Journal of Siberian Federal University. Mathematics & Physics 2019, 12(3), 392–401

УДК 517.946

Ill-posed Boundary-value Problem for a System of Partial Differential Equations with Two Degenerate Lines

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Received 20.09.2018, received in revised form 18.01.2019, accepted 26.03.2019

This paper is devoted to the investigation of ill-posed boundary-value problem for system of parabolic type equations with changing time direction with two degenerate lines. The problem under consideration is ill-posed in the sense of J. Hadamard, namely, there is no continuous dependence of the solution on the initial data. Such equations have many different applications, for example, describe the processes of heat propagation in inhomogeneous media, the interaction of filtration flows, mass transfer near the surface of an aircraft, and the description of complex viscous fluid flows. As possible applications should also indicate the task of calculating heat exchangers, in which the counter flow principle is used. Theorems on the uniqueness and conditional stability of a solution on a set of well-posedness are proved. We construct a sequence of approximate (regularized) solutions that are stable on the set of well-posedness.

Keywords: parabolic equation with changing time direction, ill-posed problem, a priori estimate, estimation of conditional stability, uniqueness of solution, approximate solution.
DOI: 10.17516/1997-1397-2019-12-3-392-401.

The paper is devoted to the investigation of ill-posed boundary-value problem for a system of partial differential equations with two degeneration lines.

1. Introduction and preliminaries

We consider the problem of finding a solution $(u(x,y,t),\nu(x,y,t))$ of the system of equations

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - sign(x)\dfrac{\partial}{\partial x^2} - sign(y)\dfrac{\partial}{\partial y^2}\right)\nu(x,y,t) = 0, \\ \left(\dfrac{\partial}{\partial t} - sign(x)\dfrac{\partial}{\partial x^2} - sign(y)\dfrac{\partial}{\partial y^2}\right)u(x,y,t) = \nu(x,y,t) \end{cases} \tag{1}$$

in the region $\Omega = \{(x,y,t)|\,(-1<x<1)\times(-1<y<1)\times(0<t<T<\infty), x\neq 0, y\neq 0\}$ that satisfy the following conditions:

initial conditions

$$u(x,y,0) = f(x,y),\ \ \nu(x,y,0) = \varphi(x,y),\ \ (x,y)\in \prod = \{\,-1\leqslant x\leqslant 1,\quad -1\leqslant y\leqslant 1\}\ , \tag{2}$$

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boundary conditions

$$\begin{aligned} &u(-1,y,t)=u(+1,y,t)=0, && -1\leqslant y\leqslant 1,\ 0\leqslant t\leqslant T,\\ &u(x,-1,t)=u(x,+1,t)=0, && -1\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\ &\nu(-1,y,t)=\nu(+1,y,t)=0, && -1\leqslant y\leqslant 1,\ 0\leqslant t\leqslant T,\\ &\nu(x,-1,t)=\nu(x,+1,t)=0, && -1\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T, \end{aligned} \tag{3}$$

and gluing conditions

$$\begin{aligned} &u(-0,y,t)=u(+0,y,t), && u_x(-0,y,t)=u_x(+0,y,t), && -1\leqslant y\leqslant 1,\ 0\leqslant t\leqslant T,\\ &u(x,-0,t)=u(x,+0,t), && u_y(x,-0,t)=u_y(x,+0,t), && -1\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\ &\nu(-0,y,t)=\nu(+0,y,t), && \nu_x(-0,y,t)=\nu_x(+0,y,t), && -1\leqslant y\leqslant 1,\ 0\leqslant t\leqslant T,\\ &\nu(x,-0,t)=\nu(x,+0,t), && \nu_y(x,-0,t)=\nu_y(x,+0,t), && -1\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T. \end{aligned} \tag{4}$$

In many cases, when solving applied problems one uses the methods of mathematical physics. By solving theoretical and practical problems, one can often meet mixed type equations, which are one of the main objects of the chapters of the theory of modern differential equations. In this paper we consider an ill-posed (in the sense of J. Hadamard) boundary-value problem for a system of parabolic type equations with a changing time direction.

