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The Spaces of Meromorphic Prym Differentials on Finite Tori

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In this article we construct all kinds of elementary Prym differentials for arbitrary characters on a variable torus with a finite numbers of punctures and find the dimensions of two important quotient spaces. As a consequence, this yields the dimension of the first holomorphic de Rham cohomology group of Prym differentials for arbitrary characters on torus. Also, we construct explicit bases in these quotient spaces.

Keywords: Prym differentials for arbitrary characters, the Gunning cohomological bundle over the Teichmuller space torus with a finite numbers of punctures.

Introduction

The theory of multiplicative functions and Prym differentials in the case of special characters on a compact Riemann surface has found applications in the geometric function theory of complex variable, the analytic number theory and in the mathematical physics [1–7]. In [1–3] the development of the general theory of multiplicative functions and Prym differentials on a compact Riemann surface of genus $g \geqslant 2$ for arbitrary characters has been started. The function theory on compact Riemann surfaces differs substantially from that on finite Riemann surfaces even for the class of single-valued meromorphic functions and abelian differentials. A number of basic spaces of functions and differentials on a finite Riemann surface F' of type (g, m), with $g \geqslant 1, m > 0$, are infinite-dimensional.

In this article we start constructing the general function theory on a variable torus with a finite numbers of punctures for multiplicative meromorphic functions and differentials with arbitrary characters. We construct all kinds of elementary Prym differentials for arbitrary characters on such surfaces and find the dimensions of two important quotient spaces. As a consequence, this yields the dimension of the first holomorphic de Rham cohomology group of Prym differentials for arbitrary characters on torus. Also, we construct explicit bases in these quotient spaces.

1. Preliminaries

Fix a smooth compact oriented surface F of genus $g = 1$, with a marking $\{a, b\}$, which is an ordered tuple of generators for $\pi_1(F)$. Let F_0 be a fixed complex analytic structure on F. From now on, for brevity, the Riemann surface $(F; F_0)$ will be denoted by F_0. Fix distinct points $P_1, \dots, P_m \in F$. Suppose that $F' = F \backslash \{P_1, \dots, P_m\}$ is a surface of type $(1, m)$, with $m \geq 1$. Denote by Γ' the Fuchsian group of the first kind that acts invariantly on the disk $U = \{z \in \mathbf{C} : |z| < 1\}$ and uniformizes the surface F_0', i.e. F_0' is conformally equivalent to U/Γ'. This group has the representation $\Gamma' = \langle A, B, \gamma_1, \dots, \gamma_m : [A, B]\gamma_1 \dots \gamma_m = I\rangle$, where $[A, B] = ABA^{-1}B^{-1}$ for $A, B \in \Gamma'$, and I is the identity mapping [6].

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Every complex analytic structure on F' is determined by some Beltrami differential μ on F'_0; i. e., an expression of the form $\mu(z)d\overline{z}/dz$ independent of the choice of a local parameter on F'_0, where $\mu(z)$ is a complex function on F'_0 and $\|\mu\|_{L_\infty(F'_0)} < 1$. Denote this structure on F' by F'_μ. It is clear that $\mu = 0$ corresponds to F'_0. Let $M(F')$ be the set of all complex analytic structures on F' endowed with the topology of C^∞ convergence on F'_0, $Diff^+(F')$ be the group of all orientation-preserving smooth diffeomorphisms of F' leaving all punctures fixed, and $Diff_0(F')$ be the normal subgroup of $Diff^+(F')$ consisting of all diffeomorphisms homotopic to the identity diffeomorphism of F'_0. The group $Diff^+(F')$ acts on $M(F')$ as $\mu \to f^*\mu$, where $f \in Diff^+(F')$, $\mu \in M(F')$. Then the Teichmuller space $\mathbf{T}_{1,m}(F') = \mathbf{T}_{1,m}(F'_0)$ is the quotient space $M(F')/Diff_0(F')$ [6].

Since the mapping $U \to F'_0 = U/\Gamma'$ is a local diffeomorphism, every Beltrami differential μ on F'_0 lifts to a Beltrami Γ'-differential μ on U; thus, $\mu \in L_\infty(U)$, $\|\mu\|_{L_\infty(U)} = \operatorname*{esssup}_{z\in U} |\mu(z)| < 1$, and $\mu(T(z))\overline{T'(z)}/T'(z) = \mu(z)$, $z \in U$, $T \in \Gamma'$.

Extend the Γ'-differential μ on U to $\overline{\mathbf{C}}\backslash U$ by putting $\mu = 0$. Then there exists a unique quasiconformal homeomorphism $w^\mu : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ with fixed points $+1, -1, i$, which is a solution to the Beltrami equation $w_{\overline{z}} = \mu(z)w_z$. The mapping $T \to T_\mu = w^\mu T(w^\mu)^{-1}$ determines an isomorphism of Γ' onto the quasi-Fuchsian group $\Gamma'_\mu = w^\mu\Gamma'(w^\mu)^{-1} = \langle A^\mu, B^\mu, \gamma_1^\mu, \dots, \gamma_m^\mu : [A^\mu, B^\mu]\gamma_1^\mu \dots \gamma_m^\mu = I\rangle$.

The classical results of Ahlfors and Bers [6], and other authors assert that

1) $\mathbf{T}_{1,m}(F')$ is a complex manifold of dimension m for $m \geqslant 1$; 2) $\mathbf{T}_{1,m}(F')$ carries a unique complex analytic structure such that the natural mapping $\Psi : M(F') \to \mathbf{T}_{1,m}(F')$ is holomorphic; furthermore, Ψ has only local holomorphic sections; 3) the elements of Γ'_μ depend holomorphically on the moduli $[\mu]$ of finite Riemann surfaces F'_μ.

Two Beltrami Γ'-differentials μ and ν are conformally equivalent if and only if $w^\mu T(w^\mu)^{-1} = w^\nu T(w^\nu)^{-1}$, with $T \in \Gamma'$. It is natural that the choice of generators $\{a, b\} \cup \{\gamma_1, \dots, \gamma_m\}$ in $\pi_1(F')$ is equivalent to the choice of a systems of generators $\{a^\mu, b^\mu\} \cup \{\gamma_1^\mu, \dots, \gamma_m^\mu\}$ in $\pi_1(F'_\mu)$, and $\{A^\mu, B^\mu\} \cup \{\gamma_1^\mu, \dots, \gamma_m^\mu\}$ in Γ'_μ for every $[\mu]$ in $\mathbf{T}_{1,m}$. This implies the identifications $M(F')/Diff_0(F') = \mathbf{T}_{1,m}(F') = \mathbf{T}_{1,m}(\Gamma')$. Furthermore, there is a bijective correspondence between the classes of Beltrami differentials $[\mu]$, the classes of conformally equivalent marked finite Riemann surfaces $[F'_\mu; \{a^\mu, b^\mu\} \cup \{\gamma_1^\mu, \dots, \gamma_m^\mu\}]$ and marked quasi-Fuchsian groups Γ'_μ [5, 6].

