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Groups Satisfying the Minimal Condition for Non-abelian Non-normal Subgroups

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In the present paper, we establish that in a great many large and extremely large classes of groups, the non-abelian groups satisfying the mentioned condition are exactly the non-abelian Chernikov groups and the non-abelian solvable groups with normal non-abelian subgroups.

Keywords: minimal conditions, non-normal subgroups, non-abelian, Chernikov, Artinian, Dedekind, Shunkov, solvable, periodic groups, weakly, binary, primitive, locally graded groups.

To the memory of my dear teacher Professor Vladimir Petrovich Shunkov, who was an outstanding mathematician and the founder of the well-known large and powerful algebraic school.

Introduction

Introduce the definition:

Definition. The group G will be called weakly graded, if for every $g, h \in G$, the subgroup $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$ whenever g is an element of infinite order, or g is a p -element $\neq 1$ with some odd prime p and also $[g^p, h] = 1$ and the subgroup $\langle g, h \rangle$ is periodic.

The same as in this definition groups were first considered in [1] (without a name).

Remind the following. A group, in which every finitely generated (respectively 2-generator) subgroup $\neq 1$ possesses a subgroup of finite index $\neq 1$, is called locally (respectively binary) graded (S.N. Chernikov, see [2, 3] and respectively [4, P. 20]). The class of weakly graded groups, clearly, includes the classes of locally and binary graded groups and (by the way) the classes of locally, binary, residually finite groups, the classes of locally, binary, residually solvable groups. It includes the classes of 2-groups, RN -groups, groups that have a series with binary finite factors, Further, remind that in view of Mal'cev's Theorem [5, Theorems 7, 8], finitely generated linear groups are residually finite. Therefore obviously all linear groups belong to the class of locally graded groups and, at the same time, to the class of weakly graded groups.

Remind that the group G is called Shunkov, if for any its finite subgroup K , every subgroup of the factor group $N_G(K)/K$, generated by two conjugate elements of prime order, is finite (V. D. Mazurov, 1998). Now let G be a periodic Shunkov group, $g, h \in G$ and g be a p -element $\neq 1$ with some odd p such that $[g^p, h] = 1$. Then conjugate elements $g < g^p >$ and $(g < g^p >)^{h < g^p >}$ of the factor group $N_G(\langle g^p \rangle) / \langle g^p \rangle$ generate its finite subgroup. At the same time, the

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subgroup $\langle g, g^h \rangle$ of the group $N_G(\langle g^p \rangle)$ is finite. So G is weakly graded. Thus the class of weakly graded groups includes the class of periodic Shunkov groups.

Thus, the class of weakly graded groups is extremely large.

A series of deep and bright results are connected with the minimal condition for non-abelian subgroups (briefly, with $\min - \overline{ab}$). S. N. Chernikov's [6], V. P. Shunkov's [7], N. S. Chernikov's [8] Theorems assert that a non-abelian group with $\min - \overline{ab}$ is Chernikov, if it has a series with finite factors, it is locally finite, it is binary finite respectively. The known N. S. Chernikov's Theorem (see [1, Theorem B]) establishes that a non-abelian weakly graded group with $\min - \overline{ab}$ is necessarily Chernikov. (Clearly, every abelian group satisfies $\min - \overline{ab}$.)

Further, the group G is called primitive graded, if for $g, h \in G$, the subgroup $\langle g, g^h \rangle$ possesses a subgroup of finite index $\neq 1$ whenever g is a p -element $\neq 1$ with some odd prime p and also $[g^p, h] = 1$ and the subgroup $\langle g, h \rangle$ is periodic (N. S. Chernikov [9]).

Every weakly graded group is, of course, primitive graded, and the class of periodic primitive graded groups coincides with the class of periodic weakly graded groups. But any Ol'shanskiy's infinite 2-generator simple torsion-free group with all proper subgroups cyclic (see, for instance, [10] or [11, Theorem 28.3]) is primitive graded and is not weakly graded. Thus, the class of weakly graded groups is a proper subclass of the class of primitive graded groups.