For parabolic type equations with changing time directions problems were first considered by M. Jevrey [1]. Various boundary value problems for such equations have been the subject of research by several scientists, such as S. D. Pagani [2], G. Talenti, S. A. Tersenov [3], V. N. Vragov [4], V. K. Romanko [5], A. M. Nakhushev [6] and their students. To study these problems, in these papers such classical methods as various analogues of the Green's function, potential theory, integral equations were used. In the work of S. A. Tersenov [3] the results of research using classical methods are summed up.

From further research it became clear that parabolic equations with a changing time direction can be considered as a special case of equations of a mixed type. In the works of N. V. Kislov [7], the generalized solvability of boundary value problems for abstract type equation was proved. A great contribution to the study of boundary value problems for equations of mixed type was made by A. I. Kozhanov [8], I. E. Egorov [9], S. G. Pyatkov [10, 11], A. A. Kerefov [12], I. S. Pulkin [13], K. B. Sabitov [14], and others.

The problem under investigation belongs to the class of ill-posed problems in mathematical physics, namely, in this problem there is no continuous dependence of the solution on the initial data. Ill-posed problems for such equations was considered in the works of H. Levine, A. L. Buchkheim, K. S Fayazov [15], K. S. Fayazov and I. O. Khazhiev [16], K. S. Fayazov and Y. K. Khudayberganov [17] and others.

Such equations have many different applications, for example, the description the processes of heat propagation in inhomogeneous media, the interaction of filtration flows, mass transfer near the surface of an aircraft, and the description of complex viscous fluid flows. As possible applications should also indicate the problem of calculating heat exchangers, in which the principle of counter flow is used (see [3, 18]).

In the previous works were examined correct and incorrect problems for the classic and mixed-type equations with one degenerate lines. Correct problems for Laplas equation with two degenerate lines was considered by A. A. Gamaltdinova in the work [19]. In this paper, based on the idea of A. N. Tikhonov, the conditional correctness of problem (1)–(4) is investigate, namely, theorems on uniqueness and conditional stability were proved, and approximate solutions that are stable on the set of well-posedness are constructed.

2. Spectral problem

Find the values of λ for which the following problem

$$sign(x)\vartheta_{xx}(x,y)+sign(y)\vartheta_{yy}(x,y)=-\lambda\vartheta(x,y),\ -1<x<1,\ -1<y<1,\ x\neq 0,\ y\neq 0\,, \quad (5)$$

$$\begin{gathered}\vartheta(-1,y)=\vartheta(+1,y)=0,\quad -1\leqslant y\leqslant 1,\\ \vartheta(-0,y)=\vartheta(+0,y),\ \vartheta_x(-0,y)=\vartheta_x(+0,y),\quad -1\leqslant y\leqslant 1,\\ \vartheta(x,-1)=\vartheta(x,+1)=0\,,\quad -1\leqslant x\leqslant 1,\\ \vartheta(x,-0)=\vartheta(x,+0),\qquad \vartheta_y(x,-0)=\vartheta_y(x,+0),\quad -1\leqslant x\leqslant 1,\end{gathered} \quad (6)$$

has nontrivial solution.

According to the result of [11], this problem (5)–(6) has a no decreasing sequence of $\lambda_{k,l}^{++}$, $\lambda_{k,l}^{+-}$, $\lambda_{k,l}^{-+}$, $\lambda_{k,l}^{--}$ eigenvalues and the corresponding Eigne functions $\left\{\vartheta_{k,l}^{++}\right\}_{k,l=1}^{\infty}$, $\left\{\vartheta_{k,l}^{+-}\right\}_{k,l=1}^{\infty}$, $\left\{\vartheta_{k,l}^{-+}\right\}_{k,l=1}^{\infty}$, $\left\{\vartheta_{k,l}^{--}\right\}_{k,l=1}^{\infty}$(see also [17]).

