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A Representation of Solution of the Identification Problem of the Coefficients at Second Order Operator in the Multi-Dimensional Parabolic Equations System

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An identification problem of the coefficients at differential operator of second order and sum of lowest terms in system of multidimensional parabolic equations with Cauchy data was studied in this article. The theorems of existence and uniqueness of the solution for direct and inverse problems have been proved. The method of weak approximation is used to the proof existence of solutions.

Keywords: inverse problem, identification problem, method of weak approximation, system of equations in partial derivatives, existence and uniqueness of the solution.

Introduction

In the investigation of coefficient inverse problems for partial differential equations, using some additional information on the solution, the original problem is reduced to a certain auxiliary direct problem. As a rule integrodifferential or non-classical "loaded" equation is obtained [3,5]. The following investigation method was proposed in [1]: initial inverse problem is reduced to two auxiliary direct problems, one of which contains an expression for the unknown coefficient. This approach was used to reduce inverse problem to auxiliary direct problems in [6], in which were proved theorems of existence and uniqueness of assigned problems. At international conference "Inverse and Ill-Posed Problems of Mathematical Physics" was submitted the result of two twodimensional parabolic equation system of a similar type. Theses are publicized in [4].

The case of system of m multidimensional parabolic equations ($m \ge 2$ — any finite number) with Cauchy data was obtained in this article. To prove existence of solutions of given problems method of weak approximation [2,7] was used which firstly proposed in the works of N. N. Yanenko and A. A. Samarskiy.

1. Formulation and reduction of the problem to the direct problems

Consider in the domain $\Gamma_{[0,T]} = \{(t, x, z) \mid x \in \mathbb{R}^n, z \in \mathbb{R}, 0 \leq t \leq T\}$ the Cauchy problem for a system of parabolic equations $(i = \overline{1, m})$

$$u_t^i = a^i(t)u_{zz}^i(t,x,z) + b(t)\Delta_x u^i(t,x,z) + \lambda^i(t,z) \left(B_z^i(u^i) + \sum_{k=1}^m g_i^k(t)u^k \right),$$
(1)

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where $B_{z}^{i}(u^{i}) = c_{1}^{i}(t)u_{zz}^{i}(t,x,z) + c_{2}^{i}(t)u_{z}^{i}(t,x,z) + c_{3}^{i}(t)u^{i}(t,x,z)$, with initial data

$$u^{i}(0,x,z) = u^{i}_{0}(x,z).$$
⁽²⁾

Let the continuous functions $a^i(t), b(t), c^i_l(t), g^k_i(t), (l = 1, 2, 3, i, k = \overline{1, m})$ be bounded on [0, T] and $a^i(t) \ge a_0 > 0$, $b(t) \ge b_0 > 0$, $c^i_1(t) \ge c_0 > 0$. Let the functions $u^i_0(x, z)$ be defined as the real-valued and be defined on \mathbb{R}^{n+1} . The functions $\lambda^i(t, z)$ are to be determined simultaneously with the solution $u^i(t, x, z)$ of problem (1), (2).

Suppose the overdetermination conditions are given

$$u^{i}(t,0,z) = \psi^{i}(t,z),$$
(3)

and consistency conditions are $u_0^i(0, z) = \psi^i(0, z)$.

Assume that the following conditions are fulfilled

$$\left| B_z^i(\psi^i) + \sum_{k=1}^m g_i^k(t)\psi^k \right| \ge \mu^i > 0, \quad \mu^i \text{ are const.}$$

$$\tag{4}$$

Theorem 1.1. If there are solutions $\varphi(t, x)$ and $f^i(t, z)$ of the following Cauchy problems

$$\frac{\partial \varphi}{\partial t} = b(t)\Delta_x \varphi,
\varphi(0,x) = w_0(x),$$
(5)

$$\frac{\partial f^{i}}{\partial t} = a^{i}(t)f^{i}_{zz} + \frac{\psi^{i}_{t}(t,z) - a^{i}(t)\psi^{i}_{zz}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B^{i}_{z}(\psi^{i}) + \sum_{k=1}^{m} g^{k}_{i}(t)\psi^{k}} \left(B^{i}_{z}(f^{i}) + \sum_{k=1}^{m} g^{k}_{i}(t)f^{k}\right),$$

$$f^{i}(0,z) = v^{i}_{0}(z),$$
(6)

the functions $u^i(t, x, z)$ and $\lambda^i(t, z)$ defined by

$$u^{i}(t, x, z) = \varphi(t, x)f^{i}(t, z),$$

$$\lambda^{i}(t, z) = \frac{\psi_{t}^{i}(t, z) - a^{i}(t)\psi_{zz}^{i}(t, z) - f^{i}(t, z)\varphi_{t}(t, 0)}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}},$$

are the solution of the inverse problem (1)–(3), in the assumption that

$$u_0^i(x,z) = w_0(x)v_0^i(z).$$
(7)

Proof. We verify the theorem by direct substitution in the equation of (1), (2) expressions for the unknown functions.

