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On Solvability of the Cauchy Problem for a Loaded System

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In this work we investigated the Cauchy problem for a loaded Burgers-type system. Example of mathematical physics inverse problem leading to problem being investigated is given. Sufficient conditions for existence of solution in continuously differentiable class are obtained.

Keywords: Cauchy problem, inverse problem, Burgers' equation, non-linear system, weak approximation method.

Inverse problems of mathematical physics play important role in science and applications today [1]. Coefficient inverse problems for parabolic equations are problems of finding solutions of differential equation with one (or more) unknown coefficients. These problems often reduce to problems for loaded equations. Loaded differential equations (see [2]) are ones with functionals of solution (e.g. values of solution or its derivatives on lesser-dimensional manifolds) as coefficients or right-hand side.

Existense of solution to special class of loaded two-dimensional parabolic equations has been proved by I. V. Frolenkov and Yu. Ya. Belov (see [3]). Problem being considered in this article arises during generalization of preceding results.

1. Problem formulation

We consider the initial-value problem for loaded system

$$\frac{\partial \bar{u}}{\partial t} = \mu(t, \bar{\omega}(t))\Delta\bar{u} + \nu(\bar{u}\cdot\nabla)\bar{u} + \bar{f}(t, x, \bar{u}, \bar{\omega}(t)), \tag{1}$$

$$\bar{u}(0, x) = \bar{\varphi}(x) \tag{2}$$

in domain $\Pi_{[0,T]} = \{(t, x) | 0 \leqslant t \leqslant T, x \in \mathbb{R}^n\}$, where $\bar{u} = \big(u_1(t, x), \dots, u_n(t, x)\big)$ are unknown functions. Let $\bar{\omega}(t) = \big(u_i(t, x^j), D^\alpha u_i(t, x^j)\big)$; $i = 1, \dots, n$; $j = 1, \dots, r$; $|\alpha| = 0, \dots, p_0$ be a vector function, with traces of unknown functions and their partial derivatives with respect to spartial variables of order up to p_0 at points $x^1, \dots, x^r \in \mathbb{R}^n$ as its components.

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdot \dots \cdot \partial^{\alpha_n} x_n}$$

is partial differential operator, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$. Functions $\mu(t, \bar{\omega}(t))$, $\bar{f} = \big(f_1, \dots, f_n\big)$, $\bar{\varphi} = \big(\varphi_1(x), \dots, \varphi_n(x)\big)$ are given ones, $\nu \in \mathbb{R}$ is given coefficient.

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We will use following notation:

$$C^{q,s}(\Pi_{[0,T]}) = \left\{ \bar{u} = (u_1(t,x), \dots, u_n(t,x)) \left| \frac{\partial^j u_i}{\partial t^j}, D^\alpha u_i(t,x) \in C(\Pi_{[0,T]}); \right.\right.$$
$$\left.\left| \frac{\partial^j u_i}{\partial t^j} \right| \leqslant K, |D^\alpha u_i(t,x)| \leqslant K; \quad i = 1, \dots, n; \ j \leqslant q; \ |\alpha| \leqslant s; \ q, s \in \mathbb{Z}; \ K \text{is const} \right\}$$

is class of bounded, continuously differentiable functions,

$$U^i_\alpha(0) = \sup_{x\in\mathbb{R}^n} |D^\alpha \varphi_i(x)|,$$
$$U^i_\alpha(t) = \sup_{\xi\in[0,t]} \sup_{x\in\mathbb{R}^n} |D^\alpha u_i(\xi,x)|,$$
$$U^i(t) = \max_{|\alpha|\leqslant p+2} U^i_\alpha(t), \quad U(t) = 1 + \sum_{i=1}^n U^i(t)$$

are nondecreasing nonnegative functions.

Let $p \geqslant \max(p_0, 2)$, function $\bar{\varphi}$ satisfies

$$\varphi_i(x) \in C^{p+2}(\mathbb{R}^n), \quad |D^\alpha \varphi_i(x)| \leqslant K_1; \quad i = 1, \dots, n; \quad |\alpha| \leqslant p+2, \tag{3}$$

μ и $\bar{f}$ are continuous in all variables and the following relations are valid for any function $\bar{u}(t,x) \in C^{1,p+2}(\Pi_{[0,T]})$:

$$\begin{aligned} &\mu(t, \bar{\omega}(t)) \geqslant \mu_0 > 0, \quad \forall \bar{u}(t,x) \in C^{1,p+2}(\Pi_{[0,T]}) \\ &|D^\alpha f_i(t,x,\bar{u},\bar{\omega})| \leqslant K_2\big(1+U(t)+U(t)^2\big), \quad |\alpha| \leqslant p+2. \end{aligned} \tag{4}$$