Let $\|u\|^2=(u,u)$ be the scalar product of $(u,\upsilon)=\int\limits_{-1}^{1}\int\limits_{-1}^{1}u\upsilon dxdy$. Besides

$$\left(sign(x)sign(y)\vartheta_{k,l}^{\pm\pm},\vartheta_{i,j}^{\pm\pm}\right)=\delta_{k,i}\cdot\delta_{l,j},$$

$$\left(sign(x)sign(y)\vartheta_{k,l}^{\pm\mp},\vartheta_{i,j}^{\pm\mp}\right)=-\delta_{k,i}\cdot\delta_{l,j},\ \delta_{k,i}=\begin{cases}1,\ k=i,\\ 0,\ k\neq i\,,\end{cases}$$

$$\left(sign(x)sign(y)\vartheta_{kl}^{\pm\mp},\vartheta_{ij}^{\mp\pm}\right)=0,$$

where $k,l,i,j=1,2,3,\dots$.

We represent the spectral projection -$P^{\pm}$ in the following form

$$\begin{aligned}P^{+}\phi&=\sum_{k,l=1}^{\infty}(sign(x)sign(y)\phi,\vartheta_{k,l}^{++})\vartheta_{k,l}^{++}+(sign(x)sign(y)\phi,\vartheta_{k,l}^{--})\vartheta_{k,l}^{--},\\ P^{-}\phi&=\sum_{k,l=1}^{\infty}(sign(x)sign(y)\phi,\vartheta_{k,l}^{+-})\vartheta_{k,l}^{+-}+(sign(x)sign(y)\phi,\vartheta_{k,l}^{-+})\vartheta_{k,l}^{-+}.\end{aligned}$$

See [10]

$$(P^{+}-P^{-})\phi=\phi,\quad (sign(x)sign(y)(P^{+}-P^{-})\phi,\phi)=\|\phi\|^2,$$

$$(sign(x)sign(y)P^{\pm}\phi,\psi)=(sign(x)sign(y)\phi,P^{\pm}\psi),\quad \phi,\psi\in \mathrm{H}_0=L_2(-1,1)\times(-1,1),$$

$$\begin{aligned}\|\vartheta\|_0^2=\sum_{k,l=1}^{\infty}\Big\{&\left|(sign(x)sign(y)\vartheta,\vartheta_{k,l}^{++})\right|^2+\left|(sign(x)sign(y)\vartheta,\vartheta_{k,l}^{+-})\right|^2+\\ &+\left|(sign(x)sign(y)\vartheta,\vartheta_{k,l}^{-+})\right|^2+\left|(sign(x)sign(y)\vartheta,\vartheta_{k,l}^{--})\right|^2\Big\}.\end{aligned}$$

According to the results of [11], the Eigen functions of problem (5)–(6) form a Rises basis.

3. A priori estimate

Definition 3.1. *By the solution of the problem we mean a pair of functions $(u(x,y,t),\nu(x,y,t))$ having continuous derivatives entering to the equation that satisfies the system of equations (1) and conditions (2)–(4).*

Lemma 3.1. *Let the function $\nu(x,y,t)$ satisfy the equation $\nu_t(x,y,t) = sign(x)\nu_{xx}(x,y,t)+$ $+sign(y)\nu_{yy}(x,y,t)$ in the domain Ω and conditions*

$$\nu(-1,y,t)=\nu(+1,y,t)=0,\quad \nu(-0,y,t)=\nu(+0,y,t),\quad \nu_x(-0,y,t)=\nu_x(+0,y,t),$$

$$\nu(x,-1,t)=\nu(x,+1,t)=0,\quad \nu(x,-0,t)=\nu(x,+0,t),\quad \nu_y(x,-0,t)=\nu_y(x,+0,t).$$

Then for $\nu(x,y,t)$ we have the estimate

$$\|\nu(x,y,t)\|_0 \leqslant \sqrt{2}\|\nu(x,y,0)\|_0^{\frac{T-t}{T}} \cdot \|\nu(x,y,T)\|_0^{\frac{t}{T}}.$$

Proof of this Lemma 3.1 you can find in [17] Theorem 3.1.

Remark. For the easy presentation without limiting generality, we assume that in the expansion of functions $f(x,y)$ and $\varphi(x,y)$ in series of the eigenfunctions of the spectral problem, the coefficients in front of the eigenfunctions $\vartheta_{k,l}^{+-}$ and $\vartheta_{k,l}^{-+}$ for any $k,l\in N$ are identically zero.