We substitute in equations of systems (1) the expressions $u^i(t, x, z) = \varphi(t, x)f^i(t, z)$, $\lambda^i(t, z) = \frac{\psi_t^i(t, z) - a^i(t)\psi_{zz}^i(t, z) - f^i(t, z)\varphi_t(t, 0)}{B_z^i(\psi^i) + \sum_{k=1}^m g_i^k(t)\psi^k}$ and obtain

$$\begin{split} \frac{\partial \varphi}{\partial t} f^i &+ \frac{\partial f^i}{\partial t} \varphi = a^i(t) \varphi f^i_{zz} + b(t) f^i \Delta_x \varphi + \\ &+ \frac{\psi^i_t(t,z) - a^i(t) \psi^i_{zz}(t,z) - f^i(t,z) \varphi_t(t,0)}{B^i_z(\psi^i) + \sum\limits_{k=1}^m g^k_i(t) \psi^k} \left(B^i_z(\varphi f^i) + \sum\limits_{k=1}^m g^k_i(t) \varphi f^k \right). \end{split}$$

The following is true by reason of operator's linearity B_z^i

$$\begin{split} \frac{\partial \varphi}{\partial t} f^i &+ \frac{\partial f^i}{\partial t} \varphi = a^i(t) \varphi f^i_{zz} + b(t) f^i \Delta_x \varphi + \\ &+ \frac{\psi^i_t(t,z) - a^i(t) \psi^i_{zz}(t,z) - f^i(t,z) \varphi_t(t,0)}{B^i_z(\psi^i) + \sum\limits_{k=1}^m g^k_i(t) \psi^k} \left(B^i_z(f^i) + \sum\limits_{k=1}^m g^k_i(t) f^k \right) \varphi. \end{split}$$

We group relative to f^i and φ ,

$$\begin{split} \left(\frac{\partial\varphi}{\partial t} - b(t)\Delta_x\varphi\right)f^i &= \left(\frac{\partial f^i}{\partial t} - a^i(t)f^i_{zz} - \left(B^i_z(f^i) + \sum_{k=1}^m g^k_i(t)f^k\right) \times \right. \\ & \times \frac{\psi^i_t(t,z) - a^i(t)\psi^i_{zz}(t,z) - f^i(t,z)\varphi_t(t,0)}{B^i_z(\psi^i) + \sum_{k=1}^m g^k_i(t)\psi^k}\right)\varphi. \end{split}$$

If $\varphi(t, x)$ is solution of problem (5) and $f^i(t, z)$ is solution of system (6), this be identical $\forall i = \overline{1, m}.$

If conditions (7) are valid, the functions $u^i(t, x, z) = \varphi(t, x) f^i(t, z)$ satisfy to initial data (2)

$$u^{i}(0, x, z) = \varphi(0, x) f^{i}(0, z) = w_{0}(x) v_{0}^{i}(z) = u_{0}^{i}(x, z)$$

We test execution overdetermination conditions (3). Let $A^{i}(t,z) = u^{i}(t,0,z) - \psi^{i}(t,z)$. We can proof $A^{i}(t,z) = 0$. Consider the system of equations (1) in $x = (0,0,\ldots,0)$. Here and further we understand that x = 0 such as *n*-dimensional vector x = (0, 0, ..., 0).

$$\begin{split} u_t^i(t,0,z) &= a^i(t)u_{zz}^i(t,0,z) + b(t)\Delta_x u^i(t,0,z) + \lambda^i(t,z) \left(B_z^i(A^i(t,z)) + \sum_{k=1}^m g_i^k(t)A^k(t,z) \right) + \\ &+ \frac{\psi_t^i(t,z) - a^i(t)\psi_{zz}^i(t,z) - f^i(t,z)\varphi_t(t,0)}{B_z^i(\psi) + \sum_{k=1}^m g_i^k(t)\psi^k} \left(B_z^i(\psi^i) + \sum_{k=1}^m g_i^k(t)\psi^k \right). \end{split}$$

$$\begin{aligned} u_t^i(t,0,z) &= a^i(t)(u_{zz}^i(t,0,z) - \psi_{zz}^i(t,z)) + b(t)\Delta_x u^i(t,0,z) + \\ &+ \psi_t^i(t,z) - f^i(t,z)\Delta_x \varphi(t,0)b(t) + \lambda^i(t,z) \left(B_z^i(A^i(t,z)) + \sum_{k=1}^m g_i^k(t)A^k(t,z) \right). \end{aligned}$$

We derive Cauchy problem for a system of parabolic equations with homogeneous initial data

$$\begin{aligned} A_t^i(t,z) &= a^i(t) A_{zz}^i(t,z) + \lambda^i(t,z) \left(B_z^i(A^i(t,z)) + \sum_{k=1}^m g_i^k(t) A^k(t,z) \right), \\ A^i(0,z) &= 0. \end{aligned}$$

The unique solution of this problem is $A^i(t,z) = 0$. Therefore $u^i(t,0,z) = \psi^i(t,z)$, because $A^{i}(t,z) = u^{i}(t,0,z) - \psi^{i}(t,z)$. Thus overdetermination conditions are attested.

The theorem is proved.