Here and further, K_i are constants depending only on the initial data. We will prove

Theorem 1.1. *Let the initial data of problem (1), (2) satisfy (3), (4) for some p. Then constant t^* exists ($t^* \in (0,T]$) for which a solution of problem (1), (2) exists and lies in $C^{1,p}(\Pi_{[0,t^*]})$ class.*

2. An example

We have investigated inverse problem involving finding functions $u(t,x), g(t)$ in Cauchy problem for the Burgers-type equation

$$u_t(t,x) = \mu(t)u_{xx} + A(t)uu_x + B(t)u + g(t)f(t,x),$$
$$u(0,x) = u_0(x), \qquad 0 \leqslant t \leqslant T, \quad x \in \mathbb{R},$$

which reduces (using overdetermination condition $u(t,x_0) = \psi(t)$) to Cauchy problem for loaded parabolic equation

$$u_t(t,x) = \mu(t)u_{xx} + A(t)uu_x + B(t)u + F(t,u), \tag{5}$$

$$u(0,x) = u_0(x), \qquad 0 \leqslant t \leqslant T, \quad x \in \mathbb{R}, \tag{6}$$

where

$$F(t,u) = \frac{f(t,x)}{f(t,x_0)} \left(\psi'(t) - B(t)\psi(t) - \mu(t)u_{xx}(t,x_0) - A(t)\psi(t)u_x(t,x_0)\right)$$

is functional depending on traces of unknown function and its derivatives at point x_0. The problem (5), (6) is the particular case of problem (1), (2) for $n = 1$, $\bar{u} = u(t,x)$, $\bar{f} = F(t,u)$, $\bar{\varphi} = u_0(x)$.

Let the initial data of problem the (5), (6) satisfies

$$u_0(x) \in C^{p+2}(\mathbb{R}), \quad \left|\frac{\partial^k u_0}{\partial x^k}\right| \leqslant K_0 - const, \quad k = 0, \dots, p+2, \tag{7}$$

$$\begin{gathered} A(t), B(t) \in C([0,T]),\ \psi(t) \in C^1([0,T]),\ |f(t,x_0)| \geqslant \frac{1}{K_0}, \\ |A(t)| + |B(t)| + |\psi(t)| + |\psi'(t)| \leqslant K_0, \\ \mu(t) \geqslant \mu_0 > 0,\ \frac{\partial^k f}{\partial x^k} \in C([0,T] \times \mathbb{R}),\ \left|\frac{\partial^k f}{\partial x^k}\right| \leqslant K_0,\ k = 0, \dots, p+2 \end{gathered} \tag{8}$$

for some $p \geqslant 2$. Conditions (3) of Theorem 1.1 are fulfilled by (7). We can check fulfillment of (3) provided (8) are valid:

$$\begin{aligned} \forall u(t,x) \in C^{1,p+2}([0,T] \times \mathbb{R})\ \forall k = 0, \dots, p+2\ \left|\frac{\partial^k}{\partial x^k} F(t,u)\right| \leqslant \\ \leqslant K_2 \left(1 + U_2^1(t) + U_1^1(t)\right) \leqslant K_2 \left(1 + U(t)\right). \end{aligned}$$

Thus in is the particular case one can use Theorem 1.1 to prove existence of solution of problem (5), (6) in $C^{1,p}(\Pi_{[0,t^*]})$ class.

3. Auxiliary theorem

Theorem 3.1. *Let $u(t,x)$ be solution of*

$$\begin{gathered} u_t = \sum_{i=1}^{n} b_i(t,x) \frac{\partial u(t,x)}{\partial x_i}, \\ u(0,x) = u_0(x), \qquad x \in E_n \end{gathered}$$

in domain $G_{[0,T]} = \{(t,x) | 0 \leqslant t \leqslant T, x \in E_n\}$ of $C^{1,p}(G_{[0,T]})$ class. Let the conditions

$$\begin{gathered} |D^\alpha b_i(t,x)| \leqslant M(p), \quad |\alpha| \leqslant p, i = 1, \dots, n; \\ |D^\alpha u_0(x)| \leqslant C(p), \quad |\alpha| \leqslant p \end{gathered}$$

are valid. Then $u(t,x)$ satisfies

$$|D^\alpha u(t,x)| \leqslant C(p) e^{l(p)M(p)T}, \qquad |\alpha| \leqslant p, \tag{9}$$

where $l(p) > 0$ depends only on p and does not depend on the initial data.