Remind: a group with all subgroups normal is called Dedekind. According to the known R. Baer's Theorem [12], the Dedekind groups are exactly all abelian groups and all groups $G = Q \times T \times R$ with a quaternion subgroup Q (of order 8), an elementary abelian 2-subgroup T and an abelian subgroup R with all its element of odd order.

S. N. Chernikov's Theorem [13] asserts that every group having a series with finite factors (in particular, every RN -group, every almost solvable group) satisfies the minimal condition for (on) non-normal subgroups (briefly, $\min - \overline{n}$) iff it is Chernikov or Dedekind. Groups with such series are obviously primitive graded. The new N. S. Chernikov's Theorem [9] establishes that a primitive graded group satisfies $\min - \overline{n}$ iff it is Chernikov or Dedekind.

Let $\min - \overline{ab\overline{n}}$ be the minimal condition for (on) non-abelian non-normal subgroups. The infinite non-abelian groups with normal infinite non-abelian subgroups (which present, in fact, a special case of groups satisfying $\min - \overline{ab\overline{n}}$) are called \overline{TH} -groups (S. N. Chernikov, see [2] and also [3]). S. N. Chernikov has obtained a lot of principal results relating to the \overline{TH} -groups (see [2, 14] and also [4, §2], [3, Chapter 6] and [15, §§ 3-5]). The known S. N. Chernikov's Theorem [2], [14, Theorem 1.1] establishes that locally graded \overline{TH} -groups are solvable. N. S. Chernikov ([16] and also [15, Theorem 3.2]) gives an affirmative answer to the following S. N. Chernikov's question (see [4, P. 20]): Are the binary graded \overline{TH} -group solvable? N. S. Chernikov ([16] and also [15, Theorem 3.2]) establishes that the weakly graded \overline{TH} -groups are solvable.

Taking into account the data set forth above, it is natural to investigate the weakly graded groups with $\min - \overline{ab\overline{n}}$.

1. The main results and some corollaries

The following three theorems are the main results of the present paper.

Clearly Chernikov groups and non-abelian groups with normal non-abelian subgroups satisfy $\min - \overline{ab\overline{n}}$.

Theorem 1. *For the weakly graded non-abelian group G the following statements are equivalent:*

- (i) *G satisfies the minimal condition for non-abelian non-normal subgroups.*

(ii) G is a Chernikov group or a solvable group with normal non-abelian subgroups.

A great many Ol'shanskiy's groups (see [11]) are infinite simple groups with $\min - \overline{ab}$. For any such group G the statement (i) of Theorem 1 is valid but (ii) is not. Thus, in the Theorem 1, the condition: " G is weakly graded" is essential.

Theorem 2. *Let G be a periodic non-abelian Shunkov group. Then the following statements hold:*

- (i) G satisfies the minimal condition for non-normal subgroups iff it is Chernikov or Hamiltonian.
- (ii) G satisfies the minimal condition for non-abelian non-normal subgroups iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.

Theorem 2 is announced [17].

Note that in view of Theorem 1 [9], in the statement (i) of Theorem 2, the restriction of periodicity of G is not essential. However, in the statement (ii) this restriction is essential. For instance, the Ol'shanskiy's infinite 2-generator simple torsion-free group with all proper subgroups cyclic (see [10] or [11, Theorem 28.3]) satisfies the corresponding minimal condition. It is Shunkov, but it is neither Chernikov, nor solvable.

Theorem 3. *Let \mathfrak{L} be the minimal local class of groups closed with respect to forming subgroups and series (at the same time, to forming subcartesian products) and containing the class \mathfrak{G} of all weakly graded groups. Let G be a non-abelian group belonging to \mathfrak{L} . The group G satisfies the condition $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

The following eight propositions are immediate consequences of Theorem 1.