Lemma 3.2. *Let $u(x,y,t)$ be a function satisfying equation*

$$u_t(x,y,t) = sign(x)u_{xx}(x,y,t) + sign(y)u_{yy}(x,y,t) + \nu(x,y,t) \tag{7}$$

in the region Ω and the conditions:

$$\begin{array}{ll} u(-1,y,t)=u(+1,y,t)=0, & u(x,-1,t)=u(x,+1,t)=0,\\ u(-0,y,t)=u(+0,y,t), & u(x,-0,t)=u(x,+0,t),\\ u_x(-0,y,t)=u_x(+0,y,t), & u_y(x,-0,t)=u_y(x,+0,t). \end{array} \tag{8}$$

Then for $u(x,y,t)$ the following inequality holds

$$\|u(x,y,t)\|_0 \leqslant \sqrt{2}(\|u(x,y,0)\|_0+\|\gamma\|_0)^{\frac{T-t}{T}}\cdot(\|u(x,y,T)\|_0+\|\gamma\|_0)^{\frac{t}{T}}+\|\gamma\|_0,$$

where $\|\gamma\|_0 = \left(\int\limits_0^T \|\nu(x,y,t)\|_0^2 dt\right)^{\frac{1}{2}}$.

Proof. We present the solutions of (7) in the form

$$u(x,y,t) = \varpi(x,y,t) + \omega(x,y,t),$$

here $\omega(x,y,t)$ is the solution of the homogeneous equation

$$\omega_t(x,y,t) = sign(x)\omega_{xx}(x,y,t) + sign(y)\omega_{yy}(x,y,t), \tag{9}$$

and $\varpi(x,y,t)$ is a solution of a non-homogeneous equation.

$$\varpi_t(x,y,t) = sign(x)\varpi_{xx}(x,y,t) + sign(y)\varpi_{yy}(x,y,t) + \nu(x,y,t). \tag{10}$$

We introduce the notation $\lambda_{k,l}^{\pm\pm} = \lambda_{k,l}^{\pm}$, $\vartheta_{k,l}^{\pm\pm} = \vartheta_{k,l}^{\pm}$.

The functions $\omega(x,y,t)$, $\varpi(x,y,t)$ satisfy conditions (8). We represent the solution of problems (9–10) in the form

$$\omega(x,y,t) = \sum_{k,l=1}^{\infty} \omega_{k,l}^{+}(t)\vartheta_{k,l}^{+} + \sum_{k,l=1}^{\infty} \omega_{k,l}^{-}(t)\vartheta_{k,l}^{-}\,,$$

$$\varpi(x,y,t) = \sum_{k,l=1}^{\infty} \varpi_{k,l}^{+}(t)\vartheta_{k,l}^{+} + \sum_{k,l=1}^{\infty} \varpi_{k,l}^{-}(t)\vartheta_{k,l}^{-}\,,$$

here the functions $\omega_{k,l}^{\pm}(t), \varpi_{k,l}^{\pm}(t)$ are solutions of the following problems

$$\begin{cases} (\varpi_{k,l}^{+}(t))_t + \lambda_{k,l}^{+}\varpi_{k,l}^{+}(t) = \nu_{k,l}^{+}(t), \\ \varpi_{k,l}^{+}(0) = 0, \end{cases} \qquad \begin{cases} (\varpi_{k,l}^{-}(t))_t + \lambda_{k,l}^{-}\varpi_{k,l}^{-}(t) = \nu_{k,l}^{-}(t), \\ \varpi_{k,l}^{-}(0) = -\displaystyle\int_0^T e^{\lambda_{k,l}^{-}\tau}\nu_{k,l}^{-}(\tau)d\tau, \end{cases}$$

$$\begin{cases} (\omega_{k,l}^{+}(t))_t + \lambda_{k,l}^{+}\omega_{k,l}^{+}(t) = 0, \\ \omega_{k,l}^{+}(0) = f_{k,l}^{+}, \end{cases} \qquad \begin{cases} (\omega_{k,l}^{-}(t))_t + \lambda_{k,l}^{-}\omega_{k,l}^{-}(t) = 0, \\ \omega_{k,l}^{-}(0) = f_{k,l}^{-} + \displaystyle\int_0^T e^{\lambda_{k,l}^{-}\tau}\nu_{k,l}^{-}(\tau)d\tau, \end{cases} \tag{11}$$

where

$$\nu_{k,l}^{\pm}(t) = \pm\int_{-1}^{1}\int_{-1}^{1} sign(x)sign(y)\nu(x,y,t)\vartheta_{k,l}^{\pm}(x,y)dxdy,$$

$$f_{k,l}^{\pm}(t) = \pm\int_{-1}^{1}\int_{-1}^{1} sign(x)sign(y)f(x,y)\vartheta_{k,l}^{\pm}(x,y)dxdy.$$