4. Proof of Theorem 1.1

We will prove existence of solution of problem (1), (2) using weak approximation method (see [4]). We split the problem into three fractional steps and make time shift by $\tau/3$ in traces of unknown functions and nonlinear terms. This leads to equation system

$$\frac{\partial u_i^\tau}{\partial t} = 3\mu(t, \bar{\omega}(t -^\tau /_3))\Delta u_i^\tau, \quad t \in \left(m\tau, \left(m +^1 /_3\right)\tau\right], \tag{10}$$

$$\frac{\partial u_i^\tau}{\partial t} = 3\nu(\bar{u}^\tau(t -^\tau /_3) \cdot \nabla)u_i^\tau, \quad t \in \left(\left(m +^1 /_3\right)\tau, \left(m +^2 /_3\right)\tau\right], \tag{11}$$

$$\frac{\partial u_i^\tau}{\partial t} = 3f_i(t -^\tau /_3, x, \bar{u}^\tau(t -^\tau /_3, x), \bar{\omega}(t -^\tau /_3)), \tag{12}$$

$$t \in \left(\left(m +^2 /_3\right)\tau, \left(m + 1\right)\tau\right],$$

$$u_i^\tau(t, x)\big|_{t\leqslant 0} = \varphi_i(x); \quad i = 1, \ldots, n; \quad m = 0, \ldots, M - 1; \quad M\tau = T. \tag{13}$$

Let us introduce the following notation

$$U_\alpha^{i\tau}(t) = \sup_{\xi\in[0,t]} \sup_{x\in\mathbb{R}_n} |D^\alpha u_i^\tau(\xi, x)|,$$

$$U^{i\tau}(t) = \max_{|\alpha|\leqslant p+2} U_\alpha^{i\tau}(t), \quad U^\tau(t) = 1 + \sum_{i=1}^{n} U^{i\tau}(t).$$

Zeroth whole step ($m = 0$) is considered. In first fractional step system (10), (13) is representing n Cauchy problems for parabolic equations, for which the maximum principle can be applied. We differentiate (10), (13) with respect to spartial variables up to $(p + 2)$ times, thus obtaining

$$U_\alpha^{i\tau}(t) \leqslant U_\alpha^i(0), \quad U^\tau(t) \leqslant U(0), \quad |\alpha| \leqslant p + 2, \quad t \in (0,^\tau /_3]. \tag{14}$$

In second fractional step (11), (13) is n separate linear first-order partial differential equations

$$\frac{\partial u_i^\tau}{\partial t} = 3\nu u_1^\tau(t -^\tau /_3, x)\frac{\partial u_i^\tau}{\partial x_1} + \cdots + 3\nu u_n^\tau(t -^\tau /_3, x)\frac{\partial u_i^\tau}{\partial x_n},$$
$$u_i^\tau|_{t=\frac{\tau}{3}} = u_i^\tau(^\tau/_3, x), \quad i = 1, \ldots, n,$$

solutions of which satisfy Theorem 3.1, giving us estimate (with K_3 equals to $l(p + 2)$ arising in Theorem 3.1)

$$|D^\alpha u_i^\tau(t, x)| \leqslant U^{i\tau}(^\tau/_3)e^{\tau K_3 U^\tau(^\tau/_3)}, \quad |\alpha| \leqslant p + 2, \quad t \in (^\tau/_3,^{2\tau} /_3],$$

leading to

$$U^\tau(t) \leqslant U^\tau(^\tau/_3)e^{\tau K_3 U^\tau(^\tau/_3)}, \quad t \in (^\tau/_3,^{2\tau} /_3]. \tag{15}$$

In third fractional step $u_i^\tau(t, x)$ are solutions to n separate Cauchy problems for ordinary differential equations with known right-hand sides. Thus $u_i^\tau(t, x)$ and their derivatives can be expressed explicitly

$$D^\alpha u_i^\tau(t, x) = D^\alpha u_i^\tau(\frac{2\tau}{3}, x) + \int_{^{2\tau}/_3}^{t} 3D^\alpha f_i\left(\xi - \frac{\tau}{3}, x, \bar{u}^\tau(\xi - \frac{\tau}{3}, x), \bar{\omega}(\xi - \frac{\tau}{3})\right) d\xi,$$
$$|\alpha| \leqslant p + 2, \quad t \in (^{2\tau}/_3, \tau],$$

and using (4) can be estimated[‡] by

$$U^\tau(t) \leqslant U^\tau(^{2\tau}/_3)e^{\tau K_4 U^\tau(^{2\tau}/_3)}, \quad t \in (^{2\tau}/_3, \tau]. \tag{16}$$

[‡]For detailed derivation of (16), see Appendix 5.