In [6, p. 99] Bers constructed a holomorphic abelian differential $\zeta[\mu] = dz$ on F_μ for every $[\mu] \in \mathbf{T}_1$, with the condition $\int_\xi^{A^\mu(\xi)} \zeta([\mu], w)dw = 1$, $\xi \in \mathbf{C}$, and it depends holomorphically on the moduli $[\mu]$ for F_μ. Moreover, the b-period on F_μ is the complex number $\mu = \int_\xi^{B^\mu(\xi)} \zeta([\mu], w)dw$, $\xi \in \mathbf{C}$, and it depends holomorphically on the moduli $[\mu]$.

Define for arbitrary fixed $[\mu] \in \mathbf{T}_1$ and $\xi_0 \in \mathbf{C}$ the classical Jacobi mapping $\varphi : \mathbf{C} \to \mathbf{C}$ by $\varphi(\xi) = \int_{\xi_0}^{\xi} \zeta([\mu], w)dw$. The quotient space $J(F_\mu) = \mathbf{C}/L(F_\mu)$ is called the marked Jacobian variety for F_μ, where $L(F_\mu)$ is the lattice over $\mathbf{Z}$ generated by the elements 1 and μ. The universal Jacobian variety for torus is a fibration over $\mathbf{T}_1$, whose fiber over $[\mu] \in \mathbf{T}_1$ is the Jacobian $J(F_\mu)$ of the surface F_μ [5, 7].

Next, given an integer $n > 1$, there exists a fibration over $\mathbf{T}_1$ whose fiber over $[\mu] \in \mathbf{T}_1$ is the space of all degree n integer divisors on F_μ. The holomorphic sections of this bundle determine on every F_μ a degree n integer divisor D^μ, which holomorphically depends on $[\mu]$. Also there exists a holomorphic mapping φ_n from this bundle into the universal Jacobian bundle, $n \geqslant 1$, whose restriction to the fibres extends the Jacobi mapping $\varphi : F_\mu \to J(F_\mu) \cong F_\mu$. We can obtain local holomorphic sections of these bundles over a neighborhood $U([\mu_0]) \subset \mathbf{T}_1$ (for every $n \geqslant 1$) from local holomorphic Earle sections s of $\Psi : M(F) \to \mathbf{T}_1$ over $U([\mu_0])$ [7].

A *character* ρ for F'_μ is every homomorphism $\rho : (\pi_1(F'_\mu), \cdot) \to (\mathbf{C}^*, \cdot)$, $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. Each character is uniquely determined by an ordered tuple $(\rho(a^\mu), \rho(b^\mu), \rho(\gamma_1^\mu), \ldots, \rho(\gamma_m^\mu)) \in (\mathbf{C}^*)^{2+m}$.

Definition 1.1. *A multiplicative function f on F'_μ for a character ρ is a meromorphic function f on $w^\mu(U)$ such that $f(Tz) = \rho(T)f(z)$, $z \in w^\mu(U)$, $T \in \Gamma'_\mu$.*

Definition 1.2. *A Prym q-differential with respect to a Fuchsian group Γ' for ρ, or (ρ, q)-differential, is a differential $\omega(z)dz^q$ such that $\omega(Tz)(T'z)^q = \rho(T)\omega(z)$, $z \in U$, $T \in \Gamma'$, $\rho : \Gamma' \to \mathbf{C}^*$, $q \in \mathbb{N}$.*

If f_0 is a multiplicative function on F_μ for ρ without zeros and poles then $f_0(P) = f_0(P_0) \exp \int_{P_0[\mu]}^{P} 2\pi i c([\mu], \rho)\zeta([\mu])$, where $P_0[\mu] = f^{s[\mu]}(P_0) \in F_\mu$, $c([\mu], \rho) \in \mathbf{C}$, c depend holomorphically on $[\mu]$ and ρ. Furthermore, integration is performed from a fixed point $P_0[\mu]$ to the current point P on the variable surface F_μ, and $s[\mu]$ is the Earle section [7] over $U([\mu_0]) \subset \mathbf{T}_1$. We deduce that the character ρ for f_0 is of the form $\rho(a^\mu) = \exp 2\pi i c([\mu], \rho)$, $\rho(b^\mu) = \exp(2\pi i c([\mu], \rho)\mu)$. Refer to these characters ρ as *unessential*, while to f_0 with this character, as a *unit*. The characters which are not unessential we call *essential* on $\pi_1(F_\mu)$. Denote by $Hom(\Gamma, \mathbf{C}^*)$ the group of all characters on Γ with the natural multiplication. The unessential characters constitute a subgroup L_1 of $Hom(\Gamma, \mathbf{C}^*)$ [1].

Definition 1.3. *ϕ on $F' = U/\Gamma'$ for ρ is called multiplicatively exact, whenever $\phi = df(z)$ and $f(Tz) = \rho(T)f(z)$, $T \in \Gamma'$, $z \in U$; thus, f is a multiplicative function on F' for ρ.*

Given $\rho \in Hom(\Gamma'_\mu, \mathbf{C}^*)$, denote by $Z^1(\Gamma'_\mu, \rho)$ the set of all mappings $\phi : \Gamma'_\mu \to \mathbf{C}$ such that $\phi(ST) = \phi(S) + \rho(S)\phi(T)$, $S, T \in \Gamma'_\mu$ [1].

Let ϕ be a closed Prym differential on F'_0 for ρ. Integrating this differential, we obtain $f(Tz) - f(Tz_0) = \rho(T)(f(z) - f(z_0))$, where $\phi = df(z)$, $z \in U$, $f(z)$ is a Prym integral on the disk U for ϕ, which is determined up to an additive term. Hence, $T \in \Gamma'$ satisfies $f(Tz) = \rho(T)f(z) + \phi_{f,z_0}(T)$, where $\phi_{f,z_0}(T) = f(Tz_0) - \rho(T)f(z_0)$. Therefore, the period mapping $\phi_{f,z_0} : \Gamma' \to \mathbf{C}$ for ϕ is defined. It depends on the choice of a Prym integral $f(z)$ on U and a base point z_0. Given another Prym integral $f_1(z) = f(z) + c$ for ϕ, we have $\phi_{f_1,z_0}(T) = \phi_{f,z_0}(T) + c\sigma(T)$, $\sigma(T) = 1 - \rho(T)$, $T \in \Gamma'$. It is easy to verify that both mappings ϕ_{f,z_0} and ϕ_{f_1,z_0} satisfy the cocycle relation $\phi(ST) = \phi(S) + \rho(S)\phi(T)$, $S, T \in \Gamma'$. They belong to the space $Z^1(\Gamma', \rho)$ and represent the same class of periods $[\phi]$ in $H^1(\Gamma', \rho) = Z^1(\Gamma', \rho)/B^1(\Gamma', \rho)$ for the Prym differential ϕ for ρ on F', where $B^1(\Gamma', \rho)$ is generated by σ.

