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Limit Cycles for a Class of Polynomial Differential Systems Via Averaging Theory

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In this paper, we consider the limit cycles of a class of polynomial differential systems of the form

$$\begin{cases} \dot{x} = y - \varepsilon(g_{11}(x)\, y^{2\alpha+1} + f_{11}(x)\, y^{2\alpha}) - \varepsilon^2(g_{12}(x)\, y^{2\alpha+1} + f_{12}(x)\, y^{2\alpha}), \\ \dot{y} = -x - \varepsilon(g_{21}(x)\, y^{2\alpha+1} + f_{21}(x)\, y^{2\alpha}) - \varepsilon^2(g_{22}(x)\, y^{2\alpha+1} + f_{22}(x)\, y^{2\alpha}), \end{cases}$$

where m, n, k, l and α are positive integers, $g_{1\kappa}$, $g_{2\kappa}$, $f_{1\kappa}$ and $f_{2\kappa}$ have degree n, m, l and k, respectively for each $\kappa = 1, 2$, and ε is a small parameter. We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first and second order.

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Introduction

One of the main problems in the qualitative theory of real planar differential equations is to determinate the number of limit cycles for a given planar differential system. As we all know, this is a very difficult problem for a general polynomial system. Therefore, many mathematicians study some systems with special conditions. To obtain the number of limit cycles as many as possible for a planar differential system, we usually take in consideration of the bifurcation theory. In recent decades, many new results have been obtained (see [9, 10]).

The number of medium amplitude limit cycles bifurcating from the linear center $\dot{x} = y$, $\dot{y} = -x$ for the following three kind of generalized polynomial Liénard differential systems

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x - g_2(x) + f_2(x)\, y \end{cases}, \quad \begin{cases} \dot{x} &= y - g_1(x), \\ \dot{y} &= -x - g_2(x) + f_2(x)\, y \end{cases}, \quad \begin{cases} \dot{x} &= y - f_1(x)\, y, \\ \dot{y} &= -x - g_2(x) + f_2(x)\, y \end{cases},$$

where studied in the papers [1, 2, 4–6] and [13, 14], respectively.

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In [12] the authors use the averaging theory of first and second order to study the system

$$\begin{cases} \dot{x} = y - \varepsilon(g_{11}(x) + f_{11}(x)\, y) - \varepsilon^2(g_{12}(x) + f_{12}(x)\, y), \\ \dot{y} = -x - \varepsilon(g_{21}(x) + f_{21}(x)\, y) - \varepsilon^2(g_{22}(x) + f_{22}(x)\, y), \end{cases}$$

where $g_{1i}, f_{1i}, g_{2i}, f_{2i}$ have degree l, k, m, n respectively for each $i = 1, 2$ and ε is a small parameter. They provided an accurate upper bound of the maximum number of limit cycles that the above system can have bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$.

In [7], using the averaging theory of first and second order, the authors studied the number of medium amplitude limit cycles bifurcating from the linear center $\dot{x} = y, \dot{y} = -x$ of the more generalized polynomial Liénard differential systems.

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(p_1(x)\, y + q_1(x)\, y^2) - \varepsilon^2(p_2(x)\, y + q_2(x)\, y^2), \end{cases}$$

where p_1, q_1, p_2 and q_2 have degree n.

By using the averaging theory we shall study in this work the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial differential equations

$$\begin{cases} \dot{x} = y - \varepsilon(g_{11}(x)\, y^{2\alpha+1} + f_{11}(x)\, y^{2\alpha}) - \varepsilon^2(g_{12}(x)\, y^{2\alpha+1} + f_{12}(x)\, y^{2\alpha}), \\ \dot{y} = -x - \varepsilon(g_{21}(x)\, y^{2\alpha+1} + f_{21}(x)\, y^{2\alpha}) - \varepsilon^2(g_{22}(x)\, y^{2\alpha+1} + f_{22}(x)\, y^{2\alpha}), \end{cases} \tag{1}$$

where m, n, k, l and α are positive integers $g_{1\kappa}$, $g_{2\kappa}, f_{1\kappa}$ and $f_{2\kappa}$ have degree n, m, l and k, respectively for each $\kappa = 1, 2$, and ε is a small parameter.

Let $[x]$ denotes the integer part function of $x \in \mathbb{R}$. Our main result is the following one.

Theorem 1. *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial differential systems* (1) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory*
(a) of first order is

$$\lambda_1 = \max\left\{\left[\tfrac{l-1}{2}\right], \left[\tfrac{m}{2}\right]\right\};$$

(b) of second order is

$$\begin{aligned} \lambda \;=\; & \max\left\{\left[\tfrac{m}{2}\right], \left[\tfrac{l}{2}\right] + \left[\tfrac{n-1}{2}\right] + \alpha, \left[\tfrac{k}{2}\right] + \left[\tfrac{m-1}{2}\right] + \alpha, \left[\tfrac{k-1}{2}\right] + \mu + \alpha, \right. \\ & \left. \left[\tfrac{l-1}{2}\right], \left[\tfrac{n-1}{2}\right] + \left[\tfrac{m-1}{2}\right] + 1 + \alpha, \left[\tfrac{l}{2}\right] + \left[\tfrac{k}{2}\right] - 1 + \alpha, \left[\tfrac{n}{2}\right] + \mu + \alpha\right\}, \end{aligned}$$

where $\mu = \min\left\{\left[\tfrac{l-1}{2}\right], \left[\tfrac{m}{2}\right]\right\}$.

The proof of the above theorem is given in Section 2.

1. Averaging theory

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper. The averaging theory up to second order for studying specifically periodic orbits was developed in [3, 11]. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}$. Assume that the following hypotheses hold.

(i) $F_1(t,\cdot) \in C^2(D), F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}, F_1, F_2, R$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε. We define $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(x) &= \frac{1}{T}\int_0^T F_1(s,x)ds, \\ F_{20}(x) &= \frac{1}{T}\int_0^T \frac{\partial F_1(s,x)}{\partial x} y_1(s,x) + F_2(s,x)ds, \end{aligned}$$

where

$$y_1(s,x) = \int_0^s F_1(t,x)dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon f, \varepsilon f)\backslash\{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$.