Let t^* be nonnegative constant satisfying

$$e^{6t^* K_5 U(0)} \leqslant 2, \qquad K_5 = \max(K_3, K_4). \tag{17}$$

We will prove that derivatives $\{D^\alpha u_i^\tau\}, |\alpha| \leqslant p+2$ are bounded uniformly on τ in some time interval $0 \leqslant t \leqslant t^*$. Here and further τ be arbitrary small ($\tau \ll t^*$) and for some integer $M' = M'(\tau)$ equality $M'\tau = t^*$ is valid. From (17)

$$e^{(2i-1)\cdot 3\tau K_5 U(0)} \leqslant 2, \quad i = 1, \ldots, M'. \tag{18}$$

Using (18) we express from (14)–(16) estimate valid in $t \in [0, \tau]$

$$U^\tau(t) \leqslant U(0) e^{3\tau K_5 U(0)}. \tag{19}$$

We will prove the inequality

$$U^\tau(i\tau) \leqslant U(0)\exp((2i-1)3\tau K_5 U(0)) = K_6, \quad i = 1, \ldots, M', \tag{20}$$

by induction. For $i = 1$ (20) is valid by (19). Let (20) be valid for some $i < M'$. Applying our reasoning as in zeroth whole step, we deduce

$$\begin{gathered} U^\tau((i+1)\tau) \leqslant U^\tau(i\tau) e^{3\tau K_5 U^\tau(i\tau)} \leqslant \\ \leqslant U(0)\exp((2i-1)\cdot 3\tau K_5 U(0)) \exp(3\tau K_5 U(0) e^{(2i-1)3\tau K_5 U(0)}) \leqslant \\ \leqslant U(0)\exp((2i+1)\cdot 3\tau K_5 U(0)) = U(0)\exp((2(i+1)-1)\cdot 3\tau K_5 U(0)), \end{gathered}$$

thus validating (20) for $i+1$. It holds for all $i < M'$ by mathematical induction principle.

Since $U^\tau(t)$ is monotonic, from (20) we have

$$U^\tau(t) \leqslant U^\tau(M'\tau) = K_6 - const, \quad t \in [0, t^*].$$

From the previous inequality it follows that uniform on τ

$$|D^\alpha u_i^\tau(t,x)| \leqslant K_6, (t,x) \in \Pi_{[0,t^*]}, |\alpha| \leqslant p+2, \tag{21}$$

where $\Pi_{[0,t^*]} = \{(t,x) | 0 \leqslant t \leqslant t^*, x \in \mathbb{R}^n\}$.

Derivatives

$$\frac{\partial}{\partial t} D^\alpha \bar{u}^\tau(t,x), \frac{\partial}{\partial x_i} D^\alpha \bar{u}^\tau(t,x), \quad (t,x) \in \Pi^{M_0}_{[0,t^*]}, \quad |\alpha| \leqslant p, \quad i = 1, \ldots, n,$$

where $\Pi^{M_0}_{[0,t^*]} = \{(t,x), t \in [0,t^*], |x_i| \leqslant M_0\}$, are bounded uniformly on τ from (21) and equations (10)–(12), which implies uniform boundedness and uniform equicontinuity (for any $M_0 > 0$) of function sets $\{D^\alpha \bar{u}^\tau\}, |\alpha| \leqslant p$ in $\Pi^{M_0}_{[0,t^*]}$.

Applying Arzelà–Ascoli theorem about compactness, we show existence of the subsequence $\bar{u}^{\tau_k}(t,x)$ of sequence $\bar{u}^\tau(t,x)$, which converges to some vector function $\bar{u}(t,x)$ with its derivatives $D^\alpha \bar{u}(t,x)$, $|\alpha| \leqslant p$. Under the theorem about weak approximation method convergence [4] the vector function $\bar{u}(t,x)$ is a solution (of $C^{1,p}(\Pi^{M_0}_{[0,t^*]})$ class) to (1), (2) in $|x_i| \leqslant M_0$, and

$$\|D^\alpha \bar{u}^\tau - D^\alpha \bar{u}\|_{C(\Pi^{M_0}_{[0,t^*]})} \to 0, \quad |\alpha| \leqslant p$$

for $\tau \to 0$.