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2. Proof the solvability of the direct problem (6)

To prove existence of the solution of direct problem (6) we consider in domain $G_{[0,T]} = \{(t,z) \mid z \in \mathbb{R}, 0 \leq t \leq T\}$ auxiliary problem

$$\frac{\partial f^{i}}{\partial t} = a^{i}(t)f^{i}_{zz} + S_{\delta^{i}}\left(\frac{\beta^{i}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B^{i}_{z}(\psi^{i}) + \sum_{k=1}^{m}g^{k}_{i}(t)\psi^{k}}\right) \left(B^{i}_{z}(f^{i}) + \sum_{k=1}^{m}g^{k}_{i}(t)f^{k}\right),$$

$$f^{i}(0,z) = v_{0}(z),$$
(8)

here $\beta^i(t,z) = \psi^i_t(t,z) - a^i(t)\psi^i_{zz}(t,z)$ are known functions, and the patch function $S_{\delta^i}(\vartheta)$, $\forall i = \overline{1,m}$ is defined in \mathbb{R} . Patch function is an arbitrary continuously differentiable function with properties:

$$S_{\delta^{i}}(\vartheta) \ge \frac{\delta^{i}}{3} > 0, \quad \vartheta \in \mathbb{R} \text{ and } S_{\delta^{i}}(\vartheta) = \begin{cases} \vartheta, & \vartheta \ge \frac{\delta^{i}}{2}, \\ \frac{\delta^{i}}{3}, & \vartheta \le \frac{\delta^{i}}{3}, \end{cases}, \quad \frac{d^{l}}{dz^{l}} \left(S_{\delta^{i}}(\vartheta)\right) \le 2, \ (l = 1, \dots, 4).$$

The functions $f^i(t, z)$ are determined. The functions $v_0^i(z)$ are defined in the real-valued and be defined on \mathbb{R} .

To prove the existence of a solution supporting the direct problem (8) we use the weak approximation method. We fix constant $\tau > 0$, $\tau J = T$. The problem is broken up into three fractional steps and is linearized displacement on variable t upon $\frac{\tau}{2}$.

$$f_t^{i\tau} = 3 a^i(t) f_{zz}^{i\tau}(t, z), \qquad j\tau < t \le \left(j + \frac{1}{3}\right)\tau, \tag{9}$$

$$f_t^{i\tau} = 3 \left(c_1^i(t) f_{zz}^{i\tau}(t,z) + c_2^i(t) f_z^{i\tau}(t,z) \right) S_{\delta^i}(\lambda^{i\tau}(t,z)), \quad \left(j + \frac{1}{3} \right) \tau < t \le \left(j + \frac{2}{3} \right) \tau, \tag{10}$$

$$f_t^{i\tau} = 3 \left(\left(c_3^i(t) + g^i(t) \right) f^{i\tau}(t, z) + \sum_{k=1, k \neq i}^m g_i^k(t) f^{k\tau} \left(t - \frac{\tau}{3}, z \right) \right) S_{\delta^i}(\lambda^{i\tau}(t, z)), \qquad (11)$$
$$\left(j + \frac{2}{3} \right) \tau < t \le \left(j + 1 \right) \tau,$$

$$f^{i\tau}(0,z) = v_0(z), \qquad j = 0, 1, 2, \dots, (J-1), \ J\tau = T,$$
(12)

here $\lambda^{i\tau}(t,z) = \frac{\beta^i(t,z) - f^{i\tau}(t-\frac{\tau}{3},z) \,\varphi_t(t,0)}{B_z^i(\psi^i) + \sum_{k=1}^m g_i^k(t) \psi^k}.$

Concerning the input data $v_0^i(z)$, $\psi^i(t, z)$ we suggest that they are sufficiently smooth, have all continuous derivatives occurring in the next lower relations and satisfy them for all $i = \overline{1, m}$

$$\left|\frac{d^{l_1}}{dz^{l_1}}v_0^i(z)\right| + \left|\frac{\partial}{\partial t}\frac{\partial^{l_1}}{\partial z^{l_1}}\psi^i(t,z)\right| + \left|\frac{\partial^{l_2}}{\partial z^{l_2}}\psi^i(t,z)\right| \leqslant C,$$

$$l_1 = 0, 1, \dots, 4, \ l_2 = 0, 1, \dots, 6, \ \forall i = \overline{1, m}.$$
(13)

It was proved that fixed constant $\tilde{t^*}: 0 < \tilde{t^*} \leq T$ exists, which depends on constant limiting input data (13), constant a_0, b_0, c_0 and μ_i from (4) such as in domain $G^M_{[0,\tilde{t^*}]} = \{(t,z)| 0 < t \leq \tilde{t^*}, |z| < M\}$, uniformly on τ estimates are hold.

$$\sum_{i=1}^{m} \left| \frac{\partial^{l}}{\partial z^{l}} f^{i\tau}(t,z) \right| \leqslant C, \quad l = 0, 1, \dots, 4, \quad (t,z) \in G_{[0,\tilde{t}^{*}]}.$$

$$\sum_{i=1}^m |f_t^{i\tau}(t,z)| \leqslant C, \quad (t,z) \in G_{[0,\tilde{t^*}]}.$$

The equations of problem (9)–(12) are differentiated on variable z ones or twice. Uniformly on τ estimates are hold

$$\sum_{i=1}^{m} |f_{tz}^{i\tau}(t,z)| + \sum_{i=1}^{m} |f_{tzz}^{i\tau}(t,z)| \leqslant C, \quad (t,z) \in G_{[0,t_5^*]}, \quad i = \overline{1,m}$$

with estimates

$$\sum_{i=1}^m \left| \frac{\partial^l}{\partial z^l} f^{i\tau}(t,z) \right| \leqslant C, \quad l=3,4, \quad (t,z) \in G_{[0,\tilde{t^*}]},$$

fulfillment of conditions of Arzela's theorem about compactness is guaranteed.