For a closed Prym differential ϕ we can determine the classical periods. For $T \in \Gamma'$ the corresponding classical period $\phi_{z_0}(T) = \int_{z_0}^{Tz_0} \phi$ and we have the equality $\phi_{z_0}(T) = \phi_{f,z_0}(T) - f(z_0)\sigma(T)$.

Consequently, the mappings of the form $T \to \phi_{f,z_0}(T)$ (periods in the sense of Gunning) and of the form $T \to \phi_{z_0}(T)$ (the classical periods) determine the same class of periods $[\phi] \in H^1(\Gamma', \rho)$ for a Prym differential ϕ on F' for ρ. Thus, we have a well-defined $\mathbf{C}$-linear mapping $p : \phi \to [\phi]$ from the vector space of closed Prym differentials ϕ on F' for ρ in the vector space $H^1(\Gamma', \rho)$.

Denote by $\Omega_{2,\rho}(F'_\mu)$ the space of Prym differentials of the second kind with finitely many poles on F'_μ for character ρ [5, 2]. The space $\mathbf{A}_1(\rho)$ consists of Prym differentials for ρ on F' that have finitely many poles on F' and extend meromorphically on F.

Lemma 1.1. *If the differential $\omega \in \Omega_{2,\rho}(F'_\mu) \cap \boldsymbol{A}_1(\rho)$ has the class of periods $[\omega] = 0$ in $H^1(\Gamma'_\mu, \rho)$, then ω is a multiplicatively exact differential on F'_μ for ρ.*

Proof. It suffices to verify this for a fixed surface and a fixed character. We obtain the classical periods $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k$ while going separately around the poles $Q_1, \ldots, Q_k$ of the differential ω. They all vanish, being equal to the residues at the poles of second or higher order for the branches of our multivalued differential ω the second kind.

If the class of periods $[\omega] = 0$, then the classical period $\omega_{z_0}(T) = c\sigma(T),\ c \neq 0$ for every T, where $\omega_{z_0}(T) = f(Tz_0) - f(z_0) = c(1 - \rho(T))$, while f is some Prym integral for ω. Then $\widetilde{f} = (f - c)$ is a multiplicative function for ρ and $\omega = d\widetilde{f} = d(f - c)$. Thus, the periods $\widetilde{\omega}_{z_0,\widetilde{f}_0}(a)$, $\widetilde{\omega}_{z_0,\widetilde{f}_0}(b)$, $\widetilde{\omega}_{z_0,\widetilde{f}_0}(\gamma_1), \ldots, \widetilde{\omega}_{z_0,\widetilde{f}_0}(\gamma_m)$ in the sense of Gunning all vanish for some representative of class $[\omega]$. Consequently, ω is a multiplicatively exact differential for ρ on F'_μ. The proof of Lemma 1.1 is complete. □

A divisor on F_μ is a formal product $D = P_1^{m_1} \ldots P_k^{m_k},\ P_j \in F_\mu,\ m_j \in \mathbf{Z},\ j = 1, \ldots, k$.

Theorem (Riemann-Roch's theorem for characters [5, 2]). *Let F be a compact Riemann surface of genus one. Then for every divisor D on F and every character ρ the equality $r_\rho(D^{-1}) = degD + i_{\rho^{-1}}(D)$ holds.*

Theorem (Abel's theorem for characters [5, 2]). *Let D be a divisor on a marked variable compact Riemann surface $[F_\mu, \{a^\mu, b^\mu\}]$ of genus one and ρ a character on $\pi_1(F_\mu)$. Then D is a divisor of a multiplicative function f on F_μ for $\rho \Leftrightarrow degD = 0$ and*

$$\varphi(D) = \frac{1}{2\pi i} \log \rho(b^\mu) - \frac{1}{2\pi i} \mu \log \rho(a^\mu) (\equiv \psi(\rho, [\mu]))$$

in $\mathbf{C}$ modulo the integer lattice $L(F_\mu)$ generated by the complex numbers 1 and μ.

Observe that by a theorem of Bers [6, p. 99], the mapping ψ depends locally holomorphically on ρ and $[\mu]$.

Every element $\phi \in Z^1(\Gamma', \rho)$ is unique determined by the ordered tuple of complex numbers $\phi(A)$, $\phi(B)$, $\phi(\gamma_1), \ldots, \phi(\gamma_m)$.

Lemma 1.2 ([8]). *For every $\phi \in Z^1(\Gamma', \rho)$ there holds the equality*

$$\sigma(B)\phi(A) - \sigma(A)\phi(B) + \phi(\gamma_1) + \sum_{j=1}^{m-1} \rho(\gamma_1 \ldots \gamma_j)\phi(\gamma_{j+1}) = 0. \tag{1}$$

Lemma 1.3 ([8]). *The holomorphic principal $Hom(\Gamma', \mathbf{C}^*)$-bundle $E = \bigcup_{[\mu]} Hom(\Gamma'_\mu, \mathbf{C}^*)$ is analytically equivalent to the trivial bundle $\mathbf{T}_{1,m}(F') \times Hom(\Gamma', \mathbf{C}^*)$ over the base $\mathbf{T}_{1,m}(F')$.*

The set $G' = \bigcup_{[\mu], \rho \neq 1} H^1(\Gamma'_\mu, \rho)$ is called the Gunning cohomological bundle over the base $\mathbf{T}_{1,m} \times Hom(\Gamma', \mathbf{C}^*) \backslash \{1\}$ [1]. For G' and $\rho \neq 1$ we use Gunning's isomorphism [1] between the complex vector space $H^1(\Gamma'_\mu, \rho)$ and the vector space $Hom_\rho([\Gamma'_\mu, \Gamma'_\mu], \mathbf{C})$, that consists of homomorphisms $\phi_0 : [\Gamma'_\mu, \Gamma'_\mu] \to (\mathbf{C}, +)$ with the condition $\phi_0(STS^{-1}) = \rho(S)\phi_0(T),\ T \in [\Gamma'_\mu, \Gamma'_\mu],\ S \in \Gamma'_\mu$. Here $[\Gamma'_\mu, \Gamma'_\mu]$ is the commutant group Γ'_μ. Thus, the bundle G' is isomorphic to the vector bundle $\bigcup_{[\mu], \rho \neq 1} Hom_\rho([\Gamma'_\mu, \Gamma'_\mu], \mathbf{C})$.

