4 without a circle S^1 found in the paper by Jarvis and Norbury [5]. In this parameterization the space $S^4 \setminus S^1$ is sliced by complex disks parameterized by the Riemann sphere S^2. In adiabatic limit a given Yang–Mills G-field on $S^4 \setminus S^1$ degenerates into a harmonic map $S^2 \to \Omega G$ which allows to associate with this Yang–Mills field a harmonic sphere in the loop space. Note however that the transition to adiabatic limit is based on the assumption that the Yang–Mills field reduces to flat connections on the slicing disks. This assumption is not yet proved and needs an additional justification.

2. Jarvis–Norbury parameterization

We identify the Euclidean sphere S^4 with quaternion projective line $\mathbb{HP}^1$ consisting of pairs of quaternions $[q_1, q_2]$ defined up to multiplication from the right by nonzero quaternions. The affine part of this line is identified with the set

$$U_0 = \{[q, 1] : \ q = a + bj \in \mathbb{H},\, a, b \in \mathbb{C}\}.$$

The restriction of the standard spherical metric on U_0 is given by the formula

$$ds^2 = 4\, \frac{d\bar{a}\, da + d\bar{b}\, db}{(1 + |a|^2 + |b|^2)^2}\, .$$

For the description of Jarvis–Norbury parameterization we need to employ two subsets of $S^4 = \mathbb{HP}^1$. The first of them is a 2-dimensional sphere S^2_∞, identified with the closure of the set $\{(a, b) \in \mathbb{C}^2 : \ b = 0\}$, and the second is the circle $S^1_0 = \{(a, b) \in \mathbb{C}^2 : a = 0, |b| = 1\}$. The subset $S^4 \setminus S^1_0$ of the sphere S^4 is sliced by the disks with the common boundary S^1_0 over the base S^2_∞. The bundle $S^4 \setminus S^1_0 \to S^2_\infty$ is trivial, i.e. isomorphic to the direct product $S^2_\infty \times D$ where D is the unit disk in $\mathbb{C}$. Denote by z the complex parameter in the fibre $D_w := \{w\} \times D$ over the point $w \in S^2_\infty$ so that

$$D_w = \{(w, z) \in S^2_\infty, z \in \mathbb{C}, |z| < 1\}.$$

The boundary of the disk D_w coincides with the circle $\partial D_w = \{(w, z) : \ z \in \mathbb{C}, |z| = 1\} = S^1_0$ so that all disk D_w have the joint boundary S^1_0 and intersect with S^2_∞ only in one point $(w, 0)$ (Fig. 1).

It is convenient to consider w in the sequel as the stereographic coordinate on S^2_∞ so that the metric on $S^4 \setminus S^1_0$ in coordinates (w, z) takes the form

$$ds^2 = \left(\frac{1 - |z|^2}{1 + |z|^2}\right)^2 \frac{4 d\bar{w}\, dw}{(1 + |w|^2)^2} + \frac{4 d\bar{z}\, dz}{(1 + |z|^2)^2}\, .$$

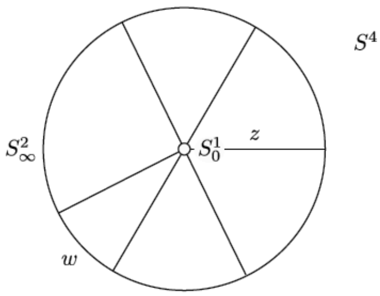

Fig. 1

This metric is conformally equivalent to the metric

$$ds^2 = \frac{4d\bar{w}\,dw}{(1+|w|^2)^2} + \frac{4d\bar{z}\,dz}{(1-|z|^2)^2} =: ds^2_w + ds^2_z \qquad (1)$$

coinciding with the metric of direct product of the spherical metric on S^2_∞ and hyperbolic metric on the disk D.

3. Adiabatic limit in Yang–Mills equations

Let G be a compact Lie group. By Uhlenbeck theorem any Yang–Mills G-field with finite action on $\mathbb{R}^4$ extends to a Yang–Mills field with values in some G-bundle over S^4. So we shall suppose from the beginning that it is given a Yang–Mills G-field on S^4. Such field is determined by a gauge potential $\mathcal{A} = A_\mu dx^\mu$ with smooth coefficients taking values in the Lie algebra $\mathfrak{g}$ of the Lie group G where we assume the summation in the repeated index μ with $\mu = 1, \dots, 4$. We provide the Lie algebra with an invariant (under the adjoint representation) inner product denoted by Tr.