Corollary 1. *A solvable non-abelian group satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a group with normal non-abelian subgroups.*

Corollary 2. *Let G be a non-abelian locally or binary or residually solvable group. Then G satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

Corollary 3. *Let G be a non-abelian locally or binary or residually finite group. Then G satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

Corollary 4. *A non-abelian group, having a series with binary finite factors, satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

Corollary 5. *A non-abelian 2-group satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

Corollary 6. *A linear non-abelian group satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

Corollary 7. *A non-abelian RN -group satisfies $\min - \overline{abn}$ iff it is a solvable Chernikov group or a solvable group with normal non-abelian subgroups.*

Corollary 8. *A non-abelian locally or binary graded group satisfies $\min - \overline{abn}$ iff it is a Chernikov group or a solvable group with normal non-abelian subgroups.*

The following two propositions are consequences of Corollary 8 and Theorem 1 respectively.

Corollary 9. *Let \mathfrak{H} be the maximal class of groups in which every non-abelian group with $\min - \overline{abn}$ is a Chernikov group or a solvable group with normal non-abelian subgroups. Then \mathfrak{H} is local and closed with respect to forming series and subcartesian products.*

Proof. First, note that Chernikov and solvable groups are locally graded, and the class of locally graded groups is local and closed with respect to forming series and subcartesian products. Therefore taking into account that the class of all groups with $\min - \overline{abn}$ is closed with respect to forming subgroups and factor groups, it is easy to see: if some non-abelian group G with $\min - \overline{abn}$ has a local system of \mathfrak{H} -subgroups or a series with \mathfrak{H} -factors or is a subcartesian product of \mathfrak{H} -groups, then G is locally graded. At the same time, in view of Corollary 8, G is a Chernikov group or a solvable group with normal non-abelian subgroups.

The proof is complete. \square

Corollary 10. *Let \mathfrak{H} be from Corollary 9, and G be a group such that for every $g, h \in G$, the subgroup $\langle g, g^h \rangle$ has a homomorphic image $K \in \mathfrak{H}$ and $K \neq 1$, whenever g is an element of infinite order, or g is a p -element $\neq 1$ with some odd p and also $[g^p, h] = 1$ and $\langle g, h \rangle$ is periodic. Then $G \in \mathfrak{H}$.*

Proof. Let G satisfy $\min - \overline{abn}$. Then K also satisfies $\min - \overline{abn}$. So K is Chernikov or solvable. Therefore the 2-generator group K and, at the same time, the group $\langle g, g^h \rangle$ have a subgroup of finite index $\neq 1$. Thus, G is weakly graded. In view of Theorem 1, G is a Chernikov group or a solvable group with normal non-abelian subgroups, if it is non-abelian.

The proof is complete. \square

In conclusion, note that the non-abelian groups with normal non-abelian subgroups are sometimes called metahamiltonian. The metahamiltonian groups are studied, for instance, in [18–23]. The structure of solvable metahamiltonian groups to some extent is roughly revealed in [23].

2. Preliminary results

Proposition 1. *Every residually finite group G with $\min - \overline{abn}$ is almost abelian.*

Proof. Let each subgroup of finite index of G be non-abelian. It is easy to see: G necessarily contains some subgroup H of finite index with normal in G subgroups of finite index; for some subgroup K of finite index of H , H/K is non-abelian. Take any subgroup L of finite index of K . Then all subgroups of the factor group H/L are normal in it. Therefore in view of R. Baer's Theorem [12] mentioned above, $|(H/L)'| = 2$. So because of $(H/L)/(K/L) (\simeq H/K)$ is non-abelian, K/L is abelian. Consequently, the (residually finite) subgroup K is abelian, which is a contradiction.

The proof is complete. \square

Lemma 1. *Let G be a group with $\min - \overline{abn}$, and $N \trianglelefteq G$. Then G/N satisfies $\min - \overline{abn}$, and in the case of non-abelian N , G/N satisfies $\min - \bar{n}$.*

Proof. Indeed, let

$$G_1/N \supset G_2/N \supset \dots \supset G_j/N \supset G_{j+1}/N \supset \dots, \quad j = 1, 2, 3, \dots, \quad (1)$$

be a descending chain of subgroups of the factor group G/N .

If all G_j/N are non-abelian, then all G_j are non-abelian and so almost all terms of the chain

$$G_1 \supset G_2 \supset \dots \supset G_j \supset G_{j+1} \supset \dots, \quad j = 1, 2, 3, \dots, \quad (2)$$

of G are normal in it. At the same time, almost all terms of (1) are normal in G/N .