It is not difficult to see that [20]

$$\varpi_{k,l}^{+}(t) = \int_0^t e^{\lambda_{k,l}^{+}(\tau-t)}\nu_{k,l}^{+}(\tau)d\tau, \quad \varpi_{k,l}^{-}(t) = -\int_t^T e^{\lambda_{k,l}^{-}(\tau-t)}\nu_{k,l}^{-}(\tau)d\tau,$$

and

$$\|\varpi(x,y,t)\|_0^2 = \sum_{k,l=1}^{\infty}\{\varpi_{k,l}^{+}(t)\}^2 + \sum_{k,l=1}^{\infty}\{\varpi_{k,l}^{-}(t)\}^2 \leqslant \int_0^T \|\nu(x,y,t)\|_0^2\,dt.$$

The solution of the equation (11) can be presented as

$$\omega_{k,l}^{+}(t) = f_{k,l}^{+}e^{-\lambda_{k,l}^{+}t},\ \omega_{k,l}^{-}(t) = f_{k,l}^{-}e^{-\lambda_{k,l}^{-}t} + \int_0^T e^{\lambda_{k,l}^{-}(\tau-t)}\nu_{k,l}^{-}(\tau)d\tau.$$

According to Lemma 3.1 for the solution $\omega(x,y,t)$ the following estimate holds

$$\|\omega(x,y,t)\|_0 \leqslant \sqrt{2}(\|\omega(x,y,0)\|_0)^{\frac{T-t}{T}}\cdot(\|\omega(x,y,T)\|_0)^{\frac{t}{T}}.$$

Therefore, for the functions $u(x,y,t) = \omega(x,y,t) + \varpi(x,y,t)$, we obtain the estimate

$$\|u(x,y,t)\|_0 \leqslant \sqrt{2}(\|u(x,y,0)\|_0 + \|\gamma\|_0)^{\frac{T-t}{T}}\cdot(\|u(x,y,T)\|_0 + \|\gamma\|_0)^{\frac{t}{T}} + \|\gamma\|_0,$$

where $\|\gamma\|_0 = \left(\int\limits_0^T \|\nu(x,y,t)\|_0^2 dt\right)^{\frac{1}{2}}$. □

4. Uniqueness and conditional stability

Let

$$M = \{(u,\nu) : \|u(x,y,T)\|_0 + \|\nu(x,y,T)\|_0 \leqslant m, m < \infty\}$$

be the set of correctness of problem (1)–(4).

Theorem 4.1. *Let the solution of problem (1)–(4) exist and $(u(x,y,t),\nu(x,y,t))$. Then the solution of problem (1)–(4) is unique.*

Proof. Let the pairs of functions $(u_1(x,y,t),\nu_1(x,y,t))$, $(u_2(x,y,t),\nu_2(x,y,t))$ be solutions of problem (1)–(4). We introduce the notation $u(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$ and $\nu(x,y,t) = \nu_1(x,y,t) - \nu_2(x,y,t)$ then the pair of functions $(u(x,y,t),\nu(x,y,t)) \in M$. satisfies the problem (1)–(4) with $u(x,y,0) = 0$, $\nu(x,y,0) = 0$. By Lemma 3.1 and Lemma 3.2, we have $\|\nu(x,y,t)\|_0 = 0$ and $\|u(x,y,t)\|_0 = 0$. Hence for any $(x,y,t) \in \Omega$ is true $u_1(x,y,t) \equiv u_2(x,y,t)$, $\nu_1(x,y,t) \equiv \nu_2(x,y,t)$. The Theorem 4.1 proved. □

Let $U(x,y,t) = u(x,y,t) - u_\varepsilon(x,y,t)$, $V(x,y,t) = \nu(x,y,t) - \nu_\varepsilon(x,y,t)$, where $(u(x,y,t),\nu(x,y,t))$ is solution of problem (1)–(4) with exact data, $(u_\varepsilon(x,y,t),\nu_\varepsilon(x,y,t))$ is solution of problem (1)–(4) with approximate data.

