# Elasto-Plastic Bending of a Beam 

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Special type of conservation laws for the equations which describe the bending of a beam is proposed in this paper. These laws are used to determine the elastic-plastic boundary.

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It is generally agreed today that problems of solid mechanics with unknown boundaries are very complicated. The problem with unknown boundary between the elastic and plastic regions is an example of such problem. As a rule, problems with unknown boundaries are solved by the semi-inverse methods. The other technique is to reduce an unknown boundary to a circle by Legendre transformation. This approach has been developed by B. D. Annin [1]. Unfortunately this technique allows one to prove only an existence theorem but it does not provide an algorithm for constructing solutions.

The authors of the paper successfully use symmetries and conservation laws for solving various problems of solid mechanics. It is well known that the symmetries admitted by the differential equations help to find a wide class of exact solutions and they are very effective semi-inverse methods. The conservation laws allow one to solve not only the systems of hyperbolic equations but also the systems of elliptic equations $[2,3]$.

Special type of conservation laws for the equations which describe the bending of a beam are proposed in this paper. These laws are used to obtain a method for determining the elasticplastic boundary. The method is suitable not only for beams with smooth cross-section contours but also for beams with piecewise - smooth cross-section contours.

Let us consider bending of a beam with a constant cross section contour $\Gamma$. The beam is under the influence of force $P$. The force is imposed to one of the beam ends and it is parallel to one of the principal axes of the cross section (Fig. 1).

Let us take the origin of the coordinates at the center of gravity of the beam fixed end. The $O z$-axis coincides with the center line of beam. The $O x$ and $O y$-axes coincide with the principal axes of the cross section.

[^0]

Fig. 1
The distribution of the component of the stress tensor $\sigma_{z}$ is taken as in the case of pure bending

$$
\sigma_{z}=-\frac{p(l-z) x}{l}
$$

Let the stress tensor components be $\sigma_{x}=\sigma_{y}=\sigma_{x y}=0$. Then the other components of the stress tensor are determined from the following equations

$$
\begin{equation*}
\frac{\partial \tau_{x z}}{\partial z}=\frac{\partial \tau_{y z}}{\partial z}=0 \quad \frac{\partial \tau_{x z}}{\partial x}+\frac{\partial \tau_{y z}}{\partial y}=-\frac{p x}{l} . \tag{1}
\end{equation*}
$$

Usually equations (1) are complemented by the compatibility equations for strains

$$
\Delta \tau_{x z}=-\frac{p}{l(1+\nu)}, \quad \Delta \tau_{y z}=0
$$

where $\Delta$ is the Laplace operator and $\nu$ is the Poisson's ratio. The system is solved with the use of the semi-inverse Saint-Venant method.

Let us write system (1) in terms of the displacement vector $(u, v, w)$. Taking into account appropriate boundary conditions, we will solve the system with the use of special conservation laws. They will be introduced later.

In order to find the components of the displacement vector we have the following system of equations

$$
\begin{align*}
\sigma_{x} & =\lambda e+2 \mu \frac{\partial u}{\partial x}=0 \\
\sigma_{y} & =\lambda e+2 \mu \frac{\partial v}{\partial y}=0 \\
\sigma_{z} & =\lambda e+2 \mu \frac{\partial w}{\partial z}=-\frac{p(l-z) x}{l}=\sigma \\
\tau_{x y} & =\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0  \tag{2}\\
\tau_{x z} & =\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\tau_{1} \\
\tau_{y z} & =\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=\tau_{2}
\end{align*}
$$

where $\lambda$ and $\mu$ are the constant Lame coefficients, $e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \tau_{1}, \tau_{2}$ are unknown function of $x$ and $y$.

From the first three equations of system (2) we obtain

$$
\begin{align*}
& \frac{\partial u}{\partial x}=A_{1} x z+B_{1} x \\
& \frac{\partial v}{\partial y}=A_{2} x z+B_{2} x  \tag{3}\\
& \frac{\partial w}{\partial z}=A_{3} x z+B_{3} x
\end{align*}
$$

where the constants $A_{i}$ and $B_{i}$ can be easily determined in terms of the constants $\lambda, \mu, p$ and $l$. Then we have the following formulas