In the case of non-abelian N , almost all terms of (2) are normal in G . Therefore almost all terms of (1) are normal in G/N .

The proof is complete. \square

Proposition 2. *Every finitely generated almost solvable group G with $\min - \overline{abn}$ is almost abelian.*

Proof. For some solvable subgroup $H \trianglelefteq G$, $|G : H| < \infty$. Suppose that G is not almost abelian. Then for some term K of the derived series of H , G/K is almost abelian but G/K' is not. Let L/K be an abelian subgroup of finite index of G/K . In view of Schreier's Theorem, L is finitely generated. Since L/K' is finitely generated metabelian, by virtue of Ph. Hall's Theorem (see for instance [24, Theorem 9.51]), it is residually finite. Also L/K' satisfies $\min - \overline{abn}$ (Lemma 1). Therefore in view of Proposition 1, L/K' is almost abelian. At the same time, G/K' is almost abelian, which is a contradiction.

The proof is complete. \square

Proposition 3. *Let G be a weakly graded non-abelian group with $\min - \overline{abn}$. Then G contains some normal Chernikov non-abelian subgroup, or some normal finitely generated non-periodic subgroup, which is solvable, non-abelian and almost abelian.*

Proof. First, let G satisfy $\min - \overline{ab}$. Then in view of N.S. Chernikov's Theorem [1] mentioned above, it is Chernikov. Therefore in what follows assume: G does not satisfy $\min - \overline{ab}$.

It is easy to see: G contains some subgroup H , which does not satisfy $\min - \overline{ab}$, such that every subgroup of H is normal in G or satisfies $\min - \overline{ab}$; H has some descending series

$$H = H_0 \supset H_1 \supset \dots \supset H_\gamma \supseteq 1 \quad (3)$$

of normal in G subgroups such that for each $\alpha < \gamma$, H_α is non-abelian, $H_\gamma = \cap_{\alpha < \gamma} H_\alpha$ and H_γ satisfies $\min - \overline{ab}$. Then again in view of N.S. Chernikov's Theorem [1], H_γ is Chernikov or abelian. Since in (3) all subgroups H_α , $\alpha < \gamma$, do not satisfy $\min - \overline{ab}$, factor groups H/H_α , $\alpha < \gamma$, are Dedekind. Consequently in view of R. Baer's Theorem [12] mentioned above, $|(H/H_\alpha)'| \leq 2$ for $\alpha < \gamma$. Therefore obviously $|(H/H_\gamma)'| \leq 2$.

If H_γ is non-abelian, then it is normal Chernikov.

Let H_γ be abelian. Then H is solvable.

In the case of non-periodic H , let K be a finitely generated non-abelian and non-periodic subgroup of H . Then with regard to mentioned S.N. Chernikov's Theorem [6] on groups with $\min - \overline{ab}$, K does not satisfy $\min - \overline{ab}$. So $K \trianglelefteq G$. Since K is solvable and finitely generated, it is almost abelian (Proposition 2).

Let H be periodic. Then in view of Proposition 1.1 [3], H is locally finite. Let F be a finite non-abelian subgroup of H . It is easy to see:

$$\cap_{\alpha < \gamma} FH_\alpha = F(\cap_{\alpha < \gamma} H_\alpha). \quad (4)$$

Since G satisfies $\min - \overline{abn}$, for some ordinal $\beta < \gamma$, $FH_\alpha \trianglelefteq G$ whenever $\beta < \alpha < \gamma$. Then, taking into consideration (4) we obtain: $F(\cap_{\alpha < \gamma} H_\alpha) \trianglelefteq G$.

If $\cap_{\alpha < \gamma} H_\alpha$ is Chernikov, then $F(\cap_{\alpha < \gamma} H_\alpha)$ is a normal Chernikov non-abelian subgroup of G .