By Arzela's theorem some subsequence $f^{i\tau_k}(t,z)$ of sequence $f^{i\tau}(t,z)$ $(\forall i = \overline{1,m})$ of problem solutions (9)–(12) converges with derivatives on z till second order to functions $f^i(t,z) \in C^{0,2}_{t,z}(G_{[0,\tilde{t}^*]})$. By the convergence theorem of the weak approximation method $f^i(t,z)$ are solution of problem (8) and $f^i(t,z) \in C^{1,2}_{t,z}(G_{[0,\tilde{t}^*]})$, here

$$C_{t,z}^{1,2}(G_{[0,\tilde{t}^*]}) = \left\{ f^i(t,z) \middle| f^i_{t}(t,z), \frac{\partial^l}{\partial z^l} f^i(t,z) \in C(G_{[0,\tilde{t}^*]}), l = 0, 1, 2, i = \overline{1,m} \right\},$$

and

$$\sum_{i=1}^{m} \left| \frac{\partial^{l}}{\partial z^{l}} f^{i}(t, z) \right| \leqslant C, \quad l = 0, 1, 2.$$
(14)

Let the following conditions satisfy with $t \in [0, \tilde{t}^*]$

$$\frac{\beta^{i}(t,z) - v^{i}_{0}(z)\varphi_{t}(t,0)}{B^{i}_{z}(\psi^{i}) + \sum_{k=1}^{m} g^{k}_{i}(t)\psi^{k}} \ge \delta^{i}, \quad \forall i = \overline{1,m}.$$
(15)

To show that the solution of problem (8) equals to solution of direct problem (6), we can prove fulfilment with $t \in [0, \tilde{t^*}]$

$$\frac{\beta^i(t,z) - f^i(t,z)\varphi_t(t,0)}{B^i_z(\psi^i) + \sum\limits_{k=1}^m g^k_i(t)\psi^k} \geqslant \frac{\delta^i}{2}, \quad \forall \, i = \overline{1,m}.$$

The system of problem (8) is integrated on temporary variable in the range from 0 to t:

$$f^{i}(t,z) = v_{0}^{i}(z) + \int_{0}^{t} \Psi^{i}(\eta,z)d\eta, \quad i = \overline{1,m},$$

here

$$\Psi^{i}(t,z) = a^{i}(t)f^{i}_{zz} + S_{\delta^{i}}\left(\frac{\beta^{i}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B^{i}_{z}(\psi^{i}) + \sum_{k=1}^{m}g^{k}_{i}(t)\psi^{k}}\right) \left(B^{i}_{z}(f^{i}) + \sum_{k=1}^{m}g^{k}_{i}(t)f^{k}\right).$$

As conditions (4) are complied, therefore equalities are true

$$\frac{\beta^{i}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}} = \frac{\beta^{i}(t,z) - \psi^{i}_{0}(z)\varphi_{t}(t,0))}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}} - \frac{\varphi_{t}(t,0)\int_{0}^{t}\Psi^{i}(\eta,z)d\eta}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}}, \quad \forall i = \overline{1,m}.$$

The conditions (13), (15) are hold and (14) is valid, therefore equalities will be true with $t \in \left[0, \frac{\delta^i}{2A^i(\delta^i)}\right]$

$$\frac{\beta^{i}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}} \ge \delta^{i} - A^{i}(\delta^{i})t \ge \frac{\delta^{i}}{2}, \quad \forall i = \overline{1,m}.$$

Here $A^i(\delta^i)$ are some positive constants, which estimate the input data and depend on δ^i , constant C from (13), and are also constant limiting coefficients $a^i(t)$.

By definition of patch function $S_{\delta^i}(\theta)$ we have

$$S_{\delta^i}(\lambda^i(t,z)) = \lambda^i(t,z), \text{ with } t \in [0,t^*], \text{ here } t^* = \min\left(\widetilde{t^*}, \frac{\delta^i}{2A^i(\delta^i)}\right), \forall i = \overline{1,m}.$$

Thus we prove the existence solution $f^i(t, z)$ of problem (6) in class $C_{t,z}^{1,2}(G_{[0,t^*]})$.

The unique solution of the problem is obtained by the instrumentality of proof that the difference of two putative solutions comes to nought. The proof isn't adduce in this paper.

Theorem 2.1 (Unique existence). Let the conditions (4), (13), (14) are hold. Fixed constant $\tilde{t^*}: 0 < \tilde{t^*} \leq T$ exists, which depends on constants limiting input data (13), constant a_0 , b_0 , c_0 and μ_i from (4). Then in the class

$$C^{1,2}_{t,z}(G_{[0,t^*]}) = \left\{ f^i(t,z) \middle| f^i_{\ t}(t,z), \frac{\partial^l}{\partial z^l} f^i(t,z) \in C(G_{[0,t^*]}), \ l = 0, 1, 2, \ i = \overline{1,m} \right\},$$

there exists a unique solution $f^{i}(t,z)$ ($\forall i = \overline{1,m}$) of problem (6), which satisfies the following relation

$$\sum_{l=0}^{2} \sum_{k=1}^{m} \left| \frac{\partial^{l}}{\partial z^{l}} f^{k}(t, z) \right| \leqslant C,$$
(16)

3. Proof existence of the solution of the inverse problem

We consider the problem (5) in domain $\Pi_{[0,T]} = \{(t,x)|0 \leq t \leq T, x \in \mathbb{R}^n\}$. The unique existence solution conditions are formulated in following theorem.