Moreover, we can determine the transition matrices of this bundle in terms of two coordinate neighborhoods $U_1 = \{\rho : \rho(A) \neq 1\}$, $U_2 = \{\rho : \rho(B) \neq 1\}$, which cover the base $Hom(\Gamma', \mathbf{C}^*) \backslash \{1\}$ provided that $\rho(\gamma_j) = 1,\ \ j = 1, \ldots, m$. For the neighborhood U_1 we have $\sigma(A^\mu) \neq 0$. Every element $\phi_0 \in Hom_\rho([\Gamma'_\mu, \Gamma'_\mu], \mathbf{C})$ for $\rho \in U_1$ can be defined as $\phi_0 = \phi_1^\mu|_{[\Gamma'_\mu, \Gamma'_\mu]}$ for $\phi_1^\mu \in Z^1(\Gamma'_\mu, \rho)$ such that $\phi_1(A^\mu) = 0$ and $\phi_1(T) = \sigma(A^\mu)^{-1}\phi_0([T, A^\mu]),\ T \in \Gamma'_\mu$ [1].

Theorem ([1, 8]). *The Gunning cohomological bundle G' over $\mathbf{T}_{1,m}(F') \times (Hom(\Gamma', \mathbf{C}^*) \backslash \{1\})$ is a holomorphic vector bundle of rank m for $m \geqslant 1$.*

2. Elementary Prym differentials

In the construction of a general theory of single-valued and multiplicative differentials, an important role is played by the elementary differentials [3, 5] of arbitrary order with the minimal number of poles, either one pole of order $\geqslant 1$, or two simple poles, depending holomorphically on the character ρ and the moduli $[\mu]$ of Riemann surfaces. In this section we find the general form of elementary (ρ, q)-differentials on F'_μ.

Proposition 2.1 ([2]). *A degree 0 divisor D is a divisor of a meromorphic (ρ, q)-differential ω on a compact Riemann surface F of genus $g = 1$ for a character ρ with $q \geqslant 1$ if and only if $\varphi(D) = \psi(\rho)$ in $J(F)$.*

The proof proceeds as in the case $q = 1$ considered in [3, 5], taking into account that $-2K = 0$ in $J(F)$ for the torus F.

Theorem 2.1. *Given a point Q, a character ρ on F'_μ of type $(1, m)$, $m \geqslant 1$, and natural numbers $n \geqslant 2$, $q \geqslant 1$ there exists an elementary (ρ, q)-differential $\tau^{(n)}_{\rho,q;Q}$ of the class $\boldsymbol{A}_1(\rho)$ with a unique pole Q of order exactly n on F'_μ. The general form of its divisor is $(\tau^{(n)}_{\rho,q;Q}) = \frac{R_1 \dots R_N}{Q^n} \frac{1}{P_1^{k_1} \dots P_m^{k_m}}$, where $\varphi(R_1) = \varphi(Q^n) - \varphi(R_2 \dots R_N) + \varphi(P_1^{k_1}) + \dots + \varphi(P_m^{k_m}) + \psi(\rho)$, $k_j \geqslant 0$, $k_j \in \mathbf{N}$, $j = 1, \dots, m$. The points $R_2, \dots, R_N$ are chosen as a local holomorphic section of divisors of degree $N - 1$ on $F'_\mu \backslash \{Q\}$, and $N = n + k_1 + \dots + k_m$. Moreover, these differentials depend locally holomorphically on $[\mu]$ and ρ, and for an essential character we have $\tau^{(n)}_{\rho,q;Q} = \left(\frac{1}{z^n} + O(1)\right) dz^q, z(Q) = 0$.*

Proof. Given $q \geqslant 1$, find the general form of (ρ, q)-differentials of the second kind with a unique pole at the point Q of order exactly $n \geqslant 1$ on F'_μ.

The Riemann-Roch theorem for (ρ, q)-differentials on F_μ [3] yields the dimension $i_{\rho,q}\left(\frac{1}{Q^n P_1^{k_1} \dots P_m^{k_m}}\right) = \dim_{\mathbf{C}} \Omega^q_\rho \left(\frac{1}{Q^n P_1^{k_1} \dots P_m^{k_m}}\right)$, where $k_j \geqslant 0, j = 1, \dots, m$. We have $i_{\rho,q}(D) = -\deg D + r\left(\frac{(f[\mu])}{D}\right)$, where $D = \frac{1}{Q^n P_1^{k_1} \dots P_m^{k_m}}$, $f[\mu]$ is an arbitrary multiplicative function for ρ on F_μ, which locally holomorphically depends on $[\mu]$ and ρ [3]. Hence $i_{\rho,q}\left(\frac{1}{Q^n P_1^{k_1} \dots P_m^{k_m}}\right) = n + k_1 + \dots + k_m \geqslant 1$. Here $r\left(\frac{(f[\mu])}{D}\right) = 0$, since $\deg\left(\frac{(f[\mu])}{D}\right) > 0$ under our assumptions. Indeed, $\deg(f[\mu]) = 0$ and $\deg\left(\frac{1}{D}\right) \geqslant m > 0$. We can prove this fact in a different way. If there exists a function $g \neq 0$ for ρ on F_μ satisfying $(g) \geqslant Q^n P_1^{k_1} \dots P_m^{k_m} (f[\mu])$, then $0 = \deg(g) \geqslant \deg(Q^n P_1^{k_1} \dots P_m^{k_m} (f[\mu])) \geqslant 2$; this is a contradiction.

It is clear that $i_{\rho,q}\left(\frac{1}{Q^n P_1^{k_1} \dots P_m^{k_m}}\right) = i_{\rho,q}\left(\frac{1}{Q^{n-1} P_1^{k_1} \dots P_m^{k_m}}\right) + 1$. Consequently, there exists a (ρ, q)-differential $\tau^{(n)}_{\rho,q;Q}$ with a pole of order exactly n at the point Q on F_μ. Therefore, $(\tau^{(n)}_{\rho,q;Q}) = \frac{R_1 \dots R_N}{Q^n P_1^{k_1} \dots P_m^{k_m}}$ on F_μ. Thus $(\tau^{(n)}_{\rho,q;Q}) = \frac{R_1 \dots R_N}{Q^n}$ on F'_μ.

These (ρ, q)-differentials $\omega = \tau^{(n)}_{\rho,q;Q}$ from $\mathbf{A}_1(\rho)$ onto F'_μ are determined non uniquely on F'_μ because of their zeros and poles: $(\omega) = \frac{R_1 \dots R_N}{Q^n} \frac{1}{P_1^{k_1} \dots P_m^{k_m}}$, $k_j \geqslant 0$, $j = 1, \dots, m$. Fix $k_1, \dots, k_m$, as the orders of possible poles at the points $P_1, \dots, P_m$. Furthermore, $\deg(\omega) = 0$ on F_μ. This implies that $N = n + k_1 + \dots + k_m$.