Let the subgroup $\cap_{\alpha < \gamma} H_\alpha$ be non-Chernikov. Then clearly F normalizes some its countable subgroup $L = \{g_1, g_2, g_3, \dots\}$, generated by elements of prime order. Further, for some subgroup L_1 of L , $L = \langle g_1 \rangle \times L_1$, for some subgroup L_2 of L_1 , $L_1 = (\langle g_1 \rangle \times L_2) \times L_3$, for some

subgroup L_3 of L_2 , $L_2 = \langle g_1 \rangle \langle g_2 \rangle \langle g_3 \rangle \cap L_2 \times L_3, \dots$. Let N_i be the intersection of all subgroups L_i^u with $u \in F$, $i = 1, 2, \dots$. In view of Poincaré's Theorem (see, for instance, [25]), $|L : N_i| < \infty$. It is easy to see:

$$F = \bigcap_{i=1}^{\infty} FN_i. \quad (5)$$

Since FN_i is periodic almost abelian and non-abelian and non-Chernikov, by virtue of Lemma 1 [6], it does not satisfy $\min - \overline{ab}$. So $FN_i \trianglelefteq G$. Therefore, with regard to (5), F is a finite normal non-abelian subgroup of G .

The proof is complete. \square

Proposition 4. *Let G be a weakly graded group, N its central subgroup. Let G/N be Artinian. Then G/N is Chernikov.*

Proof. Let G/N be Artinian non-Chernikov. Then G/N contains some non-Chernikov subgroup H/N such that every proper subgroup of H/N is Chernikov. Since the factor group $(H/N)/(H/N)'$ is abelian Artinian, it is Chernikov (A.G. Kurosh, see, for instance, [26, Proposition 4.2.11]). Further, since the class of Chernikov groups is closed with respect to forming extensions (S.N. Chernikov, see for instance, [3, Theorem 1.4]) and $(H/N)/(H/N)'$ is Chernikov, $(H/N)'$ is non-Chernikov and so $(H/N)' = H/N$.

At the same time, $H = H'N$. Therefore since $N \subseteq Z(G)$, $H' = (H'N)' = H''$. Thus

$$H' = H'' \quad (6).$$

Also $H/N = H'N/N \simeq H'/H' \cap N$.

Thus H' is a weakly graded group, $H' \cap N \subseteq Z(H')$, $H'/H' \cap N$ is Artinian non-Chernikov with all proper subgroups Chernikov and also (6) holds. Taking this into account, we may assume without loss of generality: G/N is Artinian non-Chernikov with all proper its subgroup Chernikov and also

$$G = G'. \quad (7)$$

Let K be a subgroup of G . If $G = KN$, then, with regard to (7), $G = (KN)' = K'$ and so $G = K$. If $|G : K| < \infty$, then $|G/N : KN/N| < \infty$. Therefore because of G/N is non-Chernikov, KN/N is non-Chernikov too. So $KN/N = G/N$ and $G = KN = K$.

Now remind: in view of Theorem B [1], an Artinian weakly graded group is Chernikov. Thus, G/N is not weakly graded. Further, G/N is clearly periodic. Consequently, for some $h \in G$ and $g \in G$ such that gN is a p -element $\neq 1$ with $p \neq 2$ of the factor group G/N , the subgroup $\langle gN, (gN)^{hN} \rangle$ of the factor group G/N has no subgroups of finite index $\neq 1$ and also

$$[(gN)^p, hN] = 1. \quad (8)$$

Consequently, $\langle gN, (gN)^{hN} \rangle$ is non-Chernikov and, at the same time, it coincides with G/N .

So $G = \langle g, g^h \rangle N$. Therefore

$$G = \langle g, g^h \rangle \quad (9)$$

(see above).