Theorem 3.1 (Unique existence). Let $\omega_0 \in C(\mathbb{R}^n)$ be bound. Then in the class

$$C_{t,x}^{1,2}(\Pi_{[0,T]}) = \{\varphi(t,x) | \varphi_t(t,x), D_x^{\alpha} \varphi(t,x) \in C(\Pi_{[0,T]}), \ |\alpha| \leqslant 2\},$$

there exists a unique solution $\varphi(t, x)$ of problem (6) and the following relation is valid.

$$\sum_{|\alpha| \leqslant 2} |D_x^{\alpha} \varphi_t(t, x)| \leqslant C.$$

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The solution of inverse problem (1)–(3) are considered in domain $\Gamma_{[0,T]} = \Pi_{[0,T]} \cup G_{[0,T]}$. In view of the fact that the functions $u^i(t, x, z)$, $\lambda^i(t, z)$ are expressed in terms of known functions notably $u^i(t, x, z) = (c(t, x))f^i(t, z)$

$$\begin{aligned} u(t,x,z) &= \varphi(t,x) f(t,z), \\ \lambda^{i}(t,z) &= \frac{\psi^{i}_{t}(t,z) - a^{i}(t)\psi^{i}_{zz}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B^{i}_{z}(\psi^{i}) + \sum_{k=1}^{m}g^{k}_{i}(t)\psi^{k}}, \end{aligned}$$

where $\varphi(t,x) \in C_{t,x}^{1,2}(\Pi_{[0,T]}), f^{i}(t,z) \in C_{t,z}^{1,2}(G_{[0,t^{*}]})$ are solutions of problems (5), (6), the estimate is valid

$$\sum_{i=1}^{m} \sum_{l=0}^{2} \sum_{|\alpha| \leq 2} \left| \frac{\partial^{l}}{\partial z^{l}} D_{x}^{\alpha} u^{i}(t,x,z) \right| + \sum_{i=1}^{m} \sum_{l=0}^{2} \left| \frac{\partial^{l}}{\partial z^{l}} \lambda^{i}(t,z) \right| \leq C.$$

$$(17)$$

On the account of theorems 1.1–3.1, the following theorem is true

Theorem 3.2 (Existence). Let the conditions of theorems 1.1–3.1 are valid. Then in the class

$$Z(t^*) = \left\{ u^i(t,x,z), \lambda^i(t,z) \mid u^i(t,x,z) \in C^{1,2,2}_{t,x,z}(\Gamma_{[0,t^*]}), \, \lambda^i(t,z) \in C^{1,2}_{t,z}(G_{[0,t^*]}), i = \overline{1,m} \right\},$$

there exists a solution $u^i(t, x, z)$, $\lambda^i(t, z)$ of inverse problem (1)–(3) and the relation (17) are defined.

4. Proof uniqueness of the solution of the inverse problem

Let us the conditions of theorem 3.2 are true. We use the proof by contradiction.

Let $u_1^i(t, x, z), \lambda_1^i(t, z) \bowtie u_2^i(t, x, z), \lambda_2^i(t, z), (i = \overline{1, m})$ are two classical solution of inverse problem (1), (2). Here the functions $u_1^i(t, x, z), \lambda_1^i(t, z)$ are the solution, which defined theorem 1.1 and satisfied the condition (3), and the functions $u_2^i(t, x, z), \lambda_2^i(t, z)$ are an another solution of problem (1), (2), which satisfied the condition (17).

Then the relations are valid

$$\begin{split} u_{1t}^{i} &= a^{i}(t)u_{1zz}^{i}(t,x,z) + b(t)\Delta_{x}u_{1}^{i}(t,x,z) + \lambda_{1}^{i}(t,z) \left(B_{z}^{i}(u_{1}^{i}) + \sum_{k=1}^{m}g_{k}^{k}(t)u_{1}^{k}\right), \\ u_{2t}^{i} &= a^{i}(t)u_{2zz}^{i}(t,x,z) + b(t)\Delta_{x}u_{2}^{i}(t,x,z) + \lambda_{2}^{i}(t,z) \left(B_{z}^{i}(u_{2}^{i}) + \sum_{k=1}^{m}g_{k}^{k}(t)u_{2}^{k}\right), \\ u_{1}^{i}(0,x,z) &= u_{0}^{i}(x,z), \quad u_{2}^{i}(0,x,z) = u_{0}^{i}(x,z), \\ u_{1}^{i}(t,0,z) &= \psi^{i}(t,z), \quad u_{2}^{i}(t,0,z) = \psi^{i}(t,z). \end{split}$$

The differences $u_1^i(t, x, z) - u_2^i(t, x, z) = u^i(t, x, z)$, $\lambda_1^i(t, z) - \lambda_2^i(t, z) = \lambda^i(t, z)$ are the solution of the problem

$$u_{t}^{i} = a^{i}(t)u_{zz}^{i}(t,x,z) + b(t)\Delta_{x}u^{i}(t,x,z) + \lambda_{1}^{i}(t,z)\left(B_{z}^{i}(u^{i}) + \sum_{k=1}^{m}g_{i}^{k}(t)u^{k}\right) + \lambda^{i}(t,z)\left(B_{z}^{i}(u_{2}^{i}) + \sum_{k=1}^{m}g_{i}^{k}(t)u_{2}^{k}\right), \quad (18)$$

$$u^{i}(0, x, z) = 0, \quad u^{i}(t, 0, z) = 0, \quad i = \overline{1, m}.$$
 (19)

Assuming that x = 0 in system (18), we express coefficients $\lambda^{i}(t, z)$ through (19) and substitution in (18).