Since M_0 is arbitrary constant, the vector function $\bar{u}(t,x)$ is a solution to (1), (2) in whole $\Pi_{[0,t^*]}$ domain. Theorem 1.1 proved.

5. Derivation of inequality (15)

We are given with

$$D^\alpha u_i^\tau(t,x) = D^\alpha u_i^\tau(\frac{2\tau}{3},x) + \int_{2\tau/3}^{t} 3D^\alpha f_i\left(\xi - \frac{\tau}{3}, x, \bar{u}^\tau(\xi - \frac{\tau}{3}, x), \bar{\omega}(\xi - \frac{\tau}{3})\right) d\xi,$$
$$|\alpha| \leqslant p+2, \quad t \in ({}^{2\tau}/_3, \tau],$$

Taking absolute value of both sides of the previous equality and using (4) we have

$$|D^\alpha u_i^\tau(t,x)| \leqslant \left|D^\alpha u_i^\tau(\frac{2\tau}{3},x)\right| + \int_{2\tau/3}^{t} 3K_2\left(1 + U^\tau(\xi - {}^\tau/_3) + U^\tau(\xi - {}^\tau/_3)^2\right) d\xi.$$

Since ${}^{2\tau}/_3 \leqslant \xi \leqslant t \leqslant \tau$ and $U(t)$ is nondecreasing function, it is true that $U(\xi - {}^\tau/_3) \leqslant U({}^{2\tau}/_3)$:

$$|D^\alpha u_i^\tau(t,x)| \leqslant \left|D^\alpha u_i^\tau({}^{2\tau}/_3,x)\right| + \int_{2\tau/3}^{t} 3K_2\left(1 + U^\tau({}^{2\tau}/_3) + U^\tau({}^{2\tau}/_3)^2\right) d\xi.$$

Integrand in the previous inequality does not depend on the integration variable. $\int_{2\tau/3}^{t} d\xi \leqslant {}^\tau/_3$. As $U^\tau(t) \geqslant 1$, it is obvious that $U^\tau({}^{2\tau}/_3)^2 \geqslant U^\tau({}^{2\tau}/_3) \geqslant 1$. Thus

$$|D^\alpha u_i^\tau(t,x)| \leqslant \left|D^\alpha u_i^\tau({}^{2\tau}/_3,x)\right| + 3\tau K_2 U^\tau({}^{2\tau}/_3)^2.$$

We apply $\sup_{x\in\mathbb{R}^n}$ first, then $\sup_{[0,t]}$ to both parts of the previous inequality:

$$U_\alpha^{i\tau}(t) \leqslant U_\alpha^{i\tau}({}^{2\tau}/_3) + 3\tau K_2 U^\tau({}^{2\tau}/_3)^2.$$

Taking $\max_\alpha$ for $|\alpha| \leqslant p+2$, and calculating sum for $i = 1,\dots,n$, we obtain

$$U^\tau(t) \leqslant U^\tau({}^{2\tau}/_3) + 3n\tau K_2 U^\tau({}^{2\tau}/_3)^2.$$

Let K_4 be equal $3nK_2$. We factor out $U^\tau({}^{2\tau}/_3)$:

$$U^\tau(t) \leqslant U^\tau({}^{2\tau}/_3) \cdot \left(1 + \tau K_4 U^\tau({}^{2\tau}/_3)\right).$$

Using $1 + x \leqslant e^x$ we finally get

$$U^\tau(t) \leqslant U^\tau({}^{2\tau}/_3) \cdot e^{\tau K_4 U^\tau({}^{2\tau}/_3)}.$$

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О разрешимости задачи Коши для системы нагруженных уравнений

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В работе рассмотрена задача Коши для системы нагруженных уравнений типа Бюргерса. Приведен пример обратной задачи математической физики, сводящейся к рассматриваемой задаче. Получены достаточные условия существования решения задачи в классе гладких ограниченных функций.

Ключевые слова: задача Коши, обратные задачи, уравнение Бюргерса, система нелинейных уравнений, метод слабой аппроксимации.