$$
\begin{aligned}
& \Delta=\left|\begin{array}{lll}
\lambda+2 \mu & \lambda & \lambda \\
\lambda & \lambda+2 \mu & \lambda \\
\lambda & \lambda & \lambda+2 \mu
\end{array}\right|=(\lambda+2 \mu)^{2}+2 \lambda^{3}-3 \lambda^{2}(\lambda+2 \mu)=12 \lambda \mu^{2}+8 \mu^{3}, \\
& \Delta_{1}=\sigma\left(\lambda^{2}-\lambda(\lambda+2 \mu)\right) \\
& \Delta_{2}=\sigma \cdot 2 \mu, \\
& \Delta_{3}=\sigma\left(\lambda^{2}-\lambda(\lambda+2 \mu)\right)=-\sigma \cdot 2 \mu, \\
& \frac{\partial u}{\partial x}=\frac{\Delta_{1}}{\Delta}=\frac{\sigma}{\sigma \lambda \mu+4 \mu^{2}}=\frac{1}{\sigma \lambda \mu+4 \mu^{2}}\left(z x \frac{p}{l}-p x\right), \\
& \frac{\partial v}{\partial y}=-\frac{2 \sigma \mu}{\Delta}=\frac{\sigma}{\sigma \lambda \mu+4 \mu^{2}}=-\frac{1}{\sigma \lambda \mu+4 \mu^{2}}\left(z x \frac{p}{l}-p x\right), \\
& \frac{\partial w}{\partial z}=\frac{\sigma(\lambda+2 \mu)}{\sigma \lambda \mu+4 \mu^{2}}\left(z x \frac{p}{l}-p x\right) .
\end{aligned}
$$

From equations (3) we obtain

$$
\begin{aligned}
& w=\frac{A_{3} x z^{2}}{2}+B_{3} x z+\omega(x, y) \\
& u=-\frac{A_{3} z^{3}}{\sigma}-\frac{B_{3} z^{2}}{2}+\left(\tau_{1} / \mu-\omega_{x}\right) z+U(x, y) \\
& v=\left(\tau_{2} / \mu-\omega_{y}\right)+V(x, y)
\end{aligned}
$$

where $\omega, U, V$ are some functions.
Taking into account that $\tau_{x y}=0$ we obtain

$$
\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\left(\tau_{1} y / \mu-\omega_{x y}\right) z+U_{y}+\left(\tau_{2} x / \mu-\omega_{x y}\right)+V_{x}=0
$$

From the equations

$$
\begin{aligned}
& \frac{\partial u}{\partial x}=-\left(\partial \tau_{1 x} / \mu-\omega_{x x}\right) z+U_{x}=A_{1} x z+B_{1} x \\
& \frac{\partial v}{\partial y}=\left(\partial \tau_{2 y} / \mu-\omega_{y y}\right) z+U_{y}=A_{2} x z+B_{2} x
\end{aligned}
$$

we obtain

$$
\tau_{1}=\mu\left(\omega_{x}+\frac{A_{1} x^{2}}{2}\right), \quad \tau_{2}=\mu\left(\omega_{y}+A_{2} x y\right)
$$

Substituting these expressions into equation (1) gives

$$
\mu\left(\omega_{x x}+A_{1} x\right)+\mu\left(\omega_{y y}+A_{2} x\right)=-\frac{P x}{l}
$$

or in simpler form

$$
\omega_{x x}+\omega_{y y}=a x, \quad a=-\left(\frac{P}{l}+A_{1}+A_{2}\right) .
$$

Boundary conditions. We assume that the lateral surface of the beam is stress-free. It means that

$$
\begin{equation*}
\tau_{x z} n_{2}-\tau_{y z} n_{1}=0 \tag{4}
\end{equation*}
$$

where $\left(n_{1}, n_{2}\right)$ are the components of the external normal to the contour $\Gamma$. We also assume that the plastic flow occurs on the lateral surface of the beam under the action of the force $P$ and the von Mises yield criterion

$$
\begin{equation*}
\tau_{x z}^{2}+\tau_{y z}^{2}=k^{2} \tag{5}
\end{equation*}
$$

is satisfied, where $k$ is the yield stress. Solving system (4) and (5) we obtain

$$
\tau_{x z}= \pm n_{1} k, \quad \tau_{y z}= \pm n_{2} k
$$

or

$$
\mu\left(\omega_{x}+\frac{A_{1} x^{2}}{2}\right)= \pm n_{1} k, \quad \mu\left(\omega_{y}+A_{2} x y\right)= \pm n_{2} k
$$