Then, for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $x(t,\varepsilon)$ of the system such that $x(0,\varepsilon) \to a_\varepsilon$ as $\varepsilon \to 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition in order that this inequality holds is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order.*

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order.*

2. Proof of Theorem 1

2.1. Proof of statement (a) of Theorem 1

For the proof we shall use the first order averaging theory as it was stated in Section 1. We write

$$\begin{aligned} g_{11}(x) &= \sum_{i=0}^{n} a_i x^i,\ g_{21}(x) = \sum_{i=0}^{m} c_i x^i, f_{11}(x) = \sum_{i=0}^{l} b_i x^i,\ f_{21}(x) = \sum_{i=0}^{k} d_i x^i, \\ g_{12}(x) &= \sum_{i=0}^{n} A_i x^i,\ g_{22}(x) = \sum_{i=0}^{m} C_i x^i,\ f_{12}(x) = \sum_{i=0}^{l} B_i x^i,\ f_{22}(x) = \sum_{i=0}^{k} D_{i,j} x^i. \end{aligned}$$

Then in polar coordinates (r,θ) given by $x = r\cos\theta$ and $y = r\sin\theta$, the differential system (1) becomes

$$\begin{cases} \dot{r} = -\varepsilon G_1(r,\theta) - \varepsilon^2 H_1(r,\theta), \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r} G_2(r,\theta) - \dfrac{\varepsilon^2}{r} H_2(r,\theta), \end{cases}$$

where

$$\begin{aligned} G_1(r,\theta) = &\left(\sum_{i=0}^{l} b_i r^{i+2\alpha} \cos^{i+1}\theta \sin^{2\alpha}\theta + \sum_{i=0}^{n} a_i r^{i+2\alpha+1} \cos^{i+1}\theta \sin^{2\alpha+1}\theta + \right. \\ &\left. + \sum_{i=0}^{m} c_i r^{i+2\alpha+1} \cos^{i}\theta \sin^{2\alpha+2}\theta + \sum_{i=0}^{k} d_i r^{i+2\alpha} \cos^{i}\theta \sin^{2\alpha+1}\theta \right), \end{aligned}$$

$$\begin{aligned} H_1(r,\theta) &= \left(\sum_{i=0}^{l} B_i r^{i+2\alpha}\cos^{i+1}\theta\sin^{2\alpha}\theta + \sum_{i=0}^{n} A_i r^{i+2\alpha+1}\cos^{i+1}\theta\sin^{2\alpha+1}\theta\, + \right.\\ &\left. + \sum_{i=0}^{m} C_i r^{i+2\alpha+1}\cos^{i}\theta\sin^{2\alpha+2}\theta + \sum_{i=0}^{k} D_i r^{i+2\alpha}\cos^{i}\theta\sin^{2\alpha+1}\theta\right), \end{aligned}$$

$$\begin{aligned} G_2(r,\theta) &= \left(\sum_{i=0}^{m} c_i r^{i+2\alpha+1}\cos^{i+1}\theta\sin^{2\alpha+1}\theta - \sum_{i=0}^{l} b_i r^{i+2\alpha}\cos^{i}\theta\sin^{2\alpha+1}\theta\, - \right.\\ &\left. - \sum_{i=0}^{n} a_i r^{i+2\alpha+1}\cos^{i}\theta\sin^{2\alpha+2}\theta + \sum_{i=0}^{k} d_i r^{i+2\alpha}\cos^{i+1}\theta\sin^{2\alpha}\theta\right), \end{aligned}$$

$$\begin{aligned} H_2(r,\theta) &= \left(\sum_{i=0}^{m} C_i r^{i+2\alpha+1}\cos^{i+1}\theta\sin^{2\alpha+1}\theta - \sum_{i=0}^{l} B_i r^{i+2\alpha}\cos^{i}\theta\sin^{2\alpha+1}\theta\, - \right.\\ &\left. - \sum_{i=0}^{n} A_i r^{i+2\alpha+1}\cos^{i}\theta\sin^{2\alpha+2}\theta + \sum_{i=0}^{k} D_i r^{i+2\alpha}\cos^{i+1}\theta\sin^{2\alpha}\theta\right). \end{aligned}$$

Taking θ as the new independent variable, system (1) becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + O\left(\varepsilon^3\right), \tag{2}$$

where

$$\begin{aligned} F_1(r,\theta) &= G_1(r,\theta), \\ F_2(r,\theta) &= H_1(r,\theta) - \frac{1}{r}G_1(r,\theta)\,G_2(r,\theta). \end{aligned} \tag{3}$$

First we shall study the limit cycles of the differential equation (2) using the averaging theory of first order. Therefore, by Section 1 we must study the simple positive zeros of the function

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta)\,d\theta.$$

For every one of these zeros we will have a limit cycle of the polynomial differential system (1).

If $F_{10}(r)$ is identically zero, applying the theory of averaging of second order (see again Section 2) every simple positive zero of the function

$$F_{20}(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d}{dr}F_1(r,\theta)\left(\int_0^{\theta} F_1(r,s)ds\right) + F_2(r,\theta)\right)d\theta,$$

will provide a limit cycle of the polynomial differential System (1).

Taking into account the expression of (3), in order to obtain $F_{10}(r)$ is necessary to evaluate the integrals of the form

$$\int_0^{2\pi}\cos^i\theta\sin^j\theta d\theta.$$

In the following lemmas we compute these integrals.

Lemma 2. *Let* $B_{i,j}(\theta) = \cos^i\theta\sin^j\theta$ *and* $A_{i,j}(\theta) = \int_0^\theta B_{i,j}(s)ds$. *Then the following equalities hold:*

$$A_{i,j}(2\pi) = \begin{cases} 0 & \text{if } i \text{ is odd or } j \text{ is odd,} \\ \dfrac{(j-1)(j-3)\dots 1}{(j+i)(j+i-2)\dots(i+2)} \dfrac{1}{2^{i-1}} \dbinom{i}{\frac{i}{2}} \pi & \text{if } i \text{ and } j \text{ are even,} \end{cases}.$$

where $\dbinom{i}{\frac{i}{2}} = \dfrac{i!}{\left(\frac{i}{2}!\right)^2}$, $A_{2i+2,2\alpha}(2\pi) = \dfrac{2i+1}{2(i+\alpha+1)} A_{2i,2\alpha}(2\pi)$, $A_{2i,2\alpha+2}(2\pi) = \dfrac{2\alpha+1}{2(i+\alpha+1)} A_{2i,2\alpha}(2\pi)$, *then,*

$$A_{2i+2,2\alpha}(2\pi) = \frac{2i+1}{2\alpha+1} A_{2i,2\alpha+2}(2\pi).$$

Proof. The integral $A_{i,j}(2\pi)$ can be calculated using the integrals (11), (9), (12) and (10) of the of appendix. □

Using this lemma we shall obtain in the next proposition the integral the function $F_{10}(r)$.

Proposition 3. *We have*

$$F_{10}(r) = \frac{r^{2\alpha+1}}{4} \left(\sum_{i=0}^{\left[\frac{l-1}{2}\right]} \frac{2i+1}{i+\alpha+1} b_{2i+1} r^{2i} \xi_{2i,2\alpha} + \sum_{i=0}^{\left[\frac{m}{2}\right]} \frac{2\alpha+1}{i+\alpha+1} c_{2i} r^{2i} \xi_{2i,2\alpha} \right), \tag{4}$$

where $\xi_{2i,2\alpha} = \dfrac{(2\alpha-1)(2\alpha-3)\dots 1}{(2\alpha+2i)(2\alpha+2i-2)\dots(2i+2)} \dfrac{1}{2^{2i-1}} \dbinom{2i}{i}$.