$$u_{t}^{i} = a^{i}(t)u_{zz}^{i}(t, x, z) + b(t)\Delta_{x}u^{i}(t, x, z) + \lambda_{1}^{i}(t, z)\left(B_{z}^{i}(u^{i}) + \sum_{k=1}^{m}g_{i}^{k}(t)u^{k}\right) + \frac{b(t)\Delta_{x}u^{i}(t, 0, z)}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m}g_{i}^{k}(t)\psi^{k}}\left(B_{z}^{i}(u_{2}^{i}) + \sum_{k=1}^{m}g_{i}^{k}(t)u_{2}^{k}\right), \quad (20)$$
$$u^{i}(0, x, z) = 0, \quad i = \overline{1, m}.$$

We consider nonnegative never-decreasing function on segment $[0, t^*]$

$$g_j(t) = \sup_{\Gamma[0,t]} \left| \sum_{i=1}^m D_x^{\alpha} u^i(\xi, x, z) \right|, \quad |\alpha| < 2.$$

We will consider first equation of system (20) as parabolic equation relative to function u^1 , second one is relative to function u^2 etc, *m*-th is relative to function u^m with initial data (21).

The principle of the maximum was employed to each equation, then received estimates were added

$$\left|\sum_{i=1}^{m} u^{i}(\xi, x, z)\right| \leqslant e^{C\xi} C(g_{2}(t) + g_{0}(t))\xi, \quad (\xi, x, z) \in G_{[0,t]}, \quad 0 \leqslant t \leqslant t^{*},$$

whence we obtain estimate as nonnegative functions $g_k(t)$

$$g_0(t) \leq Ct(g_2(t) + g_0(t)) \leq C(g_2(t) + g_1(t) + g_0(t))t, \quad 0 \leq t \leq t^*.$$

We differentiate system (20), (21) on variable x ones or twice

$$\begin{split} D_x^{\alpha} u_t^i &= a^i(t) D_x^{\alpha} u^i(t,x,z)_{zz} + b(t) D_x^{\alpha} (\Delta_x u^i(t,x,z)) + \lambda_1^i(t,z) B_z^i \left(D_x^{\alpha} u^i \right) + \\ &+ \sum_{k=1}^m \left(g_i^k(t) D_x^{\alpha} u^k \right) + \frac{b(t) \Delta_x u^i(t,0,z)}{B_z^i(\psi^i) + \sum_{k=1}^m g_i^k(t) \psi^k} B_z^i \left(D_x^{\alpha} u_2^i \right) + \sum_{k=1}^m \left(g_i^k(t) D_x^{\alpha} u_2^k \right), \end{split}$$

$$D^{\alpha}_{x}u^{i}(0,x,z)=0, \quad \alpha=1,2, \quad i=\overline{1,m},$$

and we receive similar estimates

$$g_q(t) \leq Ct(g_2(t) + g_0(t)) \leq C(g_2(t) + g_1(t) + g_0(t))t, \quad q = 1, 2 \quad 0 \leq t \leq t^*.$$

All of them are added

$$g_0(t,z) + g_1(t,z) + g_2(t,z) \leqslant C(g_2(t,z) + g_1(t,z) + g_0(t,z))t, \quad 0 \leqslant t \leqslant t^*.$$

Hereof equality $g_0(t,z) + g_1(t,z) + g_2(t,z) = 0$ is true with $t \in [0,\zeta]$, where $\zeta < \frac{1}{C}$, therefore,

$$u^i(t, x, z) = 0, \quad (t, x, z) \in \Gamma_{[0,\zeta]}, \quad i = \overline{1, m}.$$

Arguments having replicated for $t \in [0, 2\zeta]$, we receive

$$u^{i}(t, x, z) = 0, \quad (t, x, z) \in \Gamma_{[0, 2\zeta]}, \quad i = \overline{1, m}.$$

In finite number of steps we obtain estimate

$$u^{i}(t,x,z) \equiv 0, \quad (t,x,z) \in \Gamma_{[0,t^{*}]}, \quad i = \overline{1,m}.$$

In consideration of $u_1^i(t, x, z) \equiv u_2^i(t, x, z), (t, x, z) \in \Gamma_{[0,t^*]}, (i = \overline{1, m})$, from (18), we receive that for $\lambda^i(t, z) = \lambda_1^i(t, z) - \lambda_2^i(t, z), (i = \overline{1, m})$ correlations exist

$$\lambda^{i}(t,z)\left(B_{z}^{i}(\psi^{i})+\sum_{k=1}^{m}g_{i}^{k}(t)\psi^{k}\right)=0,\quad i=\overline{1,m}.$$

Whence taking into account (4), it follows that

$$\lambda^i(t,z) = \lambda^i_1(t,z) - \lambda^i_2(t,z) = 0, \quad t \in [0,t^*], \quad i = \overline{1,m}.$$

Theorem 4.1 (Uniqueness). Let the conditions of theorem 3.2 are valid. Then in the class $Z(t^*)$ there exists the unique solution $u^i(t, x, z)$, $\lambda^i(t, z)$ of inverse problem (1)–(3) and the relation (17) is true.

5. The example of initial data, for which the theorems conditions are valid

We examine the following Cauchy problem for system of parabolic equations in the capacity of example.