The Yang–Mills field $\mathcal{F}$ is a 2-form on S^4 with smooth coefficients taking values in $\mathfrak{g}$ of the form

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

or in tensor notations

$$\mathcal{F} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{c}, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \; \mu, \nu = 1, \dots, 4,$$

where $\partial_\mu = \partial/\partial x^\mu$. The Yang–Mills Lagrangian has the form

$$\mathcal{L}_{\mathrm{YM}}(\mathcal{A}) = \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

where the tensor indices are raised and lowered with the help of metric tensor of the sphere S^4. The Yang–Mills equations, coinciding with the Euler–Lagrange equations for the Yang–Mills action functional given by the integral over S^4 of Yang–Mills Lagrangian, have the form

$$\mathcal{D}_\mu F^{\mu\nu} := \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0, \; \mu, \nu = 1, \dots, 4.$$

Introduce now a small parameter $\varepsilon > 0$ and consider the dilation of the original metric (1) on $S^4 \setminus S^1_0$ of the form

$$ds^2_\varepsilon = ds^2_w + \varepsilon^2 ds^2_z. \qquad (2)$$

In coordinates (w,z) the Yang–Mills tensor for the metric (1) on $S^4\setminus S^1_0$ is written in the form

$$\mathcal{F}=\frac{1}{2}\mathcal{F}_{ww}+\mathcal{F}_{wz}+\frac{1}{2}\mathcal{F}_{zz}$$

where the component $\mathcal{F}_{ww}$ (resp. $\mathcal{F}_{zz}$) contains only the dw and $d\bar{w}$ (resp. dz and $d\bar{z}$). The mixed component $\mathcal{F}_{wz}$ contains only mixed differentials of the form $d\bar{w}\wedge dz$ and $dw\wedge d\bar{z}$.

The Yang–Mills tensor $\mathcal{F}_\varepsilon$ with respect to the dilated metric ds^2_ε will have the components

$$\mathcal{F}^{ww}_\varepsilon=\mathcal{F}^{ww},\quad \mathcal{F}^{wz}_\varepsilon=\varepsilon^{-2}\mathcal{F}^{wz},\quad \mathcal{F}^{zz}_\varepsilon=\varepsilon^{-4}\mathcal{F}^{zz}$$

where the indices of the tensors are raised and lowered with the help of the metric (1). The Yang–Mills equations for the tensor $\mathcal{F}_\varepsilon$ take the form

$$\begin{cases}\varepsilon^2\mathcal{D}_w\mathcal{F}^{ww}+\mathcal{D}_z\mathcal{F}^{wz}=0,\\ \mathcal{D}_w\mathcal{F}^{wz}+\varepsilon^{-2}\mathcal{D}_z\mathcal{F}^{zz}=0.\end{cases}$$

In order to get rid of the growing term $\varepsilon^{-2}\mathrm{Tr}(\mathcal{F}_{zz}\mathcal{F}^{zz})$ in Yang–Mills Lagrangian we impose an additional condition

$$\mathcal{F}_{zz}=0,$$

i.e. suppose that the connection $\mathcal{A}$ is flat on the disks D_w.

Then in the limit for $\varepsilon\to 0$ the Yang–Mills equations will turn into the adiabatic equations

$$\begin{cases}\mathcal{D}_z\mathcal{F}^{wz}=0,\\ \mathcal{D}_w\mathcal{F}^{wz}=0.\end{cases}$$

4. Flat connections on the disk

The flat G-connections $\mathcal{A}_D$ on the disk D are represented in the form

$$\mathcal{A}_D=g^{-1}dg$$

where $g\in C^\infty(D,G)$ is a smooth map. So the space of such connections may be identified with the space of based smooth maps $\mathcal{N}:=C^\infty_0(D,G)$ consisting of smooth maps $g:D\to G$ taking value $e\in G$ at the point $1\in S^1$.

On this space we have an action of the group

$$\mathcal{G}_D=\{g\in C^\infty_0(D,G):\ g|_{S^1}=e\}.$$

The quotient $C^\infty_0(D,G)$ with respect to this action may be identified with the loop space

$$\Omega G\longrightarrow C^\infty_0(D,G)/\mathcal{G}_D,$$

and this map preserves natural Kähler structures on both manifolds (cf. [9]).