Proof. Using Lemma 2, the function $F_{10}(r)$ is given by

$$\begin{aligned} 2\pi F_{10}(r) &= \int_0^{2\pi} \left(\sum_{\substack{i=0 \\ i \text{ odd}}}^{l} b_i r^{i+2\alpha} B_{i+1,2\alpha}(\theta) + \sum_{\substack{i=0 \\ i \text{ even}}}^{m} c_i r^{i+2\alpha+1} B_{i,2\alpha+2}(\theta) \right) d\theta = \\ &= \int_0^{2\pi} \left(\sum_{i=0}^{\left[\frac{l-1}{2}\right]} b_{2i+1} r^{2\alpha+2i+1} B_{2i+2,2\alpha}(\theta) + \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i} r^{2i+2\alpha+1} B_{2i,2\alpha+2}(\theta) \right) d\theta = \\ &= \sum_{i=0}^{\left[\frac{l-1}{2}\right]} b_{2i+1} r^{2\alpha+2i+1} A_{2i+2,2\alpha}(2\pi) + \sum_{i=0}^{\left[\frac{m}{2}\right]} c_{2i} r^{2i+2\alpha+1} A_{2i,2\alpha+2}(2\pi) = \\ &= \frac{r^{2\alpha+1}}{2} \left(\sum_{i=0}^{\left[\frac{l-1}{2}\right]} \tfrac{2i+1}{i+\alpha+1} b_{2i+1} r^{2i} A_{2i,2\alpha}(2\pi) + \sum_{i=0}^{\left[\frac{m}{2}\right]} \tfrac{2\alpha+1}{i+\alpha+1} c_{2i} r^{2i} A_{2i,2\alpha}(2\pi) \right). \end{aligned}$$

This completes the proof of Proposition 3. □

From Proposition 3, the polynomial $F_{10}(r)$ has at most $\lambda_1 = \max\left\{\left[\frac{l-1}{2}\right], \left[\frac{m}{2}\right]\right\}$ positive roots, and we can choose b_{2i+1} and c_{2i} a way that $F_{10}(r)$ has exactly λ_1 simple positive roots, hence statement (a) of Theorem 1 is proved.

2.2. Proof of statement (b) of Theorem 1

Now using the results stated in Section 1 we shall apply the second order averaging theory to the previous differential equation, but for doing that we need that $F_{10}(r) \equiv 0$. Therefore, from (4) in what follows we must take

$$\left\{ \begin{array}{ll} c_{2i} = -\dfrac{2i+1}{2\alpha+1} b_{2i+1}, & i = 0, 1, 2, \ldots, \mu \\ b_{2i+1} = c_{2i} = 0, & i = \mu+1, \ldots, \lambda_1 \end{array} \right\}, \tag{5}$$

where $\mu = \min\left\{ \left[\frac{l-1}{2}\right], \left[\frac{m}{2}\right] \right\}$.

We must study the simple positive zeros of the function

$$F_{20}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{d}{dr} F_1(r,\theta) \left(\int_0^{\theta} F_1(r,s) ds \right) + F_2(r,\theta) \right) d\theta.$$

We split the computation of the function $F_{20}(r)$ in two pieces i.e. we define $2\pi F_{20}(r) = L(r) + J(r)$, where

$$L(r) = \int_0^{2\pi} \frac{d}{dr} F_1(r,\theta) \left(\int_0^{\theta} F_1(r,s) ds \right) d\theta, \qquad J(r) = \int_0^{2\pi} F_2(r,\theta)\, d\theta.$$

Lemma 4. *The integral $J(r)$ can be expressed by*

$$J(r) = r^{2\alpha+1} \left(P_1\left(r^2\right) + r^{2\alpha} P_2\left(r^2\right) \right),$$

where P_1 and P_2 are polynomials of degree

$$\lambda_1 = \max\left\{ \left[\tfrac{l-1}{2}\right], \left[\tfrac{m}{2}\right] \right\},$$

and
$\lambda_2 = \max\left\{ \left[\frac{k}{2}\right] + \left[\frac{m-1}{2}\right], \left[\frac{n-1}{2}\right] + \left[\frac{m-1}{2}\right] + 1, \left[\frac{l}{2}\right] + \left[\frac{n-1}{2}\right], \left[\frac{k-1}{2}\right] + \mu, \left[\frac{n}{2}\right] + \mu, \left[\frac{l}{2}\right] + \left[\frac{k}{2}\right] - 1 \right\}$, *respectively.*

Proof. First we calculate $\int_0^{2\pi} H_1(r,\theta)\, d\theta$.

$$\begin{aligned} \int_0^{2\pi} H_1(r,\theta)\, d\theta &= \sum_{i=0}^{l} B_i r^{i+2\alpha} \int_0^{2\pi} B_{i+1,2\alpha}(\theta)\, d\theta + \sum_{i=0}^{k} D_i r^{i+2\alpha} \int_0^{2\pi} B_{i,2\alpha+1}(\theta)\, d\theta + \\ &+ \sum_{i=0}^{m} C_i r^{i+2\alpha+1} \int_0^{2\pi} B_{i,2\alpha+2}(\theta)\, d\theta + \sum_{i=0}^{n} A_i r^{i+2\alpha+1} \int_0^{2\pi} B_{i+1,2\alpha+1}(\theta)\, d\theta = \\ &= \sum_{\substack{i=0 \\ i \text{ odd}}}^{l} B_i r^{i+2\alpha} \int_0^{2\pi} B_{i+1,2\alpha}(\theta)\, d\theta + \sum_{\substack{i=0 \\ i \text{ even}}}^{m} C_i r^{i+2\alpha+1} \int_0^{2\pi} B_{i,2\alpha+2}(\theta)\, d\theta = \\ &= \sum_{i=0}^{\left[\frac{l-1}{2}\right]} B_{2i+1} A_{2i+2,2i}(2\pi)\, r^{2i+2\alpha+1} + \sum_{i=0}^{\left[\frac{m}{2}\right]} C_{2i} A_{2i,2i+2}(2\pi)\, r^{2i+2\alpha+1} = \\ &= r^{2\alpha+1} P_1\left(r^2\right). \end{aligned}$$

where P_1 is a polynomial in the variable r^2 of degree λ_1. Finally we shall study the contribution of the second part $\int_0^{2\pi} \frac{1}{r} G_1(r,\theta) G_2(r,\theta)\, d\theta$ of $F_2(\theta, r)$ to $F_{20}(r)$. Taking into account the expression of (2), then

$$G_1(r,\theta) = \sum_{i=0}^{\mu} \left(B_{2i+2,2\alpha}(\theta) - \frac{2i+1}{2\alpha+1} B_{2i,2\alpha+2}(\theta) \right) b_{2i+1} r^{2i+2\alpha+1} +$$
$$+ \sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1} r^{2i+2\alpha+2} B_{2+1,2\alpha+2}(\theta) + \sum_{i=0}^{\left[\frac{l}{2}\right]} b_{2i} r^{2i+2\alpha} B_{2i+1,2\alpha}(\theta) +$$
$$+ \sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1} r^{2i+2\alpha+1} B_{2i+1,2\alpha+1}(\theta) + + \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1} r^{2i+2\alpha+2} B_{2i+2,2\alpha+1}(\theta) +$$
$$+ \sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i} r^{2i+2\alpha} B_{2i,2\alpha+1}(\theta) + \sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i} r^{2i+2\alpha+1} B_{2i+1,2\alpha+1}(\theta).$$