Proposition 2.1 yields the equation $\varphi_{P_0}(R_1 \dots R_N) - \varphi_{P_0}(Q^n) - \varphi_{P_0}(P_1^{k_1} \dots P_m^{k_m}) = \psi(\rho)$ in the Jacobian variety $J(F_\mu)$. Consequently, $\varphi(R_1 \dots R_N) = \varphi(Q^n) + k_1\varphi(P_1) + \dots + k_m\varphi(P_m) + \psi(\rho) = M$ or $\varphi(R_1) = M - \varphi(R_2 \dots R_N)$. Therefore, to specify the zeros of the differential we have $N - 1 = n - 1 + k_1 + \dots + k_m \geqslant 1$, for $k_1 \geqslant 1$, free parameters which can choose arbitrarily on F'_μ. Solving the Jacobi inversion problem, we find R_1, which is the unique holomorphic solution to the equation.

Show now that the point Q is a pole of order exactly n on F'_μ. The proof is by contradiction. Assume that only first points $R_1, ..., R_k$, $k \leqslant n$, coincide with Q on F'_μ. Then we have the equality

$$\varphi(R_{k+1}) = \varphi(Q^{n-k}) + \psi(\rho) + k_1\varphi(P_1) + \cdots + k_m\varphi(P_m) - \varphi(R_{k+2}...R_N) = \widetilde{M}. \qquad (*)$$

Choosing the point $R_{k+1} \in F'_\mu$ such that $\varphi(R_{k+1})$ is not equal to the constant $\widetilde{M}$ and the point R_{k+1} is not Q, we have the contradiction. Thus, the point Q is indeed a pole of order exactly n on F_μ.

Consequently, the divisor $(\tau^{(n)}_{\rho,q;Q}) = \frac{R_1}{Q^n}\frac{R_2...R_N}{P_1^{k_1}...P_m^{k_m}}$ has the most general form for the (ρ, q)-differentials $\tau^{(n)}_{\rho,q;Q}$ of class $\mathbf{A}_1(\rho)$ with a unique pole $Q \in F'_\mu$ exactly of order $n \geqslant 2$ on $F'_\mu = F_\mu\backslash\{P_1, \ldots, P_m\}$. By induction on n, taking into account the case $n = 1$ and an essential character ρ [8], we have $\tau^{(n)}_{\rho,q;Q} = \big(\frac{1}{z^n} + O(1)\big)dz^q$.

The proof of Theorem 2.1 is complete. □

We can prove the next statement in the same fashion.

Theorem 2.2. *Given distinct points Q_1, Q_2 on a surface F'_μ of type $(1, m), m \geqslant 1$, a character ρ on F'_μ and a natural number $q \geqslant 1$, there exists an elementary $(\rho, q)-$differential $\tau_{\rho,q;Q_1Q_2}$ of the third kind and class $\boldsymbol{A}_1(\rho)$, with exactly simple poles Q_1 and Q_2 on F'_μ, with the general form of divisor $(\tau_{\rho,q;Q_1Q_2}) = \frac{R_1...R_N}{Q_1Q_2}\frac{1}{P_1^{k_1}...P_m^{k_m}}$, where $\varphi(R_1) = \varphi(Q_1Q_2) - \varphi(R_2 \ldots R_N) + \varphi(P_1^{k_1}) + \cdots + \varphi(P_m^{k_m}) + \psi(\rho)$, $k_j \geqslant 0$, $j = 1, \ldots, m$. Furthermore, we choose the points $R_2, \ldots, R_N$ as locally holomorphic section of divisors of degree $N - 1$ on $F'_\mu\backslash\{Q_1, Q_2\}$, $N = 2 + k_1 + \cdots + k_m$. Also, these differentials depend locally holomorphically on $[\mu]$ and ρ.*

3. Prym differentials for an unessential characters

Given a character ρ, denote by $\Omega_{2,\rho}(F'_\mu)$ the space of meromorphic differentials of the second kind with finitely many poles on F'_μ, and by $\Omega_{e,\rho}(F'_\mu)$, the subspace of all multiplicatively exact Prym differentials for ρ on F'_μ. Let $\tau^{(n)}_{\widetilde{P}_1}$ be an abelian differential of the second kind on F_μ with a unique pole of order exactly $n \geqslant 2$ at the point $\widetilde{P}_1$, and with zero a-periods [5]. The point $\widetilde{P}_1$ is chosen using the condition $r_\rho(\frac{1}{\widetilde{P}_1}) = 1$.

For every character $\rho \neq 1$ define the mapping from $\Omega_{2,\rho}(F'_\mu)$ into $H^1(\Gamma'_\mu, \rho)$, associating to a differential ω its class of periods $[\omega]$.

Suppose that $\omega \in \Omega_{2,\rho}(F'_\mu)$ lifts to U, where $F'_\mu = U/\widetilde{\Gamma}$, and $\widetilde{\Gamma}$ is the Fuchsian group of the first kind uniformizing F'_μ on U [2, 6]. Find the classical periods $\omega_{z_0}(T) = \displaystyle\int_{z_0}^{Tz_0}\omega + n_{m+1}\int_{\gamma_{m+1}}\omega + \cdots + n_{m+k}\int_{\gamma_{m+k}}\omega$, where $n_j \in \mathbf{Z}$, $j = m+1, \ldots, m+k$. Here $\gamma_{m+1}, \ldots, \gamma_{m+k}$ stand for loops enclosing only the poles $Q_1, \ldots, Q_k$ of ω on F'_μ respectively. We take the integral $\displaystyle\int_{z_0}^{Tz_0}\omega$ along some fixed particular path in the disk U avoiding the poles of ω.

Since ω is a differential of the second kind, all residues at its poles vanish. Thus, there exists a global primitive, a meromorphic function $f(z)$ on U satisfying $\omega = df$ on $F'_\mu\backslash\{Q_1, ..., Q_k\}$. Lifting now $\omega = df$ to U, relative to $\widetilde{\widetilde{\Gamma}}$, we obtain $\omega_{z_0}(T) = \displaystyle\int_{z_0}^{Tz_0} df(z)$ for every $T \in \widetilde{\widetilde{\Gamma}}$, where $\widetilde{\widetilde{\Gamma}}$

is the Fuchsian group of the first kind on U uniformizing the surface F''_μ, which results from F'_μ by removing all k poles $Q_1,\ldots,Q_k$ of the differentials ω.

