Journal of Siberian Federal University. Mathematics & Physics 2019, 12(4), 439–443

УДК 517.55

Functions with the One-dimensional Holomorphic Extension Property

Simona G. Myslivets[*]
Institute of Mathematics and Computer Science
Siberian Federal University
Svobodny, 79, Krasnoyarsk, 660041
Russia

Received 16.03.2019, received in revised form 26.03.2019, accepted 06.05.2019

In this paper we consider different families of complex lines, sufficient for holomorphic extension the functions f, defined on the boundary of a domain $D \subset \mathbb{C}^n$, $n > 1$, into this domain, and possessing the one-dimensional holomorphic extension property along this complex lines.

Keywords: holomorphic extension.
DOI: 10.17516/1997-1397-2019-12-4-439-443.

This paper presents some results related to the holomorphic extension of functions f, defined on the boundary of a domain $D \subset \mathbb{C}^n$, $n > 1$, into this domain. It is about functions with the one-dimensional holomorphic extension property along complex lines.

The first result related to this subject was obtained M. L. Agranovsky and R. E. Valsky in [1], who studied functions with the one-dimensional holomorphic extension property into a ball. The proof was based on the automorphism group properties of a sphere.

E. L. Stout in [2] used complex Radon transformation to generalize the Agranovsky and Valsky theorem for an arbitrary bounded domain with a smooth boundary. An alternative proof of the Stout theorem was obtained by A. M. Kytmanov in [3] by applying the Bochner–Martinelli integral. The idea of using the integral representations (Bochner–Martinelli, Cauchy–Fantappiè, logarithmic residue) has been useful in the study of the functions with the one-dimensional holomorphic extension property (see review [4]).

The question of finding various families of complex lines sufficient for holomorphic extension was put in [5]. As shown in [6], a family of complex lines passing through a finite number of points, generally speaking, is not sufficient. Thus, a simple analog theorem of Hartogs should be not expected.

Various other families are given in [7–11]. In [12–16] it is shown that for holomorphic extension of continuous functions defined on the boundary of a ball there is enough $n+1$ points inside the ball that do not lie on a complex hyperplane. By the author and A. Kytmanov this result was generalized for n-circular [17] and circular domains [18].

In this paper we formulate some results about various sufficient families of complex lines sufficient for holomorphic extension.

Let D be a bounded domain in $\mathbb{C}^n$ with a smooth boundary. Consider a complex line of the form

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \zeta = z + bt, t \in \mathbb{C}\} = \{(\zeta_1, \dots \zeta_n) : \zeta_j = z_j + b_j t, j = 1, 2, \dots, n, t \in \mathbb{C}\},$$

where $z \in \mathbb{C}^n$, $b \in \mathbb{CP}^{n-1}$.

We shall say that a function $f \in \mathcal{C}(\partial D)$ has the *one-dimensional holomorphic extension property along the complex line* $l_{z,b}$, if $\partial D \cap l_{z,b} \neq \varnothing$ and there exists a function $F_{l_{z,b}}$ with the following properties:

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[*] sMyslivets@sfu-kras.ru

1) $F_{l_{z,b}} \in \mathcal{C}(\overline{B} \cap l_{z,b})$,

2) $F_{l_{z,b}} = f$ on the set $\partial D \cap l_{z,b}$,

3) function $F_{l_{z,b}}$ is holomorphic at the interior (with respect to the topology of $l_{z,b}$) points of the set $\overline{D} \cap l_{z,b}$.

Let Γ be a set in $\mathbb{C}^n$. Denote by $\mathfrak{L}_\Gamma$ the set of all complex lines $l_{z,b}$ such that $z \in \Gamma$ and $b \in \mathbb{CP}^{n-1}$, i.e., the set of all complex lines passing through $z \in \Gamma$.

We shall say that a function $f \in \mathcal{C}(\partial D)$ has the *one-dimensional holomorphic extension property along the family* $\mathfrak{L}_\Gamma$, if it has the one-dimensional holomorphic extension property along any complex line $l_{z,b} \in \mathfrak{L}_\Gamma$.

We shall say *set* $\mathfrak{L}_\Gamma$ *sufficient for holomorphic extension*, if the function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along all complex lines of the family $\mathfrak{L}_\Gamma$, and then the function f extends holomorphically into D (i.e., f is a CR-function on ∂D).

In what follows we will need the definition of a domain with the Nevanlinna property [19]. Let $G \subset \mathbb{C}$ be a simply connected domain and $t = k(\tau)$ be a conformal mapping of the unit circle $\Delta = \{\tau : |\tau| < 1\}$ on G.

Domain G is *a domain with the Nevanlinna property*, if there are bounded holomorphic functions u and v in G such that almost everywhere on $S = \partial\Delta$, the equality

$$\bar{k}(\tau) = \frac{u(k(\tau))}{v(k(\tau))}$$

holds in terms of the angular boundary values. Essentially this means

$$\bar{t} = \frac{u(t)}{v(t)} \quad \text{on} \quad \partial G.$$

Let us give a characterization of domains with the Nevanlinna property (Proposition 3.1 in [19]). A domain G is a domain with the Nevanlinna property if and only if $k(\tau)$ admits a holomorphic pseudocontinuation through S in $\overline{\mathbb{C}} \setminus \overline{\Delta}$, i.e., there are bounded holomorphic functions u_1 and v_1 such that the function $\tilde{k}(\tau) = \dfrac{u_1(\tau)}{v_1(\tau)}$ coincides almost everywhere with the function $k(\tau)$ on S.