$$G_2(r,\theta) = \sum_{p=0}^{\left[\frac{k}{2}\right]} d_{2p} r^{2p+2\alpha} B_{2p+1,2\alpha}(\theta) + \sum_{p=0}^{\left[\frac{m-1}{2}\right]} c_{2p+1} r^{2p+1+2\alpha+1} B_{2p+2,2\alpha+1}(\theta) -$$
$$- \sum_{p=0}^{\left[\frac{l}{2}\right]} b_{2p} r^{2p+2\alpha} B_{2p,2\alpha+1}(\theta) - \sum_{p=0}^{\mu} 2\frac{\alpha+1+i}{2\alpha+1} b_{2p+1} r^{2(p+\alpha)+1} B_{2i+1,2\alpha+1}(\theta) -$$
$$- \sum_{p=0}^{\left[\frac{n}{2}\right]} a_{2p} r^{p+2\alpha+1} B_{2p,2\alpha+2}(\theta) + \sum_{p=0}^{\left[\frac{k-1}{2}\right]} d_{2p+1} r^{2p+2\alpha+1} B_{2p+2,2\alpha}(\theta) -$$
$$- \sum_{p=0}^{\left[\frac{n-1}{2}\right]} a_{2p+1} r^{2p+2\alpha+2} B_{2p+1,2\alpha+2}(\theta).$$

From the 49 products between the different sums only 12 will not be zero after the integration with respect to θ between 0 and 2π. So the terms of $\int_0^{2\pi} \frac{1}{r} G_1(r,\theta) G_2(r,\theta)\, d\theta$ which will contribute to $F_{20}(r)$ are

$$\int_0^{2\pi} \frac{1}{r} G_1(r,\theta) G_2(r,\theta)\, d\theta = \sum_{i=0}^{\left[\frac{l}{2}\right]} \sum_{p=0}^{\left[\frac{k}{2}\right]} d_{2p} b_{2i} A_{2i+2p+2,4\alpha}(2\pi) r^{2p+2i+4\alpha-1} +$$
$$+ \sum_{i=0}^{\left[\frac{m-1}{2}\right]} \sum_{p=0}^{\left[\frac{k}{2}\right]} d_{2p} c_{2i+1} A_{2i+2p+2,4\alpha+2}(2\pi) r^{2p+2i+4\alpha+1} -$$
$$- \sum_{p=0}^{\left[\frac{l}{2}\right]} \sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1} b_{2p} A_{2i+2p+2,4\alpha+2}(2\pi) r^{2p+2i+4\alpha+1} -$$
$$- \sum_{p=0}^{\left[\frac{l}{2}\right]} \sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i} b_{2p} A_{2i+2p,4\alpha+2}(2\pi) r^{2p+2i+4\alpha-1} -$$

$$-\sum_{p=0}^{\left[\frac{n}{2}\right]}\sum_{i=0}^{\mu} b_{2i+1}a_{2p}\left(A_{2i+2p+2,4\alpha+2}\left(2\pi\right)-\tfrac{2i+1}{2\alpha+1}A_{2i+2p,4\alpha+4}\left(2\pi\right)\right)r^{2p+2i+4\alpha+1}-$$

$$-\sum_{p=0}^{\mu}\sum_{i=0}^{\left[\frac{k-1}{2}\right]} 2\frac{\alpha+1+i}{2\alpha+1}d_{2i+1}b_{2i+1}A_{2i+2p+2,4\alpha+2}\left(2\pi\right)r^{2p+2i+4\alpha+1}-$$

$$-\sum_{p=0}^{\mu}\sum_{i=0}^{\left[\frac{n}{2}\right]} 2\frac{\alpha+1+i}{2\alpha+1}a_{2i}b_{2i+1}A_{2i+2p+2,4\alpha+2}\left(2\pi\right)r^{2p+2i+4\alpha+1}+$$

$$+\sum_{i=0}^{\mu}\sum_{p=0}^{\left[\frac{k-1}{2}\right]} d_{2p+1}b_{2i+1}\left(A_{2i+2p+4,4\alpha}\left(2\pi\right)-\frac{2i+1}{2\alpha+1}A_{2i+2p+2,4\alpha+2}\left(2\pi\right)\right)r^{2p+2i+4\alpha+1}-$$

$$-\sum_{p=0}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{\left[\frac{l}{2}\right]} b_{2i}a_{2p+1}A_{2i+2p+2,4\alpha+2}\left(2\pi\right)r^{2p+2i+4\alpha+1}-$$

$$-\sum_{p=0}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1}a_{2p+1}A_{2i+2p+2,4\alpha+4}\left(2\pi\right)r^{2p+2i+4\alpha+3}+$$

$$+\sum_{2p+1=0}^{\left[\frac{m-1}{2}\right]}\sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1}c_{2p+1}A_{2i+2p+4,4\alpha+2}\left(2\pi\right)r^{2p+2i+4\alpha+3}+$$

$$+\sum_{2p+1=0}^{\left[\frac{m-1}{2}\right]}\sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i}c_{2p+1}A_{2i+2p+2,4\alpha+2}\left(2\pi\right)r^{2p+2i+4\alpha+1}=r^{4\alpha+1}P_2\left(r^2\right),$$

where P_2 is a polynomial in the variable r^2 of degree

$\lambda_3= \max\left\{\left[\frac{k}{2}\right]+\left[\frac{m-1}{2}\right],\left[\frac{n-1}{2}\right]+\left[\frac{m-1}{2}\right]+1,\left[\frac{l}{2}\right]+\left[\frac{n-1}{2}\right],\ \left[\frac{k-1}{2}\right]+\mu,\left[\frac{n}{2}\right]+\mu,\left[\frac{l}{2}\right]+\left[\frac{k}{2}\right]-1\right\}.$

Finally, we obtain $J(r)$ is a polynomial in the variable r^2

$$J(r)=r^{2\alpha+1}\left(P_1\left(r^2\right)+r^{2\alpha}P_2\left(r^2\right)\right),$$

of degree

$$\lambda_{J(r)}=\max\left\{\lambda_1,\lambda_3+\alpha\right\}.$$

This completes the proof of the lemma. □

In order to complete the computation of $F_{20}(r)$ we must determine the function $L(r)$. First we compute the integrals $\int_0^{2\pi} A_{i,j}(\theta)B_{p,q}(\theta)d\theta$. In the following lemmas we compute these integrals.