Define the mapping

$$\Omega_{2,\rho}(F'_\mu) \ni \omega \to [\omega] = \{\omega_{z_0}(A), \omega_{z_0}(B), \omega_{z_0}(\gamma_1), \ldots,$$

$$\omega_{z_0}(\gamma_{m-1}), \omega_{z_0}(\gamma_{m+1}), \ldots, \omega_{z_0}(\gamma_{m+k})\} \in H^1(\widetilde{\widetilde{\Gamma}}, \rho'),$$

where $\rho'(\gamma_s) = 1,\ s = m+1,\ldots,m+k$ and $\rho' = \rho$ on $\Gamma'_\mu \cong \widetilde{\Gamma}$. Since all $\omega_{z_0}(\gamma_s)$, for $s = m+1,\ldots,m+k$, vanish, $[\omega]$ is expressed in terms of only $\omega_{z_0}(A)$, $\omega_{z_0}(B)$, $\omega_{z_0}(\gamma_1),\ldots,\omega_{z_0}(\gamma_{m-1})$, satisfying (1), while for $\rho \neq 1$ we have $\omega_{z_0}(A) = 0$ for $\rho(A) \neq 1$, and $\omega_{z_0}(B) = 0$ for $\rho(B) \neq 1$. Hence, the mapping $\Omega_{2,\rho}(F'_\mu) \ni \omega \to [\omega] \in H^1(\Gamma'_\mu, \rho)$ is well-defined.

If the class of periods $[\omega] = 0$ in $H^1(\Gamma'_\mu, \rho)$ for $\omega \in \Omega_{2,\rho}(F'_\mu)$, then the differential ω is multiplicatively exact for ρ on F'_μ, and so $\omega \in \Omega_{e,\rho}(F'_\mu)$. If $\omega \in \Omega_{e,\rho}(F'_\mu)$, then, as above, $\omega_{z_0}(\gamma_s) = 0$ for $s = m+1,\ldots,m+k$, where γ_s is a loop enclosing only the pole Q_s of ω. By assumption, $\omega = df$, where f is a multiplicative meromorphic function on F'_μ, and so all periods in the sense of Gunning of ω on F'_μ vanish. So, $[\omega] = 0$ in $H^1(\Gamma'_\mu, \rho)$.

Thus, for every $\rho \neq 1$ the period mapping from $\Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu)$ into $H^1(\Gamma'_\mu, \rho)$ determined by the rule $\omega + \Omega_{e,\rho}(F'_\mu) \to [\omega + \Omega_{e,\rho}(F'_\mu)] = [\omega]$ is well-defined, bijective, and linear. Consequently, $\dim_{\mathbf{C}} \Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu) \leqslant m$ for every $\rho \neq 1$.

Theorem 3.1. *The vector bundle $E_1 = \bigcup \Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu)$ is holomorphic of rank m over the base $\mathbf{T}_{1,m} \times (L_1 \backslash \{1\})$ for $m \geqslant 2$. Furthermore, the tuples*

$$f_0 \tau^{(2)}_{\widetilde{P}_1}, f_0 \tau_{P_2 P_1}, \ldots, f_0 \tau_{P_m P_1}, \tag{2}$$

of cosets of Prym differentials constitute bases of locally holomorphic sections of this bundle, where f_0 is a multiplicative unit on F_μ for ρ, $r_\rho(\widetilde{P}_1^{-1}) = 1$ on F_μ and point $\widetilde{P}_1 \in F'_\mu$.

Proof. This bundle is well-defined over this base by Lemma 1.3. Let us establish the reverse inequality $\dim_{\mathbf{C}} \Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu) \geqslant m$ and construct a basis for this quotient space.

Verify that for $\rho \neq 1,\ \rho(A) \neq 1$, the differentials in the tuples (2) represent cosets in our quotient space which are linearly independent over $\mathbf{C}$. For $\rho_0 \neq 1$ on $\pi_1(F'_{\mu_0})$ there exists $A \in \Gamma'_{\mu_0}$ satisfying $\rho_0(A) = \exp 2\pi i c \neq 1$. Thus, $c \neq 0$ for every ρ in a sufficiently small neighborhood $U(\rho_0) \subset L_1 \backslash \{1\}$ and every $[\mu] \in U[\mu_0]$. Since $df_0 = 2\pi i c f_0 \zeta$ on F_μ, we can express $f_0\zeta$ linearly via df_0. Consequently, instead of the differential $f_0\zeta$ we can take df_0, which represents the zero coset. Suppose that there exists a linear combination with nonzero coefficients

$$\tilde{c}_1 f_0 \tau^{(2)}_{\widetilde{P}_1} + \widetilde{\widetilde{c}}_1 f_0 \tau_{P_2 P_1} + \cdots + \widetilde{\widetilde{c}}_{m-1} f_0 \tau_{P_m P_1} = df,$$

where f is multiplicative function for an unessential character ρ on F'_μ with $\rho_0(A) \neq 1$.

Go around the point P_2 along a small loop γ_2, starting from $\widetilde{P}_{2,0}$ on F'_μ. Then the expression on the left-hand side has the residue $\widetilde{\widetilde{c}}_1 f_0(\widetilde{P}_{2,0})\rho(\gamma_2)$, while this residue on the right-hand side vanishes. But $f_0(\widetilde{P}_{2,0}) \neq 0,\ \rho(\gamma_2) = 1$, so $\widetilde{\widetilde{c}}_1 = 0$. In the same fashion we calculate the residues along small loops enclosing the points $P_3,\ldots,P_m$ and obtain $\widetilde{\widetilde{c}}_2 = \cdots = \widetilde{\widetilde{c}}_{m-1} = 0$. Then we are left with the sum $\tilde{c}_1 f_0 \tau^{(2)}_{\widetilde{P}_1} = df$.

Consider the coefficient $\widetilde{c}_1$.

1) If df has removable singularities at all punctures, then this equality on F'_μ implies that there exists a meromorphic multiplicative function on F_μ with simple pole at $\widetilde{P}_1$. But this is impossible by the choice of this point and the condition $r_\rho\left(\frac{1}{\widetilde{P}_1}\right) = 1$;

2) If the continuation of df to F_μ has at least one pole or essential singularity at the punctures, then for the combination on the left-hand side this point (puncture) is not singular, while for df it is singular. This is a contradiction. Therefore, $\widetilde{c}_1 = 0$.

Thus, the differentials in the tuple (2) represent cosets in our quotient space which are linearly independent over $\mathbf{C}$.

The case $\rho_0(B) \neq 1$ can be proved analogously. The proof of Theorem 3.1 is complete. □

Denote by $\Omega_\rho\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big)$ the space of the differentials for ρ, which are multiples of the divisor $\frac{1}{Q_1 \ldots Q_s}$ on F'_μ, and by $\Omega_{e,\rho}(1; F'_\mu)$, the subspace of the holomorphic multiplicatively exact differentials for ρ on F'_μ.

Theorem 3.2. *The vector bundle* $E_2 = \bigcup \Omega_\rho\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big)/\Omega_{e,\rho}(1; F'_\mu)$ *is holomorphic of rank* $m+s$ *with the base* $\mathbf{T}_{1,m} \times (L_1 \backslash \{1\})$ *for* $m \geqslant 2$, $s \geqslant 1$. *Furthermore, the tuple*

$$f_0 \tau^{(2)}_{P_1},\ f_0 \tau_{P_2 P_1}, \ldots, f_0 \tau_{P_m P_1},\ f_0 \tau_{Q_1 P_1}, \ldots, f_0 \tau_{Q_s P_1}, \tag{3}$$

of cosets of differentials constitutes a basis for locally holomorphic sections of this bundle, where $Q_1, \ldots, Q_s$ *are distinct points on* F'_μ, *depending holomorphically on* $[\mu]$.