Since G has no subgroups of finite index $\neq 1$ (see above), $\langle g, g^h \rangle$, with regard to (9), has no such subgroups. Therefore because of G is weakly graded, g is an element of finite order. So for some p -element b of $\langle g \rangle$, $gN = bN$. Obviously, $\langle b, b^h \rangle N = \langle g, g^h \rangle N$. Therefore, by virtue of (9), $G = \langle b, b^h \rangle N$. Hence follows:

$$G = \langle b, b^h \rangle. \quad (10)$$

Further, with regard to (8), $[b^p, h] = z \in N(\subseteq Z(G))$. Put $|\langle b^p \rangle| = m$. Then $[(b^p)^m, h] = z^m = 1$. Thus $\langle b^p \rangle \langle z \rangle$ is a finite subgroup, normalized by b and h . At the same time, with regard to (10), $\langle b^p \rangle \langle z \rangle \trianglelefteq G$. Consequently $|G : C_G(\langle b^p \rangle \langle z \rangle)| < \infty$. Since also G has no subgroups of finite index $\neq 1$, $b^p \in Z(G)$. So

$$[b^p, h] = 1. \quad (11)$$

Since G is weakly graded and b is its p -element $\neq 1$ where $p \neq 2$, with regard to (11), $\langle b, b^h \rangle$ contains a subgroup of finite index $\neq 1$, if $\langle b, h \rangle$ is periodic. But with regard to (10), $\langle b, b^h \rangle$ has no such subgroups, which is a contradiction.

Thus $\langle b, h \rangle$ is non-periodic. Then, because of G/N is periodic, N has some element u of infinite order. Since also $|\langle b \rangle| < \infty$ and $u \in Z(G)$, bu is of infinite order. Further, with regard to (10), $G = \langle bu, (bu)^h \rangle N$. So $G = \langle bu, (bu)^h \rangle$ (see above). Since G is weakly graded, $\langle bu, (bu)^h \rangle$ has a subgroup of finite index $\neq 1$, which is a contradiction.

The proof is complete. \square

Proposition 5. *Let G be a weakly graded group, N its central subgroup. Let G/N satisfy the minimal condition for non-normal subgroups. Then G/N is Chernikov or Dedekind.*

Proof. Indeed, in view of Proposition 1 [9], $(G/N)'$ is Artinian. Therefore by virtue of Proposition 4, $(G/N)'$ is Chernikov. Then again in view of Proposition 1 [9], G/N is Chernikov or Dedekind.

The proof is complete. \square

Proposition 6. *Let G be a weakly graded non-abelian group with $\min - \overline{abn}$, N its normal non-abelian subgroup. Then $C_G(N)/Z(N)$ is Chernikov or Dedekind and $C_G(N)$ is almost solvable.*

Proof. Indeed,

$$C_G(N)/Z(N) = C_G(N)/(N \cap C_G(N)) \simeq NC_G(N)/N. \quad (12)$$

In view of Lemma 1, G/N satisfies $\min - \bar{n}$. Then with regard to (12), $C_G(N)/Z(N)$ satisfies $\min - \bar{n}$. Therefore by virtue of Proposition 5, $C_G(N)/Z(N)$ is Chernikov or Dedekind. Then, with regard to R. Baer's Theorem mentioned above, $C_G(N)/Z(N)$ is solvable, if it is Dedekind. Consequently, $C_G(N)$ is almost solvable.

The proof is complete. \square

Proposition 7. *The weakly graded group G with $\min - \overline{abn}$ is almost solvable.*

Proof. Let G be non-abelian. In view of Proposition 3, G has a non-abelian subgroup $N \trianglelefteq G$, which is Chernikov, or solvable non-periodic and also (finitely generated abelian)-by-finite. In view of Proposition 6, $C_G(N)$ is almost solvable.

First, let N be Chernikov. Then for some abelian subgroup $A \triangleleft G$ of N , $|N : A| < \infty$. Put $A_i = \{g : g \in A \text{ and } |\langle g \rangle| \leq i\}$, $i = 1, 2, \dots$. Clearly, $A_i \trianglelefteq G$ and A_i is finite, $i = 1, 2, \dots$, and $A = \cup_{i=1}^{\infty} A_i$. Consequently, $G/C_G(A_i)$ is finite, $i = 1, 2, \dots$, and $C_G(A) = \cap_{i=1}^{\infty} C_G(A_i)$. Therefore $G/C_G(A)$ is residually finite. Then $G/(C_G(A) \cap C_G(N/A))$ is also residually finite. Therefore in view of Lemma 1 and Proposition 1, $G/(C_G(A) \cap C_G(N/A))$ is almost abelian. Further, in view of Kaluzhnin's Theorem (see, for instance, [25]), $(C_G(A) \cap C_G(N/A))/C_G(N)$ is abelian. So $G/C_G(N)$ is almost solvable. Thus, with regard to Proposition 6, G is (almost solvable)-by-almost solvable. Therefore G is almost solvable.