Theorem 4.2. *Let $(u(x,y,t),\nu(x,y,t)) \in M$, $(u_\varepsilon(x,y,t),\nu_\varepsilon(x,y,t)) \in M$ and $\|\varphi(x,y) - \varphi_\varepsilon(x,y)\|_0 \leqslant \varepsilon$, $\|f(x,y) - f_\varepsilon(x,y)\|_0 \leqslant \varepsilon$. Then by any $t \in [0;T]$ for (U,V) the estimates*

$$\|V(x,y,t)\|_0 \leqslant \sqrt{2}(\varepsilon)^{1-\frac{t}{T}}(2m)^{\frac{t}{T}},$$

$$\|U(x,y,t)\|_0 \leqslant (\varepsilon + \|\gamma_\varepsilon\|_0)^{1-\frac{t}{T}}(2m + \|\gamma_\varepsilon\|_0)^{\frac{t}{T}} + \|\gamma_\varepsilon\|_0,$$

hold, where $\|\gamma_\varepsilon\|_0 = \left(\int\limits_0^T 2\left(\varepsilon^2\right)^{1-\frac{t}{T}}\left(4m^2\right)^{\frac{t}{T}} dt\right)^{\frac{1}{2}}$.

Proof. The pair of functions $(U(x,y,t),V(x,y,t))$ satisfies conditions (1)–(4), where

$$U(x,y,0) = f(x,y) - f_\varepsilon(x,y),$$

$$V(x,y,0) = \varphi(x,y) - \varphi_\varepsilon(x,y), \qquad -1 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 1.$$

According to the conditions of Theorem

$$\|V(x,y,0)\|_0 = \|\varphi(x,y) - \varphi_\varepsilon(x,y)\|_0 \leqslant \varepsilon, \quad \|V(x,y,T\|_0 \leqslant 2m,$$

$$\|U(x,y,0)\|_0 = \|f(x,y) - f_\varepsilon(x,y)\|_0 \leqslant \varepsilon, \quad \|U(x,y,T\|_0 \leqslant 2m.$$

Then from Lemma 3.1 and Lemma 3.2 it yields

$$\|V(x,y,t)\|_0 \leqslant \sqrt{2}(\varepsilon)^{1-\frac{t}{T}}(2m)^{\frac{t}{T}},$$

$$\|U(x,y,t)\|_0 \leqslant (\varepsilon + \|\gamma_\varepsilon\|_0)^{1-\frac{t}{T}}(2m + \|\gamma_\varepsilon\|_0)^{\frac{t}{T}} + \|\gamma_\varepsilon\|_0.$$

The theorem is proved. □

5. Approximate solution

Let in the problem (1)–(4) $f(x,y)=0$. Then the solution $(u(x,y,t),\nu(x,y,t))$ can be presented in the form

$$\nu(x,y,t)=\sum_{k,l=1}^{\infty}\varphi^{+}_{k,l}e^{-\lambda^{+}_{k,l}t}\vartheta^{+}_{k,l}+\sum_{k,l=1}^{\infty}\varphi^{-}_{k,l}e^{-\lambda^{-}_{k,l}t}\vartheta^{-}_{k,l}\,,$$

$$u(x,y,t)=\sum_{k,l=1}^{\infty}\varphi^{+}_{k,l}te^{-\lambda^{+}_{k,l}t}\vartheta^{+}_{k,l}+\sum_{k,l=1}^{\infty}\varphi^{-}_{k,l}te^{-\lambda^{-}_{k,l}t}\vartheta^{-}_{k,l}\,,$$

where $\varphi^{\pm}_{k,l}=\pm\int\limits_{-1}^{1}\int\limits_{-1}^{1}sign(x)sign(y)\varphi(x,y)\vartheta^{\pm}_{k,l}(x,y)dxdy\,.$