Proof. Consider the period mapping $\Omega_\rho\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big) \ni \omega \to [\omega] \in H^1(\Gamma'', \rho)$. The class of $[\omega]$ is determined by the tuple of classical periods $(\omega(A) = 0, \omega(B), \omega(\gamma_1), \ldots, \omega(\gamma_{m-1}), \omega(\widetilde{\gamma}_1), \ldots, \omega(\widetilde{\gamma}_s))$. Here the period $\omega(\gamma_m)$ is expressed via the remaining $m+s$ of periods, $F''_\mu = F'_\mu \backslash \{Q_1, \ldots, Q_s\} = F_\mu \backslash \{P_1, \ldots, P_m\} \cup \{Q_1, \ldots, Q_s\}$, and $F''_\mu = U/\Gamma''$.

If $\Omega\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big) \ni \omega \to [\omega] = 0$ in $H^1(\Gamma'', \rho)$, then the differential ω is multiplicatively exact on F'_μ. The points $Q_1, \ldots, Q_s$ are removable singularities for ω since $2\pi i (res_{Q_j} \omega) = \int_{\widetilde{\gamma}_j} \omega = 0$, for $j = 1, \ldots, s$. Thus, $\omega \in \Omega_{e,\rho}(1; F'_\mu)$. Consequently, the period mapping is well-defined, bijective, and takes $\Omega_\rho\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big)/\Omega_{e,\rho}(1; F'_\mu)$ linearly into $H^1(\Gamma'', \rho)$. Thus, $\dim \Omega_\rho\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big)/\Omega_{e,\rho}(1; F'_\mu) \leqslant m+s$.

Let us establish the reverse inequality for the dimension and construct a basis. The tuple of cosets of differentials in (3) is linearly independent over $\mathbf{C}$. Indeed, if

$$C_1 f_0 \tau^{(2)}_{P_1} + \widetilde{\widetilde{c}}_1 f_0 \tau_{P_2 P_1} + \cdots + \widetilde{\widetilde{c}}_{m-1} f_0 \tau_{P_m P_1} + c'_1 f_0 \tau_{Q_1 P_1} + \cdots + c'_s f_0 \tau_{Q_s P_1} = df,$$

then $\widetilde{\widetilde{c}}_1 = \cdots = \widetilde{\widetilde{c}}_{m-1} = c'_1 = \cdots = c'_s = 0$ since f is a multiplicative meromorphic function for ρ on F'_μ and its residues at the points $P_2, \ldots, Q_s$ vanish.

This yields the equality $C_1 f_0 \tau^{(2)}_{P_1} = df$. If the continuation f has in the punctures a unique pole P_1, then $(f) \geqslant \frac{1}{P_1}$ on F_μ. But this is impossible because of the condition $r_\rho\big(\frac{1}{P_1}\big) = 1$, since we have the inequality $(f) \geqslant 1 \geqslant \frac{1}{P_1}$. This is a contradiction. Hence, the dimension of the quotient space is at least $m+s$ and we have constructed a basis. The proof of Theorem 3.2 is complete. □

By Grauert's theorem, since the base is simply connected, we obtain

Corollary 3.1. *The holomorphic vector bundle (with fibers consisting of the first holomorphic de Rham cohomology groups for* ρ *on* F'_μ*)* $E'_2 = \bigcup_{[\mu], \rho \neq 1} H^1_{hol,\rho}(F'_\mu) = \bigcup \Omega_\rho(1; F'_\mu)/\Omega_{e,\rho}(1; F'_\mu)$ *is analytically equivalent to the trivial rank* m *vector bundle with the base* $\mathbf{T}_{1,m} \times (L_1 \backslash \{1\})$ *for* $m \geqslant 2$.

Define the period mapping χ from $\Omega_\rho(1; F')$ onto $H^1(\Gamma', \rho)$, associating to ω its class of periods $[\omega]$, which is determined by the tuple of classical periods $\left(\int_a \omega, \int_b \omega, \int_{\gamma_1} \omega, \ldots, \int_{\gamma_{m-1}} \omega \right)$.

Choose a representative for $[\omega]$ satisfying $\int_a \omega = \omega(A) = 0$.

Corollary 3.2. *On every surface F'_μ of type $(1,m)$, $m \geqslant 2$, given an unessential character ρ, we have an isomorphism $\Omega_\rho(1;F'_\mu) \cong Ker\chi \oplus H^1_{hol,\rho}(F'_\mu)$, where $Ker\chi = \Omega_{e,\rho}(1;F'_\mu)$ is an infinite dimensional vector space and $\dim_{\mathbf{C}} H^1_{hol,\rho}(F'_\mu) = m$.*

4. Prym differentials for an essential character

Lemma 4.1. *On a surface F'_μ of type $(1,m)$, $m \geqslant 1$, given an essential character ρ, there exists a $(\rho,1)$-differential $\tau = \tau_{\rho;Q^2P_1}$, where $Q \in F'_\mu$, and $(\tau) = \frac{R_1 \dots R_N}{Q^2 P_1 P_2^{k_2} \dots P_m^{k_m}}$ on F_μ, where $k_j \in \mathbf{N}$, $j = 2,\dots,m$, and $R_k \neq P_1, Q$, $k = 1,\dots,N$, $N = 3 + k_2 + \dots + k_m$, depending locally holomorphically on $[\mu]$ and ρ.*

Proof. Proceed as in Section 3.

Theorem 4.1. *The vector bundle $E_3 = \bigcup \Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu)$ is holomorphic of rank m with the base $\mathbf{T}_{1,m} \times Hom(\Gamma',\mathbf{C}^*)\backslash L_1$ for $m \geqslant 2$. Furthermore, the tuples : either*

$$\tau_{\rho;P_2P_1}, \dots, \tau_{\rho;P_mP_1}, \tau_{\rho;Q^2P_1} \tag{4}$$

or

$$\tau_{\rho;P_1}, \dots, \tau_{\rho;P_m}, \tag{5}$$

of cosets of differentials constitute a basis for locally holomorphic sections of this bundle, where $Q \in F'_\mu$.

Proof. Take an essential character ρ on F'_μ. Define a mapping Φ from $\Omega_{2,\rho}(F'_\mu)$ into $H^1(\Gamma'_\mu,\rho)$, by associating to a differential ω its class of periods $[\omega] \in H^1(\Gamma'_\mu,\rho)$.