Lemma 5. *Let* $S_i(\theta)=A_{2i+2,2\alpha}(\theta)-\dfrac{2i+1}{2\alpha+1}A_{2i,2\alpha+2}(\theta)$, *then*

$$\begin{aligned}S_i(\theta) &= -\frac{1}{2(\alpha+i+1)}\left(B_{2i+3,2\alpha+1}(\theta)+\sum_{l=1}^{\alpha-1}\gamma_{2i+2,2\alpha}B_{2i+3,2\alpha-2l-1}(\theta)\right)+\\ &\quad+\frac{1}{2(i+1)}\eta_{2i+2,2\alpha}\left(B_{2i+1,1}(\theta)+\sum_{l=1}^{i}\delta_{2i+2,0}B_{2i-2l+1,1}(\theta)\right)-\end{aligned}$$

$$-\frac{2i+1}{2(\alpha+i+1)(2\alpha+1)}\left(B_{2i+1,2\alpha+3}(\theta)+\sum_{l=1}^{\alpha}\gamma_{2i,2\alpha+2}B_{2i+1,2\alpha-2l+1}(\theta)\right)-$$
$$-\frac{2i+1}{2i(2\alpha+1)}\eta_{2i,2\alpha+2}\left(B_{2i-1,1}(\theta)+\sum_{l=1}^{i-1}\delta_{2i,0}B_{2i-2l-1,1}(\theta)\right),$$

where

$$\gamma_{i,2\alpha}=\frac{(2\alpha-1)(2\alpha-3)\dots(2\alpha-2l+1)}{(2\alpha+i-2)(2\alpha+i-4)\dots(2\alpha+i-2l)},\quad \eta_{i,2\alpha}=\frac{(2\alpha-1)(2\alpha-3)\dots 1}{(2\alpha+i)(2\alpha+i-2)\dots(i+2)},$$
$$\delta_{2i,0}=\frac{(2i-1)(2i-3)\dots(2i-2l+1)}{2^l(i-1)(i-2)\dots(i-l)}.$$

Moreover

$$S_i(2\pi)=0.$$

Proof. Using the integrals (11) and (9) of appendix and, taking into account that

$$\frac{(2j-1)(2j-3)\dots 1}{(2j+2i+2)(2j+2i)\dots(2i+4)}\frac{1}{2^{2(i+1)}}\binom{2i+2}{i+1}\theta-$$
$$-\frac{2i+1}{2j+1}\frac{(2j+1)(2j-1)\dots 1}{(2j++2i2)(2j+2i)\dots(2i+2)}\frac{1}{2^{2i}}\binom{2i}{i}\theta=0,$$

it follows the expression of $S_i(\theta)$. □

Lemma 6. *Let* $\varphi_{i,j}^{p,q}(2\pi)=\int_0^{2\pi}A_{i,j}(\theta)B_{p,q}(\theta)d\theta$ *Then the following equalities hold:*

a) The integral $\varphi_{2i+1,0}^{p,q}(2\pi)$ *is zero if p is odd or q is even, and equal to*

$$\frac{1}{2i+1}\left(A_{2i+p,q+1}(2\pi)+\sum_{l=0}^{i-1}\frac{2^{l+1}i(i-1)\dots(i-l)}{(2i-1)(2i-3)\dots(2i-2l-1)}A_{2i+p+2l-2,q+1}(2\pi)\right),$$

if p is even and q is odd.

b) The integral $\varphi_{2i+1,2j+1}^{p,q}(2\pi)$ *is zero if p is odd or q is odd, and equal to*

$$-\frac{1}{2(j+i+1)}\Bigg(A_{2i+p+2,2j+q}(2\pi)+$$
$$+\sum_{l=1}^{j-1}\frac{2^l j(j-1)\dots(j-l+1)}{(2j+2i)(2j+2i-2)\dots(2j+2i-2l+2)}A_{2i+p+2,2j-2l+q}(2\pi)\Bigg),$$

if p is even and q is even.

c) The integral $\varphi_{2i,2j+1}^{p,q}(2\pi)$ *is zero if p is even or q is odd, and equal to*

$$-\frac{1}{2j+2i+1}\Bigg(A_{2i+p+1,2j+q}(2\pi)+$$
$$+\sum_{l=1}^{j-1}\frac{2^l j(j-1)\dots(j-l+1)}{(2j+2i-1)(2j+2i-3)\dots(2j+2i-2l+1)}A_{2i+p+1,2j-2l+q}(2\pi)\Bigg),$$

if p is odd and q is even.

d) The integral $\varphi^{p,q}_{2i+1,2j}(2\pi)$ *is zero if* p *is odd or* q *is even, and equal to*

$$-\frac{1}{2j+2i+1}\Bigg(A_{2i+p+2,2j+q+1}\left(2\pi\right)- -\sum_{l=1}^{j-1}\frac{(2j-1)\,(2j-3)\ldots(2j-2l+1)}{(2j+2i-1)\,(2j+2i-3)\ldots(2j+2i-2l+1)}A_{2i+2+p,2j-2l+q-1}\left(2\pi\right)\Bigg)+ +\frac{(2j-1)\,(2j-3)\ldots 1}{(2j+2i+1)\,(2j+2i-1)\ldots(2i+3)}\varphi^{p,q}_{2i+1,0}(2\pi),$$

if p *is even and* q *is odd.*

e) The integral $T^{p,q}_{i,\alpha}\left(2\pi\right)=\int\limits_0^{2\pi}S_i(\theta)B_{p,q}(\theta)d\theta=\varphi^{p,q}_{2i+2,2\alpha}(2\pi)-\dfrac{2i+1}{2\alpha+1}\varphi^{p,q}_{2i,2\alpha+2}(2\pi)$ *is zero if* p *is even or* q *is even, and equal to*

$$-\frac{1}{2\alpha+2i+2}\left(A_{2i+p+3,2\alpha+q+1}(2\pi)+\sum_{l=1}^{\alpha-1}\gamma_{2i+2,2\alpha}A_{2i+p+3,2\alpha+q-2l-1}(2\pi)\right)+ +\eta_{2i+2,2\alpha}\frac{1}{2\left(i+1\right)}\left(A_{2i+p+1,q+1}(2\pi)+\sum_{l=1}^{i}\delta_{2i+2,0}A_{p+1-2l+2i,q+1}(2\pi)\right)- -\frac{2i+1}{\left(2\alpha+2+2i\right)\left(2\alpha+1\right)}\left(A_{p+2i+1,q+2\alpha+3}(2\pi)+\sum_{l=1}^{\alpha}\gamma_{2i,2\alpha+2}A_{p+2i+1,q+2\alpha-2l+1}(2\pi)\right)- -\frac{2i+1}{2i\left(2\alpha+1\right)}\eta_{2i,2\alpha+2}\left(A_{p+2i-1,q+1}(2\pi)+\sum_{l=1}^{i-1}\delta_{2i,0}A_{Pp+2i-2l-1,q+1}(2\pi)\right),$$

where

$$\gamma_{i,2\alpha}=\frac{(2\alpha-1)\,(2\alpha-3)\ldots(2\alpha-2l+1)}{(2\alpha+i-2)\,(2\alpha+i-4)\ldots(2\alpha+i-2l)},\ \eta_{i,2\alpha}=\frac{(2\alpha-1)\,(2\alpha-3)\ldots 1}{(2\alpha+i)\,(2\alpha+i-2)\ldots(i+2)},$$

$$\delta_{2i,0}=\frac{(2i-1)\,(2i-3)\ldots(2i-2l+1)}{2^l\,(i-1)\,(i-2)\ldots(i-l)}$$

if p *is odd and* q *is odd.*

f) $\tilde{T}^{p,q}_{i,\alpha}\left(2\pi\right)=\varphi^{2p+2,2\alpha}_{2i+1,2\alpha+1}(2\pi)-\dfrac{2p+1}{2\alpha+1}\varphi^{2p,2\alpha+2}_{2i+1,2\alpha+1}(2\pi)=2\dfrac{(2p+5\alpha+2i\alpha+6p\alpha+2+i)}{(2\alpha+1)\,(p+2+i)}\pi.$