Let us choose the upper sign and we obtain the following problem. It is necessary to solve the equation

$$
\begin{equation*}
\omega_{x x}+\omega_{y y}=a x \tag{6}
\end{equation*}
$$

with the following boundary conditions on $\Gamma$ :

$$
\begin{equation*}
\omega_{x}=\frac{\left(n_{1} k-\frac{A_{1} x^{2}}{2}\right)}{\mu} \quad \omega_{y}=-\frac{\left(n_{2} k-A_{2} x y\right)}{\mu} \tag{7}
\end{equation*}
$$

Remark. There are elastic and plastic regions in the beam. Points where inequality $\tau_{x z}^{2}+\tau_{y z}^{2}<1$ is satisfied belong to the elastic zone. Other points, including points on the contour $\Gamma$, belong to the plastic one.
Conservation laws. The relation of the form

$$
\begin{equation*}
\partial_{x} A+\partial_{y} B=0 \tag{8}
\end{equation*}
$$

will be called a conservation law for equation (6). Assume that (8) is valid by means of equation (6). Let conserved current has the form

$$
\begin{aligned}
& A=\alpha(x, y) \omega_{x}+\beta(x, y) \omega_{y}+\gamma(x, y) \\
& B=\alpha^{1}(x, y) \omega_{x}+\beta^{1}(x, y) \omega_{y}+\gamma^{1}(x, y)
\end{aligned}
$$

From (6) and (8) we obtain

$$
\begin{equation*}
\alpha\left(a-\omega_{y y}\right)+\alpha_{x} \omega_{x}+\beta \omega_{x y}+\beta_{x} \omega_{y}+\gamma_{x}++\alpha^{1} \omega_{x y}+\alpha_{y}^{1} \omega_{x}+\beta^{1} \omega_{y y}+\beta_{y}^{1} \omega_{y}+\gamma_{y}^{1}=0 \tag{9}
\end{equation*}
$$

Since relation (9) is valid for all solutions of equation (6) from (9) we have

$$
\alpha-\beta^{1}=0 \quad \beta+\alpha^{1}=0 \quad \alpha_{x}+\alpha_{y}^{1}=0, \quad \beta_{x}+\beta_{y}^{1}=0, \quad \alpha a+\gamma_{x}+\gamma_{y}^{1}=0
$$

or

$$
\begin{equation*}
\alpha_{x}-\beta_{y}=0, \quad \beta_{x}+\alpha_{y}=0, \quad \alpha a+\gamma_{x}+\gamma_{y}^{1}=0 \tag{10}
\end{equation*}
$$

Then the conserved current has the form

$$
A=\alpha \omega_{x}+\beta \omega_{y}+\gamma, \quad B=-\beta \omega_{x}+\alpha \omega_{y}+\gamma^{1}
$$



Fig. 2
Thus we arrive to the following theorem.
Theorem. Equations (6) admit an infinite set of conservation laws.
Using Green's formula, conservation law (8) can be written in the form (Fig. 2)

$$
0=\int_{\Gamma}-\left(\alpha \omega_{x}+\beta \omega_{y}+\gamma\right) d y+\left(-\beta \omega_{x}+\alpha \omega_{y}+\gamma^{1}\right) d x
$$

Let us consider two solutions of equation (10). The first solution is

$$
\begin{aligned}
& \alpha^{1}=\frac{x-x_{0}}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}, \quad \beta^{1}=-\frac{y-y_{0}}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}, \\
& \gamma_{x}=0, \quad \alpha^{1} x a=-\gamma_{y}^{1}, \quad \gamma^{1}=-a x \cdot \operatorname{arctg}\left(\frac{y-y_{0}}{x-x_{0}}\right) .
\end{aligned}
$$

The second solution is

$$
\begin{aligned}
& \alpha^{1}=\frac{y-y_{0}}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}}, \quad \beta^{1}=\frac{x-x_{0}}{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}} \\
& \gamma_{y}^{1}=0, \quad \alpha^{2} x a=-\gamma_{x}, \quad \gamma=-\frac{a\left(y-y_{0}\right)}{2} \ln \left(\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}\right)-x_{0} a \cdot \operatorname{arctg}\left(\frac{x-x_{0}}{y-y_{0}}\right) .
\end{aligned}
$$

From the conservation law given above we have

$$
\begin{aligned}
& \int_{\Gamma}\left(\alpha \omega_{x}+\beta \omega_{y}+\gamma\right) d y-\left(-\beta \omega_{x}+\alpha \omega_{y}+\gamma^{1}\right) d x= \\
& =-\int_{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}=R^{2}}\left(\alpha \omega_{x}+\beta \omega_{y}+\gamma\right) d y-\left(-\beta \omega_{x}+\alpha \omega_{y}+\gamma^{1}\right) d x
\end{aligned}
$$