5. Construction of harmonic spheres

As we have remarked in Sec. 3., solutions of Yang–Mills equations on $S^4\setminus S^1_0\cong S^2\times D$ in the adiabatic limit are represented by connections

$$\mathcal{A}=\mathcal{A}_z+\mathcal{A}_w$$

on $S^2 \times D$ which are solutions of adiabatic equations

$$\mathcal{D}_z \mathcal{F}^{wz} = 0, \tag{AD1}$$

$$\mathcal{D}_w \mathcal{F}^{wz} = 0 \tag{AD2}$$

with restrictions to D given by flat connections. According to the interpretation of flat connections given in Sec. 4, these solutions are given by smooth maps

$$\varphi : S^2 \longrightarrow \Omega G, \quad \varphi = (\varphi^\alpha(w)),$$

depending smoothly on the parameter $w \in S^2$ and satisfying the equations (AD).

The flat connections $\mathcal{A}_z$ depend on w as on parameter so the derivatives $\partial_w \mathcal{A}_z$, belonging to the tangent space $T_{\mathcal{A}_z}\mathcal{N} = T_{\mathcal{A}_z} C_0^\infty(D, G)$ at $\mathcal{A}_z$, should satisfy the linearized equations of zero curvature. This tangent space may be represented in the form

$$T_{\mathcal{A}_z}\mathcal{N} = \pi^* T_{\mathcal{A}_z}(\Omega G) \oplus T_{\mathcal{A}_z}\mathcal{G}$$

where $\mathcal{G}$ is the group of gauge transformations which being restricted to $D = D_w$ coincides with $\mathcal{G}_D$. According to this decomposition derivatives $\partial_w \mathcal{A}_z$ are represented in the form

$$\partial_w \mathcal{A}_z = (\partial_w \varphi^\alpha)\xi_{\alpha z} + \mathcal{D}_z \epsilon_w$$

where $\xi_\alpha = \xi_{\alpha z}$ form a local basis of tangent vector fields on $T_{\mathcal{A}}(\Omega G)$ and $\mathcal{D}_z \epsilon_w$ is local basis of tangent vector fields on $T_{\mathcal{A}}\mathcal{G}$. The fields $\mathcal{D}_z \epsilon_w$ are chosen from the gauge fixing condition

$$\mathcal{D}^z \xi_{\alpha z} = 0 \Longleftrightarrow \mathcal{D}^z \partial_w \mathcal{A}_z = \mathcal{D}^z \mathcal{D}_z \epsilon_w. \tag{3}$$

The mixed components of the curvature have the form

$$\mathcal{F}_{wz} = \partial_w \mathcal{A}_z - \mathcal{D}_z \mathcal{A}_w = (\partial_w \varphi^\alpha)\xi_{\alpha z} + \mathcal{D}_z(\mathcal{A}_w - \epsilon_w).$$

Note that we have fixed the components $\mathcal{A}_z$ with the help of condition (3) while the components $\mathcal{A}_w$ still remain free. Now we shall fix them by setting $\mathcal{A}_w = \epsilon_w$. Under this condition the formula for the mixed components of the curvature will take the form

$$\mathcal{F}_{wz} = (\partial_w \varphi^\alpha)\xi_{\alpha z} = \pi^*(\partial_w \mathcal{A}_z) \in T_{\mathcal{A}}(\Omega G).$$

Plugging this expression into the adiabatic equation (AD2), we obtain

$$\partial_w(\partial^w \varphi^\alpha)\xi_\alpha^z + (\mathcal{D}^w \xi_\alpha^z)\partial^w \varphi^\alpha = 0. \tag{4}$$

The vector fields ξ_α define a metric on the loop space ΩG given by the formula

$$\langle \xi_\alpha, \xi_\beta \rangle = \int_D \mathrm{Tr}(\xi_\alpha \xi_\beta) dz d\bar{z}. \tag{5}$$

As it is shown in [8], the equations (4) coincide with harmonic equations for the map $\varphi : S^2 \to \Omega G$ if we define the metric on ΩG by the formula (5). In other words, the Yang–Mills G-equations on $S^4 \setminus S_0^1$ in the adiabatic limit reduce to the harmonic equations for the maps $S^2 \to G$ which allows to associate with every Yang–Mills G-field on $\mathbb{R}^4$ a harmonic sphere in ΩG.

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Адиабатический предел в уравнениях Янга–Миллса на $\mathbb{R}^4$

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Целью этой работы является изложение подхода к решению гипотезы о гармонических сферах, основанного на конструкции адиабатического предела. Указанная конструкция позволяет сопоставить произвольному G-полю Янга–Миллса на евклидовом 4-мерном пространстве гармоническое отображение сферы Римана в пространство петель группы G.

Ключевые слова: доля Янга-Миллса, пространства петель, адиабатический предел, гармонические отображения.