Proof. Using the integrals of the appendix, the six equalities are easily deduced by direct calculation. □

Lemma 7. *The integral* $L(r)$ *can be expressed by*

$$L(r)=r^{1+4\alpha}P_3\left(r^2\right),$$

where P_3 *is a polynomial of degree*

$$\lambda_3=\max\left\{\mu+\left[\tfrac{k-1}{2}\right],\left[\tfrac{l}{2}\right]+\left[\tfrac{n-1}{2}\right],\left[\tfrac{m-1}{2}\right]+\left[\tfrac{n-1}{2}\right]+1,\left[\tfrac{l}{2}\right]+\left[\tfrac{k}{2}\right]-1,\left[\tfrac{m-1}{2}\right]+\left[\tfrac{k}{2}\right],\mu+\left[\tfrac{n}{2}\right]\right\}.$$

Proof. Using (2.), we get

$$\begin{aligned}F_1(r,\theta) &= \sum_{i=0}^{\mu} b_{2i+1}r^{2i+2\alpha+1}\left(B_{2i+2,2\alpha}(\theta)-\frac{2i+1}{2\alpha+1}B_{2i,2\alpha+2}(\theta)\right)+\\ &+\sum_{i=0}^{\left[\frac{l}{2}\right]} b_{2i}r^{2i+2\alpha}B_{2i+1,2\alpha}(\theta)+\sum_{i=0}^{\left[\frac{m-1}{2}\right]} c_{2i+1}r^{2i+2\alpha+2}B_{2i+1,2\alpha+2}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{k-1}{2}\right]} d_{2i+1}r^{2i+2\alpha+1}B_{2i+1,2\alpha+1}(\theta)+\sum_{i=0}^{\left[\frac{n}{2}\right]} a_{2i}r^{2i+2\alpha+1}B_{2i+1,2\alpha+1}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{k}{2}\right]} d_{2i}r^{2i+2\alpha}B_{2i,2\alpha+1}(\theta)+\sum_{i=0}^{\left[\frac{n-1}{2}\right]} a_{2i+1}r^{2i+2\alpha+2}B_{2i+2,2\alpha+1}(\theta).\end{aligned}$$

Next we calculate the terms of this integral. First we have that

$$\begin{aligned}\frac{dF_1(r,\theta)}{dr} &= \sum_{i=0}^{\mu}(2i+2\alpha+1)b_{2i+1}r^{2i+2\alpha}\left(B_{2i+2,2\alpha}(\theta)-\tfrac{2i+1}{2\alpha+1}B_{2i,2\alpha+2}(\theta)\right)+\\ &+\sum_{i=0}^{\left[\frac{l}{2}\right]}2(i+\alpha)b_{2i}r^{2i+2\alpha-1}B_{2i+1,2\alpha}(\theta)+\sum_{i=0}^{\left[\frac{m-1}{2}\right]}2(i+\alpha+1)c_{2i+1}r^{2i+2\alpha+1}B_{2i+1,2\alpha+2}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{k-1}{2}\right]}(2i+2\alpha+1)d_{2i+1}r^{2i+2\alpha}B_{2i+1,2\alpha+1}(\theta)+\sum_{i=0}^{\left[\frac{n}{2}\right]}(2i+2\alpha+1)a_{2i}r^{2i+2\alpha}B_{2i+1,2\alpha+1}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{k}{2}\right]}2(i+\alpha)d_{2i}r^{2i+2\alpha-1}B_{2i,2\alpha+1}(\theta)+\sum_{i=0}^{\left[\frac{n-1}{2}\right]}2(i+\alpha+1)a_{2i+1}r^{2i+2\alpha+1}B_{2i+2,2\alpha+1}(\theta)\end{aligned}$$

and

$$\begin{aligned}\int_0^{\theta}F_1(r,s)ds &= \sum_{i=0}^{\mu}b_{2i+1}r^{2i+2\alpha+1}S_i(\theta)+\sum_{i=0}^{\left[\frac{l}{2}\right]}b_{2i}r^{2i+2\alpha}A_{2i+1,2\alpha}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{m-1}{2}\right]}c_{2i+1}r^{2i+2\alpha+2}A_{2i+1,2\alpha+2}(\theta)+\sum_{i=0}^{\left[\frac{k-1}{2}\right]}d_{2i+1}r^{2i+2\alpha+1}A_{2i+1,2\alpha+1}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{n}{2}\right]}a_{2i}r^{2i+2\alpha+1}A_{2i+1,2\alpha+1}(\theta)+\sum_{i=0}^{\left[\frac{k}{2}\right]}d_{2i}r^{2i+2\alpha}A_{2i,2\alpha+1}(\theta)+\\ &+\sum_{i=0}^{\left[\frac{n-1}{2}\right]}a_{2i+1}r^{2i+2\alpha+2}A_{2i+2,2\alpha+1}(\theta).\end{aligned}$$

From the 49 products between the different sums only 12 will not be zero after the integration with respect to θ between 0 and 2π. So the terms of $L(r)$ which will contribute to $F_{20}(r)$ are

$$L(r)=\sum_{p=0}^{\mu}\sum_{i=0}^{\left[\frac{k-1}{2}\right]}(2p+2\alpha+1)d_{2i+1}b_{2p+1}\tilde{T}_{i,\alpha}^{p,q}(2\pi)r^{2i+2p+4\alpha+1}+$$