If $\omega \in \Omega_{2,\rho}(F'_\mu)$ satisfies $[\omega] = 0$ in $H^1(\Gamma'_\mu,\rho)$, then ω is multiplicatively exact for ρ on F'_μ, and hence $\omega \in \Omega_{e,\rho}(F'_\mu)$. It is clear also that every differential ω in $\Omega_{e,\rho}(F'_\mu)$ has the vanishing class of periods. Therefore, the kernel of Φ coincides with $\Omega_{e,\rho}(F'_\mu)$. Consequently, this mapping is well-defined on the quotient space $\Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu)$. Furthermore, Φ is bijective and linear. This implies that $\dim_{\mathbf{C}} \Omega_{2,\rho}(F'_\mu)/\Omega_{e,\rho}(F'_\mu) \leqslant m$.

Let us establish the reverse inequality for the dimension and construct two forms of basis for our quotient space.

By Theorem 2.2, there exists a tuple $\tau_{\rho;P_2P_1}, \dots, \tau_{\rho;P_mP_1}$ of elementary Prym differentials of third kind with simple poles at the points P_j and P_1, $j = 2,\dots,m$, on F'_μ respectively.

Suppose that the tuple (4) represents linearly dependent cosets in our quotient space for the essential character ρ; therefore, there exists a nontrivial linear combination with nonzero coefficients :

$$\tilde{\tilde{c}}_1 \tau_{\rho;P_2P_1} + \dots + \tilde{\tilde{c}}_{m-1} \tau_{\rho;P_mP_1} + \tilde{\tilde{c}}_m \tau_{\rho;Q^2P_1} = df,$$

for fixed $Q \in F'_\mu$, where f is multiplicative function on F'_μ, (possibly, with poles of arbitrary orders and essential singularities in punctures for the branches of this function on F_μ.)

Consider the coefficients $\tilde{\tilde{c}}_j$, $j = 1,\dots,m-1$. For $j = 2,\dots,m$, take a loop γ_j enclosing only the point P_j. Then the classical period $\int_{\gamma_j} df = c\sigma(\gamma_j)$ and, choosing instead of f the function $(f-c)$, we obtain $\int_{\gamma_j} d(f-c) = 0$. Consequently, all coefficients $\tilde{\tilde{c}}_1 = \dots = \tilde{\tilde{c}}_{m-1} = 0$.

What's left to prove is the equality $\tilde{\tilde{c}}_m \tau_{\rho;Q^2P_1} = df$. Go along $\tilde{\gamma}_1$ around the point P_1 and calculate residue at point, we obtain that $\tilde{\tilde{c}}_m = 0$.

We can prove the same statement for tuple (5) in same fashion.

The theorem 4.1 is proved. □

Theorem 4.2. *The vector bundle* $E_4 = \bigcup \Omega_\rho\big(\frac{1}{Q_1 \ldots Q_s}; F'_\mu\big)/\Omega_{e,\rho}(1, F'_\mu)$ *is holomorphic of rank* $m+s$ *with the base* $\mathbf{T}_{1,m} \times Hom(\Gamma', \mathbf{C}^*)\backslash L_1$ *for distinct points* $Q_1, \ldots, Q_s$, $s \geqslant 1$, *on the surface* F'_μ *of type* $(1, m)$, $m \geqslant 2$. *Furthermore, the tuple : either*

$$\tau_{\rho;P_2P_1}, \ldots, \tau_{\rho;P_mP_1}, \tau_{\rho;Q_1P_1}, \ldots, \tau_{\rho;Q_sP_1}, \tau_{\rho;P_2^2P_1}, \tag{6}$$

or

$$\tau_{\rho;P_1}, \ldots, \tau_{\rho;P_m}; \tau_{\rho;Q_1}, \ldots, \tau_{\rho;Q_s}, \tag{7}$$

of cosets of Prym differentials constitutes a basis for locally holomorphic sections of this bundle.

Proof. It suffices to verify only the linear independence of the cosets of differentials in (6). Suppose that there exists a nontrivial linear combination :

$$\tilde{\tilde{c}}_2\tau_{\rho;P_2P_1} + \cdots + \tilde{\tilde{c}}_m\tau_{\rho;P_mP_1} + \tilde{\tilde{c}}_{m+1}\tau_{\rho;Q_1P_1} + \cdots + \tilde{\tilde{c}}_{m+s}\tau_{\rho;Q_sP_1} + c'\tau_{\rho;P_2^2P_1} = df.$$

If f has essential singularities at the punctures, then we immediately obtain a contradiction since the left-hand side lacks those. Using the residues and periods, as in the proof of the previous theorem, we infer that $\tilde{\tilde{c}}_j = 0$, for $j = 2, \ldots, m+s$. It remains to consider the equality $c'\tau_{\rho;P_2^2P_1} = df$.

The residue at P_1 is a multiple of c', it has the form $Mc', M \neq 0$, while on the right-hand side, since df is multiplicatively exact, we can make the classical period around only point P_1 vanish. Hence, $c' = 0$. Thus the cosets of differentials in (6) constitute a basis for our quotient space.

Analogously we can prove this statement for tuple (7).

The proof of Theorem 4.2 is complete. □

Corollary 4.1. *The vector bundle* $E'_4 = \bigcup H^1_{hol,\rho}(F'_\mu) = \bigcup \Omega_\rho(1; F'_\mu)/\Omega_{e,\rho}(1; F'_\mu)$ *is holomorphic of rank* m *with the base* $\mathbf{T}_{1,m} \times (Hom(\Gamma', \mathbf{C}^*)\backslash L_1)$ *for* $m \geqslant 2$.

Remark 1. The vector bundles E_1 and E_3 are analytically equivalent to the Gunning cohomological bundle G' over their respective bases.

Remark 2. Theorems 3.2 and 4.2, taking into account theorems 2.1 and 2.2, can be generalized for vector bundles with fibers $\Omega^q_\rho\big(\frac{1}{Q_1^{\alpha_1} \ldots Q_s^{\alpha_s}}; F'_\mu\big)$, over the products $\mathbf{T}_{1,m} \times Hom(\Gamma', \mathbf{C}^*)\backslash L_1$ and $\mathbf{T}_{1,m} \times L_1\backslash 1$ for $s \geqslant 1$, $\alpha_1, \ldots, \alpha_s \in \mathbf{N}$, $q > 1$.

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Пространства мероморфных дифференциалов Прима на конечных торах

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В данной статье построены все виды элементарных дифференциалов Прима для любых характеров на переменных торах с конечным числом проколов и найдены размерности двух важных фактор-пространств мероморфных дифференциалов Прима. Как следствие, находится размерность первой голоморфной группы когомологий де Рама дифференциалов Прима для любых характеров на торе. В этих фактор-пространствах построены явные базисы.

Ключевые слова: дифференциалы Прима для произвольных характеров, когомологическое расслоение Ганнинга над пространством Тейхмюллера для тора с конечным числом проколов.