The above definition and statement will be applied to bounded domains G with a boundary of class $\mathcal{C}^2$, therefore (due to the principle of correspondence of boundaries) the function $k(\tau)$ extends to $\overline{\Delta}$ as a function of class $\mathcal{C}^1(\overline{\Delta})$ and $\tilde{k}(\tau)$ extends to $\mathbb{C}\setminus\overline{\Delta}$ as a function of class $\mathcal{C}(\mathbb{C}\setminus\overline{\Delta})$. Therefore, the function $\bar{t} = \dfrac{u(\tau)}{v(\tau)}$ is a meromorphic function in $\mathbb{C}$. Various example of domains with the Nevanlinna property are given in [19]. For example, if ∂G is real-analytic, then $k(\tau)$ is a rational function with no poles on the closure of Δ.

In our further consideration we will need the domain G to possess *the strengthened Nevanlinna property*, that is the function $u_1(\tau) \neq 0$ in $\mathbb{C} \setminus \Delta$ and $\tilde{k}$ has at infinity zero of no more than first order. If $G = \Delta$ then $\overline{\tau} = \dfrac{1}{\tau}$ on $\partial\Delta$. Therefore the meromorphic function $\dfrac{1}{\tau}$ has a zero of the first order at ∞.

For example, such domain include domains for which $k(\tau)$ is a rational function with no poles on $\overline{\Delta}$ and no zeros in $\mathbb{C} \setminus \Delta$.

We shall to say that a domain $D \subset \mathbb{C}^n$ possesses *the strengthened Nevanlinna property in the point* $z \in D$ if the section $D \cap l_{z,b}$ possesses the strengthened Nevanlinna property for any $b \in \mathbb{CP}^{n-1}$.

We now formulate some results about the different families of complex lines sufficient for holomorphic extension.

We consider families of complex lines passing through a generic manifold. The real dimension of such a manifold is at least n. Recall that a smooth manifold $\mathit{\Gamma}$ of class $\mathcal{C}^\infty$ is said to be *generic* if the complex linear span of the tangent space $T_z(\mathit{\Gamma})$ coincides with $\mathbb{C}^n$ for each point $z \in \mathit{\Gamma}$. We denote the family of all complex lines intersecting $\mathit{\Gamma}$ by $\mathfrak{L}_{\mathit{\Gamma}}$.

Theorem 1. *Let $\mathit{\Gamma}$ be a germ of a generic manifold in $\mathbb{C}^n \setminus \overline{D}$ and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathit{\Gamma}}$, then the function f extends holomorphically into the domain D.*

Here we consider a generic manifold $\mathit{\Gamma}$ lying in the domain D.

Theorem 2. *Let Γ be a germ of a generic manifold in D and a function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_\Gamma$ and the connected components of the intersection $D \cap l$ be domains with the strengthened Nevanlinna property, then the function f extends holomorphically into the domain D.*

Let $\mathit{\Gamma}$ be the germ of a complex manifold of dimension $(n-1)$ in $\mathbb{C}^n$, which lies outside $\overline{D}$. Having done the shift and unitary transformation, we can assume that $0 \in \mathit{\Gamma}$, $0 \notin \overline{D}$ and that the complex hypersurface $\mathit{\Gamma}$ in some neighborhood U of 0 has the form

$$\mathit{\Gamma} = \{z \in U : \ z_n = \varphi(z'), \ z' = (z_1, \dots, z_{n-1})\},$$

where φ is the holomorphic function in a neighborhood of zero in $\mathbb{C}^{n-1}$ and $\varphi(0) = 0$, $\dfrac{\partial \varphi}{\partial z_k}(0) = 0$, $k = 1, \dots, n-1$.

We will assume that there is a direction $b^0 \neq 0$ such that

$$\langle b^0, \bar{\zeta} \rangle \neq 0 \quad \text{for all} \quad \zeta \in \overline{D}. \tag{1}$$

Theorem 3. *Let D be a simply connected bounded domain and condition (1) be fulfilled and the function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathit{\Gamma}}$, then the function f extends holomorphically into the domain D.*

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be a unit ball in $\mathbb{C}^n$ centered at the origin and let $S = \partial B$ be the boundary of the ball.

We denote by $\mathfrak{A}$ the set of points $a_k \in D \subset \mathbb{C}^n$, $k = 1, \dots, n+1$, which do not lie on the complex hyperplane in $\mathbb{C}^n$.

Theorem 4. *Let a function $f \in \mathcal{C}(S)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$, then the function f extends holomorphically into the ball B.*

This theorem was proved for circular domains with the strengthened Nevanlinna property.

Theorem 5. *Let D be a bounded strictly convex circular domain with twice smooth boundary in $\mathbb{C}^n$ and possess the strengthened Nevanlinna property in the points from the set $\mathfrak{A}$ and a function $f \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$, then the function f extends holomorphically into the domain D.*

The research for this paper was supported by RFBR, grant 18-51-41011, Uzb-t.

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Функции со свойством одномерого голоморфного продолжения

Симона Г. Мысливец
Институт математики и фундаментальной информатики
Сибирский федеральный университет
Свободный, 79, Красноярск, 660041
Россия

В данной работе рассматриваются различные семейства комплексных прямых, достаточные для голоморфного продолжения функций f*, заданных на границе области* $D \subset \mathbb{C}^n$*,* $n > 1$*, в эту область и обладающих свойством одномерного голоморфного продолжения вдоль этих комплексных прямых.*

Ключевые слова: голоморфное продолжение.