$$+\sum_{i=0}^{\left[\frac{n}{2}\right]}\sum_{p=0}^{\mu}(2p+2\alpha+1)\,a_{2i}b_{2p+1}\tilde{T}_{i,\alpha}^{p,q}\,(2\pi)\,r^{2i+2p+4\alpha+1}+$$
$$+\sum_{p=0}^{\left[\frac{l}{2}\right]}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}2\,(p+\alpha)\,b_{2p}a_{2i+1}\varphi_{2i+2,2\alpha+1}^{2p+1,2\alpha}(2\pi)r^{2i+4\alpha+2p+1}+$$
$$+\sum_{p=0}^{\left[\frac{l}{2}\right]}\sum_{i=0}^{\left[\frac{k}{2}\right]}2\,(p+\alpha)\,b_{2p}d_{2i}\varphi_{2i,2\alpha+1}^{2p+1,2\alpha}(2\pi)r^{2i+2p+4\alpha-1}+$$
$$+\sum_{p=0}^{\left[\frac{m-1}{2}\right]}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}2\,(p+\alpha+1)\,c_{2p+1}a_{2i+1}\varphi_{2i+2,2\alpha+1}^{2p+1,2\alpha+2}(2\pi)r^{2i+2p+4\alpha+3}+$$
$$+\sum_{p=0}^{\left[\frac{m-1}{2}\right]}\sum_{i=0}^{\left[\frac{k}{2}\right]}2\,(p+\alpha+1)\,c_{2p+1}d_{2i}\varphi_{2i,2\alpha+1}^{2p+1,2\alpha+2}(2\pi)r^{2i+2p+4\alpha+1}+$$
$$+\sum_{p=0}^{\left[\frac{k-1}{2}\right]}\sum_{i=0}^{\mu}(2p+2\alpha+1)\,d_{2p+1}b_{2i+1}T_{i,\alpha}^{2p+1,2\alpha+1}\,(2\pi)\,r^{2i+2p+4\alpha+1}+$$
$$+\sum_{p=0}^{\left[\frac{n}{2}\right]}\sum_{i=0}^{\mu}(2p+2\alpha+1)\,a_{2p}b_{2i+1}T_{i,\alpha}^{2p+1,2\alpha+1}\,(2\pi)\,r^{2i+2p+4\alpha+1}+$$
$$+\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{i=0}^{\left[\frac{m-1}{2}\right]}(2p+2\alpha)\,d_{2p}c_{2i+1}\varphi_{2i+1,2\alpha+2}^{2p,2\alpha+1}(2\pi)r^{2i+2p+4\alpha+1}+$$
$$+\sum_{p=0}^{\left[\frac{k}{2}\right]}\sum_{i=0}^{\left[\frac{l}{2}\right]}2\,(p+\alpha)\,d_{2p}b_{2i}\varphi_{2i+1,2\alpha}^{2p,2\alpha+1}(2\pi)r^{2i+2p+4\alpha-1}+$$
$$+\sum_{p=0}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{\left[\frac{m-1}{2}\right]}2\,(p+\alpha+1)\,a_{2p+1}c_{2i+1}\varphi_{2i+1,2\alpha+2}^{2p+2,2\alpha+1}(2\pi)r^{2i+2p+4\alpha+3}+$$
$$+\sum_{p=0}^{\left[\frac{n-1}{2}\right]}\sum_{i=0}^{\left[\frac{l}{2}\right]}2\,(p+\alpha+1)\,a_{2p+1}b_{2i}\varphi_{2i+1,2\alpha}^{2p+2,2\alpha+1}(2\pi)r^{2i+2p+4\alpha+1}=$$
$$=r^{1+4\alpha}P_3\left(r^2\right),$$

where P_3 is a polynomial in the variable r^2 of degree

$$\lambda_3=\max\Big\{\mu+\left[\tfrac{k-1}{2}\right],\left[\tfrac{l}{2}\right]+\left[\tfrac{n-1}{2}\right],\left[\tfrac{m-1}{2}\right]+\left[\tfrac{n-1}{2}\right]+1,\left[\tfrac{l}{2}\right]+\left[\tfrac{k}{2}\right]-1,$$
$$\left[\tfrac{m-1}{2}\right]+\left[\tfrac{k}{2}\right],\mu+\left[\tfrac{n}{2}\right]\Big\}.$$

This completes the proof of the lemma. □

Finally, we obtain $F_{20}(r)$ is a polynomial in the variable r^2 of the form

$$2\pi F_{20}(r)=r^{2\alpha+1}\Big(r^{2\alpha}P_1\left(r^2\right)+P_2\left(r^2\right)+r^{2\alpha}P_3\left(r^2\right)\Big).$$

Then, to find the real positive roots of F_{20}, we must find the zeros of a polynomial in r^2 of degree $\lambda=\max\left\{\lambda_{J(r)},\lambda_3+\alpha\right\}$. This yields that F_{20} has at most λ real positive roots. Moreover, we can choose the coefficients a_i, b_i, c_i, d_i, A_i, B_i, C_i and D_i in such a way that F_{20} has exactly λ real positive roots. Hence, the statement (b) of Theorem 1 is proved.

In fact, we consider the example with $n = k = 1$ and $m = l = 3$, $\alpha = 1$

$$\begin{cases} \dot{x} = y - \varepsilon\left(\left(1+x^3\right)y^3 + \frac{49}{80}xy^2\right) - \varepsilon^2\left(\left(1+x^3\right)y^3 + \left(1 - \frac{3}{20}x\right)y^2\right) \\ \dot{y} = -x - \varepsilon\left(\left(-\frac{49}{240} + \frac{1}{9}x\right)y^3 + \left(x - \frac{16}{21}x^3\right)y^2\right) - \varepsilon^2\left(xy^3 + x^3y^2\right). \end{cases} \tag{6}$$

We have that $F_{10}(r)$ is identically zero, so to look for the limit cycles, we must solve the equation $F_{20}(r) = 0$ which is equivalent to

$$2\pi F_{02}\left(r\right) = \frac{1}{960}\pi r^9 - \frac{7}{480}\pi r^7 + \frac{49}{960}\pi r^5 - \frac{3}{80}\pi r^3.$$

This equation has exactly the three positive roots $r_1 = 1, r_2 = 2$ and $r_3 = 3$.

According with Theorem 1, that system (6) has exactly there limit cycles bifurcating from the periodic orbits of the linear differential system (6) with $\varepsilon = 0$, using the averaging theory of second order.

We consider the differential system with $n = k = 1$ and $m = l = 3$, $\alpha = 2$

$$\begin{cases} \dot{x} = y - \varepsilon\left(\left(1+x^3\right)y^5 + \left(\frac{41}{2548} - 5x\right)y^4\right) - \varepsilon^2\left(\left(1+x^3\right)y^5 + \left(1 + \frac{9}{35}x\right)y^4\right) \\ \dot{y} = -x - \varepsilon\left(\left(1 + \frac{1}{90}x\right)y^5 + \left(\frac{1929}{50\,960}x^3 - \frac{13}{2}\right)y^4\right) - \varepsilon^2\left(xy^5 + x^3y^4\right). \end{cases} \tag{7}$$

An easy computation shows that $F_{10}(r) \equiv 0$ and

$$2\pi F_{02}\left(r\right) = \frac{1}{17\,920}\pi r^{13} - \frac{3}{1792}\pi r^{11} + \frac{39}{2560}\pi r^9 - \frac{41}{896}\pi r^7 + \frac{9}{280}\pi r^5.$$

Therefore from the periodic orbits of radius $1, 2, 3$ and 4 of the linear center $\dot{x} = y$, $\dot{y} = -x$, it bifurcates three limit cycles. Consequently for system (7) we have that $\lambda = \left[\frac{n-1}{2}\right] + \left[\frac{m-1}{2}\right] + \alpha + 1 = 4$.