An approximate solution of the problem is as a sequence of functions given below

$$\nu^{N}(x,y,t)=\sum_{k,l=1}^{N}\varphi^{+}_{k,l}e^{-\lambda^{+}_{k,l}t}\vartheta^{+}_{k,l}+\sum_{k,l=1}^{N}\varphi^{-}_{k,l}e^{-\lambda^{-}_{k,l}t}\vartheta^{-}_{k,l}\,,$$

$$u^{N}(x,y,t)=\sum_{k,l=1}^{N}\varphi^{+}_{k,l}te^{-\lambda^{+}_{k,l}t}\vartheta^{+}_{k,l}+\sum_{k,l=1}^{N}\varphi^{-}_{k,l}te^{-\lambda^{-}_{k,l}t}\vartheta^{-}_{k,l}\,,$$

here N is the integer number of the parameter regularization. An approximate solution with approximate data has the form

$$\nu^{N}_{\varepsilon}(x,y,t)=\sum_{k,l=1}^{N}\varphi\varepsilon^{+}_{k,l}e^{-\lambda^{+}_{k,l}t}\vartheta^{+}_{k,l}+\sum_{k,l=1}^{N}\varphi\varepsilon^{-}_{k,l}e^{-\lambda^{-}_{k,l}t}\vartheta^{-}_{k,l}\,,$$

$$u^{N}_{\varepsilon}(x,y,t)=\sum_{k,l=1}^{N}\varphi\varepsilon^{+}_{k,l}te^{-\lambda^{+}_{k,l}t}\vartheta^{+}_{k,l}+\sum_{k,l=1}^{N}\varphi\varepsilon^{-}_{k,l}te^{-\lambda^{-}_{k,l}t}\vartheta^{-}_{k,l}\,,$$

Let $\|\varphi(x,y)-\varphi_{\varepsilon}(x,y)\|_0\leqslant\varepsilon$ and $(u(x,y,t),\nu(x,y,t))\in M$. Then we estimate the norm of the difference between the exact and approximate solution

$$\begin{aligned}\left\|\nu(x,y,t)-\nu^{N}_{\varepsilon}(x,y,t)\right\|_0&=\left\|\nu(x,y,t)-\nu^{N}(x,y,t)+\nu^{N}(x,y,t)-\nu^{N}_{\varepsilon}(x,y,t)\right\|_0\leqslant\\&\leqslant\left\|\nu(x,y,t)-\nu^{N}(x,y,t)\right\|_0+\left\|\nu^{N}(x,y,t)-\nu^{N}_{\varepsilon}(x,y,t)\right\|_0,\\\left\|u(x,y,t)-u^{N}_{\varepsilon}(x,y,t)\right\|_0&=\left\|u(x,y,t)-u^{N}(x,y,t)+u^{N}(x,y,t)-u^{N}_{\varepsilon}(x,y,t)\right\|_0\leqslant\\&\leqslant\left\|u(x,y,t)-u^{N}(x,y,t)\right\|_0+\left\|u^{N}(x,y,t)-u^{N}_{\varepsilon}(x,y,t)\right\|_0.\end{aligned}\tag{12}$$

Estimating the second term on the right-hand side of (12) we get

$$\begin{aligned}\left\|\nu^{N}(x,y,t)-\nu^{N}_{\varepsilon}(x,y,t)\right\|_0^2&=\sum_{k,l=1}^{N}e^{-2\lambda^{+}_{k,l}t}\left(\varphi^{+}_{k,l}-\varphi^{+}_{\varepsilon k,l}\right)^2+\sum_{k,l=1}^{N}e^{-2\lambda^{-}_{k,l}t}\left(\varphi^{-}_{k,l}-\varphi^{-}_{\varepsilon k,l}\right)^2\leqslant\\&\leqslant e^{-2\lambda^{-}_{N,N}t}\sum_{k,l=1}^{N}\left(\left(\varphi^{-}_{k,l}-\varphi^{-}_{\varepsilon k,l}\right)^2+\left(\varphi^{+}_{k,l}-\varphi^{+}_{\varepsilon k}\right)^2\right)\leqslant e^{-2\lambda^{-}_{N,N}t}\varepsilon^2,\end{aligned}$$

or

$$\left\|\nu^{N}(x,y,t)-\nu^{N}_{\varepsilon}(x,y,t)\right\|_0\leqslant e^{-\lambda^{-}_{N,N}t}\varepsilon.$$