Consider in domain $\Gamma_{[0,0.5]} = \{(t,x,z) \mid x \in \mathbb{R}, z \in \mathbb{R}, 0 \leq t \leq 0.5\}$ the set of equations

$$u_t^i = a^i(t)u_{zz}^i(t, x, z) + b(t)u_{xx}^i(t, x, z) + \lambda^i(t, z) \left(B_z^i(u^i) + \sum_{k=1}^m g_i^k(t)u^k \right),$$
(22)

where $B_z(u) = u_{zz}(t, x, z) + u_z(t, x, z) + u(t, x, z)$, with initial data

$$u^{i}(0,x,z) = u_{0}^{i}(x,z) = (\sin(z) + (i+2))(\sin(x) + 1), \quad i = \overline{1,m}.$$
(23)

The continuous functions $a^i(t) = b^i(t) = c_1^i(t) = c_2^i(t) = c_3^i(t) = g_i^k(t) = 1$, $k = \overline{1, m}$, $\forall i = \overline{1, m}$, are bounded on [0, T]. The functions $u_0^i(x, z)$ are defined as the real-valued and be defined on \mathbb{R}^2 . The functions $\lambda^i(t, z)$ are to be determined simultaneously with the solution $u^i(t, x, z)$ of problem (22), (23).

The overdetermination conditions are given

$$u^{i}(t,0,z) = \psi^{i}(t,z) = (t+1)(\sin(z) + (i+2)), \quad i = \overline{1,m},$$

and consistency conditions are valid

$$u_0^i(0,z) = \psi^i(0,z) = \sin(z) + (i+2).$$

The fulfillment of the following conditions is required

$$|B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}| = \left|c_{1}^{i}(t)\psi_{zz}^{i}(t,z) + c_{2}^{i}(t)\psi_{z}^{i}(t,z) + c_{3}^{i}(t)\psi^{i}(t,z) + \sum_{k=1}^{m} g_{i}^{k}(t)\psi^{k}\right| \ge \mu > 0,$$

$$\mu - \text{const.}$$

We can easily verify that these conditions are true

$$\left| B_z^i(\psi^i) + \sum_{k=1}^m g_i^k(t)\psi^k \right| = (t+1) \left| m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^m (k+2) \right| \ge \mu^i > 0,$$

here the choice μ^i depends on the amount of equation m and the number i. These inequations are fulfilled, for instance, with $\mu^i = 1$.

Consider the Cauchy problem

$$\frac{\partial \varphi}{\partial t} = \varphi_{xx}, \quad \varphi(0, x) = w_0(x) = \sin(x) + 1.$$

The solution if this problem is the function $\varphi(t, x) = e^{-t} \sin(x) + 1$. This is easily seen by substituting the function $\varphi(t, x)$ in equation

$$-e^{(-t)}\sin(x) = -e^{(-t)}\sin(x), \quad \varphi(0,x) = w_0(x) = \sin(x) + 1.$$

The function $w_0(x) = (\sin(x) + 1) \in C(\mathbb{R}^n)$ is bounded. For following problem

$$\frac{\partial f^i}{\partial t} = f_{zz} + \left(B_z^i(f^i) + \sum_{k=1}^m g_i^k(t) f^k \right) \frac{(i+2) + t\sin(z) + 2\sin(z)}{(t+1)(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^m (k+2))}, \quad (24)$$
$$f^i(0, z) = v_0^i(z) = \sin(z) + (i+2),$$

the following functions $f^i(t,z) = (t+1)(\sin(z) + (i+2))$ are solutions of problem (24) , with

$$B_z^i(f^i) + \sum_{k=1}^m g_i^k(t) f^k = (t+1) \left(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^m (k+2) \right),$$

$$\begin{split} \lambda^{i}(t,z) &= \frac{\psi^{i}_{t}(t,z) - a^{i}(t)\psi^{i}_{zz}(t,z) - f^{i}(t,z)\varphi_{t}(t,0)}{B^{i}_{z}(\psi^{i}) + \sum_{k=1}^{m}g^{k}_{i}(t)\psi^{k}} \\ &= \frac{(i+2) + t\sin(z) + 2\sin(z)}{(t+1)(m\cdot\sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m}(k+2))}. \end{split}$$

The given solution is inserted in the system of problem (24)

$$\begin{aligned} \sin(z) + (i+2) &= -(t+1)\sin(z) + \\ &+ (t+1)\left(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m} (k+2)\right) \times \\ &\times \frac{(i+2) + t\sin(z) + 2sin(z)}{(t+1)(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m} (k+2))}, \\ &\sin(z) + (i+2) = -(t+1)\sin(z) + (i+2) + t\sin(z) + 2sin(z), \end{aligned}$$

so we obtain correct identity.

The accomplishment of the following conditions is required for existence of the solution of problem (24)

$$\left| \frac{d^{l_1}}{dz^{l_1}} v_0^i(z) \right| + \left| \frac{\partial}{\partial t} \frac{\partial^{l_1}}{\partial z^{l_1}} \psi^i(t,z) \right| + \left| \frac{\partial^{l_2}}{\partial z^{l_2}} \psi^i(t,z) \right| \leqslant C,$$

$$l_1 = 0, 1, \dots, 4, \ l_2 = 0, 1, \dots, 6, \ \forall i = \overline{1, m}.$$

The given condition is true on account of the limitations of all derivatives of functions $v_0^i(z) = \sin(z) + (i+2)$ and $\psi^i(t,z) = (t+1)(\sin(z) + (i+2))$.