3. Appendix

Here we list some important formulas used in this article, for more details see [8]. For $i \geqslant 0$ and $j \geqslant 0$, we have

$$\begin{aligned} \int_0^\theta \cos^i s \sin^\alpha s ds &= \frac{\cos^{i-1}\theta \sin^{\alpha+1}\theta}{i+\alpha} + \frac{i-1}{i+\alpha}\int_0^\theta \cos^{i-2} s \sin^\alpha s ds = \\ &= -\frac{\cos^{i+1}\theta \sin^{\alpha-1}\theta}{i+\alpha} + \frac{\alpha-1}{i+\alpha}\int_0^\theta \cos^i s \sin^{\alpha-2} s ds. \end{aligned} \tag{8}$$

$$\begin{aligned} \int_0^\theta \cos^{2i} s ds = \frac{\sin\theta}{2i}\left(\cos^{2i-1}\theta + \sum_{l=1}^{i-1}\frac{(2i-1)(2i-3)\ldots(2i-2l+1)}{2^l(i-1)(i-2)\ldots(i-l)}\cos^{2i-2l-1}\theta\right) + \\ + \frac{(2i-1)(2i-3)\ldots 1}{2^i i!}\theta = \frac{1}{2^{2i-1}}\sum_{l=0}^{i-1}\binom{2i}{l}\frac{\sin 2(i-l)\theta}{2(i-l)} + \frac{1}{2^{2i}}\binom{2i}{i}\theta. \end{aligned} \tag{9}$$

$$\int_0^\theta \cos^{2i+1} s ds = \frac{\sin\theta}{2i+1}\left(\cos^{2i}\theta + \sum_{l=0}^{i-1}\frac{2^{l+1}i\,(i-1)\ldots(i-l)}{(2i-1)\,(2i-3)\ldots(2i-2l-1)}\cos^{2i-2l-2}\theta\right) = \\ = \frac{1}{2^{2i}}\sum_{l=0}^{i-1}\binom{2i+1}{l}\frac{\sin(2i-2l+1)\,\theta}{(2i-2l+1)}. \quad (10)$$

$$\int_0^\theta \cos^i s \sin^{2\alpha} s ds = \\ = -\frac{\cos^{i+1}\theta}{2\alpha+i}\left(\sum_{l=1}^{\alpha-1}\frac{(2\alpha-1)\,(2\alpha-3)\ldots(2\alpha-2l+1)}{(2\alpha+i-2)\,(2\alpha+i-4)\ldots(2\alpha+i-2l)}\sin^{2\alpha-2l-1}\theta + \sin^{2\alpha+1}\theta\right) + \\ + \frac{(2\alpha-1)\,(2\alpha-3)\ldots 1}{(2\alpha+i)\,(2\alpha+i-2)\ldots(i+2)}\int_0^\theta \cos^i s ds. \quad (11)$$

$$\int_0^\theta \cos^i s \sin^{2\alpha+1} s ds = \\ = -\frac{\cos^{i+1}\theta}{2\alpha+i+1}\left(\sin^{2\alpha}\theta + \sum_{l=1}^{\alpha-1}\frac{2^l\alpha\,(\alpha-1)\ldots(\alpha-l+1)}{(2\alpha+i-1)\,(2\alpha+i-3)\ldots(2\alpha+i-2l+1)}\sin^{2\alpha-2l}\theta\right). \quad (12)$$

References

[1] J.Alavez-Ramirez, G.Blé, J.Llibre and J. Lopez-Lopez, On the maximum number of limit cycles of a class of generalized Liénard differential systems, *Int. J. Bifurcation and Chaos*, **22**(2012), 1250063.

[2] T.R.Blows, N.G.Lloyd, The number of small-amplitude limit cycles of Liénard equations, *Math. Proc. Camb. Phil. Soc.*, **95**(1984), 359–366.

[3] A.Buica, J., Llibre, Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. Math.*, **128**(2004), 7–22.

[4] X.Chen, J.Llibre, Z.Zhang, Sufficient conditions for the existence of at least n or exactly n limit cycles for the Lienard differential systems, *J. Differential Equations*, **242**(2007), 11–23.

[5] C.J.Christopher, S.Lynch, Small-amplitude limti cycle bifurcations for Liénard systems with quadratic or cubic dapimg or restoring forces, *Nonlinearity*, **12**(1999), 1099–1112.

[6] W.A.Coppel, Some quadratic systems with at most one limit cycles, Dynamics Reported, Vol. 2, Wiley, New York, 1998, 61–68.

[7] B.Garca, J.Llibre, J.S.Pérez del Rio, Limit cycles of generalized Liénard polynomial differential systems via averaging theory, *Chaos Solitons Fractals*, **62-63**(2014), 1–9.

[8] I.S.Gradshteyn, I.M.Ryzhik, Table of Integrals, Series and Products, Academic Press, 1979.

[9] M.Han, P.Yu, Normal forms, Melnikov functions and bifurctions of limit cycles, Applied Mathematical Sciences, Vol. 181, Springer, London, 2012.

[10] J.Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **13**(2003), no. 1, 47–106.

[11] J.Llibre, A.C.Mereu, M.A.Teixeira, Limit cycles of generalized polynomial Liénard differential equations, *Math. Proc. Camb. Phil. Soc.*, (2009), 000, 1.

[12] J.Llibre, C.Valls, On the number of limit cycles of a class of polynomial differential systems, *Proc. A: R. Soc.*, **468**(2012), 2347–2360.

[13] J.Llibre, C.Valls, Limit cycles for a generalization of Liénard polynomial differential systems, *Chaos, Solitons and Fractals*, **46**(2013), 65–74.

[14] J.Llibre, C.Valls, On the number of limit cycles for a generalization of Liénard polynomial differential systems, *Int. J. Bifurcation and Chaos*, **23**(2013), 1350048.

Предельные циклы для одного класса полиномиальных дифференциальных систем, использующие теорию усреднения

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В данной работе рассматриваются предельные циклы одного класса полиномиальных дифференциальных систем вида

$$\begin{cases} \dot{x} = y - \varepsilon(g_{11}(x)\, y^{2\alpha+1} + f_{11}(x)\, y^{2\alpha}) - \varepsilon^2(g_{12}(x)\, y^{2\alpha+1} + f_{12}(x)\, y^{2\alpha}), \\ \dot{y} = -x - \varepsilon(g_{21}(x)\, y^{2\alpha+1} + f_{21}(x)\, y^{2\alpha}) - \varepsilon^2(g_{22}(x)\, y^{2\alpha+1} + f_{22}(x)\, y^{2\alpha}), \end{cases}$$

где $g_{1\kappa}$, $g_{2\kappa}, f_{1\kappa}$ *и* $f_{2\kappa}$ *имеют степень* n, m, l *и* k, *где* m, n, k, l *и являются положительными целыми числами, соответственно, для каждого* $\kappa = 1, 2$ *и* ε *— малый параметр. Мы получаем максимальное число предельных циклов, которые раздваиваются от периодических орбит линейного центра* $\dot{x} = y$, $\dot{y} = -x$, *используя теорию усреднения первого и второго порядка.*

Ключевые слова: предельные циклы, теория усреднения, лиенардовы дифференциальные системы.