We estimate the first term on the right-hand side of (12)

$$\left\|\nu(x,y,t)-\nu^N(x,y,t)\right\|_0^2=\sum_{k,l=N+1}^{\infty}\left(e^{-2\lambda_{k,l}^{+}t}\left\{\varphi_{k,l}^{+}\right\}^2+e^{-2\lambda_{k,l}^{-}t}\left\{\varphi_{k,l}^{-}\right\}^2\right) \qquad (13)$$

on condition

$$\sum_{k,l=N+1}^{\infty} e^{-2\lambda_{k,l}^{-}T}\left\{\varphi_{k,l}^{-}\right\}^2 \leqslant m^2. \qquad (14)$$

We estimate expression (13) under the condition (14) by the method of Lagrange multipliers. One can see

$$\varphi_{k,l}^{-}=\begin{cases}0, & k,l\neq N+1,\\ e^{\lambda_{k,l}^{-}T}m, & k,l=N+1.\end{cases}$$

As a result, we have

$$\sum_{k,l=N+1}^{\infty} e^{-2\lambda_{k,l}^{-}t}\left\{\varphi_{k,l}^{-}\right\}^2 \leqslant m^2 e^{2\lambda_{N+1,N+1}^{-}(T-t)}.$$

Let's assume that the series $\sum\limits_{k,l=1}^{\infty}\left(\left\{\varphi_{k,l}^{+}\right\}^2+\left\{\varphi_{k,l}^{-}\right\}^2\right)$ converges. Then it is not difficult to see

$$\sum_{k,l=N+1}^{\infty} e^{-2\lambda_{k,l}^{+}t}\left\{\varphi_{k,l}^{+}\right\}^2 \leqslant \sum_{k,l=1}^{\infty}\left\{\varphi_{k,l}^{+}\right\}^2=\alpha(N),$$

where $\alpha(N)\to 0$ for $N\to\infty$. Thus we have

$$\left\|\nu(x,y,t)-\nu^N(x,y,t)\right\|_0^2 \leqslant m^2 e^{2\lambda_{N+1,N+1}^{-}(T-t)}+\alpha(N).$$

Summing the above estimates, we have

$$0.5\left\|\nu(x,y,t)-\nu_{\varepsilon}^N(x,y,t)\right\|_0^2 \leqslant \varepsilon^2 e^{-2\lambda_{N,N}^{-}t}+m^2 e^{2\lambda_{N+1,N+1}^{-}(T-t)}+\alpha(N). \qquad (15)$$

Similarly for $u(x,y,t)$ we have

$$0.5\left\|u(x,y,t)-u_{\varepsilon}^N(x,y,t)\right\|_0^2 \leqslant t^2 e^{-2\lambda_{N,N}^{-}t}\varepsilon^2+m^2\left(\frac{t}{T}\right)^2 e^{2\lambda_{N+1,N+1}^{-}(T-t)}+t^2\alpha(N). \qquad (16)$$

Minimizing the right-hand side of (15) and (16) with respect to N,we find the corresponding regularization parameter. m is selected depending from the application problem.

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Некорректная краевая задача для системы уравнений в частных производных с двумя линиями вырождения

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Данная работа посвящена исследованию некорректной краевой задачи для системы уравнений параболического типа с меняющимся направлением времени с двумя линиями вырождения. Рассматриваемая задача некорректна по Ж. Адамару, а именно отсутствует непрерывная зависимость решения от данных задачи. Подобные уравнения имеют множество различных применений, например, описывают процессы распространения тепла в неоднородных средах, взаимодействия фильтрационных потоков, массопереноса вблизи поверхности летательного аппарата сложных течений вязкой жидкости. В качестве возможных приложений следует также указать задачи расчета теплообменников, в которых используется принцип противотока. Доказаны теоремы о единственности и условной устойчивости решения на множестве корректности. Построена последовательность приближенных (регуляризованных) решений, устойчивых на множестве корректности.

Ключевые слова: параболическое уравнение с меняющимся направлением времени, некорректная задача, априорная оценка, оценка условной устойчивости, единственность решения, приближенное решение.