Let us compute the second integral of the first and second solutions. For the first solution of (10) we have

$$
\begin{aligned}
& \quad \int_{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}=R^{2}}\left(\alpha \omega_{x}+\beta \omega_{y}+\gamma\right) d y-\left(-\beta \omega_{x}+\alpha \omega_{y}+\gamma^{1}\right) d x= \\
& =\int_{0}^{2 \pi}\left(\frac{\cos \theta}{R} \omega_{x}-\frac{\sin \theta}{R} \omega_{y}\right) R \cos \theta-\left(-\left(-\frac{\sin \theta}{R} \omega_{x}\right)+\frac{\cos \theta}{R} \omega_{y}+\gamma^{1}\right) R \sin \theta d \theta= \\
& =\int_{0}^{2 \pi} \omega_{x} d \theta=2 \pi \omega_{x}\left(x_{0}, y_{0}\right)
\end{aligned}
$$

This formula is obtained with the use of the mean value theorem and in the limit of $R \rightarrow 0$. Finally, we obtain

$$
\begin{align*}
& 2 \pi \omega_{x}\left(x_{0}, y_{0}\right)=-\int_{\Gamma}\left(\alpha^{1} \omega_{x}+\beta^{1} \omega_{y}+\gamma\right) d y-\left(-\beta^{1} \omega_{x}+\alpha^{1} \omega_{y}+\gamma^{1}\right) d x= \\
& =-\int_{\Gamma}\left(\alpha^{1}\left(n_{1} k-\frac{A_{1} x^{2}}{2}\right) / \mu+\beta^{1}\left(n_{2} k-A_{2} x y\right) / \mu+\gamma\right) d y-  \tag{11}\\
& -\left(-\beta^{1}\left(n_{1} k-\frac{A_{1} x^{2}}{2}\right) / \mu+\alpha^{1}\left(n_{2} k-A_{2} x y\right) / \mu+\gamma^{1}\right) d x
\end{align*}
$$

For the second solution of (10) we have

$$
\begin{aligned}
& \quad \int_{\left(x-x_{0}\right)^{2}+\left(y-y_{0}\right)^{2}=R^{2}}\left(\alpha \omega_{x}+\beta \omega_{y}+\gamma\right) d y-\left(-\beta \omega_{x}+\alpha \omega_{y}+\gamma^{1}\right) d x= \\
& =\int_{0}^{2 \pi}\left(\frac{\sin \theta}{R} \omega_{x}+\frac{\cos \theta}{R} \omega_{y}+\gamma\right) R \cos \theta-\left(-\left(-\frac{\cos \theta}{R} \omega_{x}\right)+\frac{\sin \theta}{R} \omega_{y}\right) R \sin \theta d \theta= \\
& =\int_{0}^{2 \pi} \omega_{y} d \theta=2 \pi \omega_{y}\left(x_{0}, y_{0}\right) .
\end{aligned}
$$

Finally, we obtain

$$
\begin{align*}
& 2 \pi \omega_{y}\left(x_{0}, y_{0}\right)=-\int_{\Gamma}\left(\alpha^{2}\left(n_{1} k-\frac{A_{1} x^{2}}{2}\right) / \mu+\beta^{2}\left(n_{2} k-A_{2} x y\right) / \mu+\gamma\right) d y- \\
& -\left(-\beta^{2}\left(n_{1} k-\frac{A_{1} x^{2}}{2}\right) / \mu+\alpha^{2}\left(n_{2} k-A_{2} x y\right) / \mu+\gamma^{1}\right) d x \tag{12}
\end{align*}
$$

Formulas (11) and (12) allow us to find the stress state at the point $\left(x_{0}, y_{0}\right)$. It means that it is possible to determine the point of cross-section in which the material is in plastic or elastic state. Hence the position of boundary between elastic and plastic regions can be accurately calculated. Preliminary test calculations confirm this conclusion.

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## Упруго-пластический изгиб бруса

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В статъе для уравнений, описывающих изгиб консоли, строятся законы сохранения специалъного вида, позволяющие в аналитическом виде находить упруго-пластическую гранииу.
Ключевые слова: симметрии,законы сохранения, поперечное сечение, граничные условия, кусочногладкая гранича, упруго-пластическая гранича.


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