By theorem 1.1 the functions $u^i(t, x, z)$ are represented in form

$$u^{i}(t,x,z) = \varphi(t,x)f^{i}(t,z) = (t+1)(e^{-t}\sin(x)+1)(\sin(z)+(i+2)).$$

Let's test whether the functions $u^i(t, x, z)$ satisfy to the system of equations. The functions

$$\begin{split} u^{i}{}_{t} &= (\sin(z) + (i+2)) \left(\sin(x)(e^{-t} - (t+1)e^{-t}) + 1 \right) = (\sin(z) + (i+2)) \left(\sin(x)e^{-t}(-t) + 1 \right), \\ u^{i}{}_{xx} &= -(\sin(z) + (i+2))(t+1) \sin(x)e^{-t}, \\ u^{i}{}_{zz} &= -(\sin(x)e^{-t} + 1)(t+1) \sin(z), \\ \lambda^{i}(t,z) &= \frac{(i+2) + t \sin(z) + 2sin(z)}{(t+1)(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m} (k+2))}, \\ B^{i}_{z}(u^{i}) &+ \sum_{k=1}^{m} g^{k}_{i}(t)u^{k} = (t+1)(\sin(x)e^{-t} + 1)(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m} (k+2)) \end{split}$$

are substituted in system (22)

$$\begin{aligned} ((\sin(z) + (i+2)) \left(\sin(x)e^{-t}(-t) + 1 \right) &= -(\sin(x)e^{-t} + 1)(t+1)\sin(z) - \\ &- (\sin(z) + (i+2))(t+1)\sin(x)e^{-t} + (t+1)(\sin(x)e^{-t} + 1) \times \\ &\times \left(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m} (k+2) \right) \times \\ &\times \frac{(i+2) + t\sin(z) + 2\sin(z)}{(t+1)(m \cdot \sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m} (k+2))}. \end{aligned}$$

The elementary transformations are reduced to

$$((\sin(z) + (i+2)) (\sin(x)e^{-t}(-t) + 1) = = -(\sin(x)e^{-t} + 1)(t+1)\sin(z) - (\sin(z) + (i+2))(t+1)\sin(x)e^{-t} + + (\sin(x)e^{-t} + 1)((i+2) + t\sin(z) + 2sin(z))$$

After cancellation we obtained identity $\forall i = \overline{1, m}$. The functions $u_0^i(x, z)$ are

$$u_0^i(x,z) = w_0(x)v_0^i(z) = (\sin(z) + (i+2))(\sin(x) + 1).$$

The conditions for existence of the solution of the problem (24) are

$$\frac{\beta^{i}(t,z) - v^{i}_{0}(z)\varphi_{t}(t,0)}{B_{z}^{i}(\psi^{i}) + \sum_{k=1}^{m} g^{k}(t)\psi^{k}} = \frac{(i+2) + t\sin(z) + 2sin(z)}{(t+1)(m\cdot\sin(z) + \cos(z) + (i+2) + \sum_{k=1}^{m}(k+2))} \ge \delta^{i},$$

here the choice δ^i depends on the amount of equation m and the number i. Due to the limited functions in relation above and also true to the fact that the m and i are final numbers, we can choose $\delta^i : 0 < \delta^i < 1$.

This example is showed, that the solution set of problem (1)-(3) is nonempty.

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References

- Yu.E.Anikonov, On the methods of study of multidimensional inverse problems for evolutional equations, *Dokl. Math.*, **331**(1993), no. 3, 409–410.
- [2] Yu. Ya. Belov, Inverse Problems for Partial Differential Equations, Utrecht, VSP, 2002.
- [3] Yu.Ya.Belov, I.V.Frolenkov, Coefficient Identification Problems for Semilinear Parabolic Equations, Dokl. Math., 72(2005), no.2, 737–739.
- [4] G.V.Romanenko, I.V.Frolenkov, On the representation of the solution of the inverse problem for a system of twodimensional parabolic equations, International Conference "Inverse and Ill-Posed Problems of Mathematical Physics", dedicated to the 80th anniversary of the birthday of Academician M. M. Lavrent'ev, Siberian Mathematical Publishers, Novosibirsk, 2012, 103–104.
- [5] I.V.Frolenkov, E.N.Kriger, An existence of the solution for identification problem of coefficient in special form at source function, Vestnik, Quart. J. of Novosibirsk State Univ., Series: Math., mech. and informatics, 13(2013) no. 1, 120–134.
- [6] I.V.Frolenkov, G.V.Romanenko, On the solution of the inverse problem for multidimensional parabolic equations, Sib. J. Math. XV, 50(2012), no. 2, 139–146.
- [7] N.N.Yanenko, Fractional Steps for Solving Multidimensional Problems of Mathematical Physics, Nauka, Novosibirsk, 1967 (in Russian).

О представлении решения задачи идентификации коэффициентов при дифференциальном операторе второго порядка в системе многомерных параболических уравнений

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Исследована обратная задача с данными Коши для системы многомерных параболических уравнений, содержащих неизвестные коэффициенты перед дифференциальным оператором второго порядка по выделенной переменной и суммой младших членов. Начальные данные имеют специальный вид и заданы в виде произведения двух функций, зависящих от разных переменных. Получены достаточные условия существования и единственности решения вспомогательной прямой и исходной обратной задач. Для доказательства используется метод слабой аппроксимации.

Ключевые слова: обратная задача, задача идентификации, коэффициентные обратные задачи, метод слабой аппроксимации, системы уравнений в частных производных.