Let N be non-Chernikov. Then obviously it is polycyclic. Therefore N is residually finite (K. A. Hirsch, see, for instance, [26, Proposition 5.4.17]). Since it is also finitely generated, in

accordance with Baumslag-Smirnov Theorem (see, for instance, [24, Corollary, P. 119]), $\text{Aut } N$ is residually finite. So $G/C_G(N)$ is residually finite. Consequently, in view of Lemma 1 and Proposition 1, $G/C_G(N)$ is almost abelian. Therefore, with regard to Proposition 6, G is almost solvable.

The proof is complete. \square

Proposition 8. *The weakly graded periodic group G with $\min - \overline{abn}$ is Chernikov-by-abelian.*

Proof. Let G be non-abelian. Then in view of Proposition 3, G has some normal Chernikov non-abelian subgroup N . By virtue of Proposition 7, G and, at the same time, G/N are almost solvable. Further, G/N satisfies $\min - \bar{n}$ (Lemma 1). In consequence of Theorem 1 [13] and Corollary [13, P. 16], G/N is Chernikov or Dedekind. In the first case, G is Chernikov (see [3, Theorem 1.4]). In the second case, by virtue of R. Baer's Theorem [12], $|(G/N)'| \leq 2$. Hence follows: G is Chernikov-by-abelian.

The proof is complete. \square

Proposition 9. *Let G be an infinite almost abelian group with $\min - \overline{abn}$. Let also G be finitely generated and non-abelian. Then every non-abelian subgroup of G is normal in it.*

Proof. Let A be a normal abelian subgroup of finite index of G . In view of Schreier's Theorem, A is finitely generated. Therefore in view of the Main Theorem on finitely generated abelian groups, $A = (\times_{i=1}^n \langle a_i \rangle) \times (\times_{i=n+1}^l \langle a_i \rangle)$ where $\langle a_i \rangle$, $i = 1, \dots, n$, are infinite, and $\langle a_i \rangle$, $i = n+1, \dots, l$, are finite. Put $m = |\langle a_{n+1} \rangle| \dots |\langle a_l \rangle|$ and $B = \{g^m : g \in A\}$ (of course, it is possible that $m = 1$). Then $B \triangleleft G$ and $B = \times_{i=1}^n \langle b_i \rangle$ with $b_i = a_i^m$. Obviously B is torsion-free and $|G : B| < \infty$.

Let p_1, p_2, \dots , be primes such that for $i \neq j$, $p_i \neq p_j$, and $p_j \nmid |G : B|$, $j = 1, 2, \dots$. Put $B_j = \times_{i=1}^n \langle b_i^{p_i \dots p_j} \rangle$, $j = 1, 2, \dots$ (in particular, $B_1 = \times_{i=1}^n \langle b_i^{p_1} \rangle$). Then $B_j \triangleleft G$, $j = 1, 2, \dots$. Since G is non-abelian and, clearly, $\cap_{j=1}^\infty B_j = 1$, G/B_k is non-abelian for some k . Fix k . Further, clearly,

$$|G : B_k| = |G : B| p_1^n \dots p_k^n \quad (13)$$

and $|B_k : B_{k+1}| = p_{k+1}^n$. At the same time, $(|G : B_k|, |B_k : B_{k+1}|) = 1$. Therefore in view of Schur's Theorem (see [25]), for some subgroup D_{k+1}/B_{k+1} of G/B_{k+1} ,

$$G/B_{k+1} = B_k/B_{k+1} \rtimes D_{k+1}/B_{k+1}. \quad (14)$$

Then

$$G = B_k D_{k+1} \quad (15)$$

and

$$B_k \cap D_{k+1} = B_{k+1}. \quad (16)$$

In particular, with regard to (15), $G/B_k \simeq D_{k+1}/D_{k+1} \cap B_k$. Consequently, $D_{k+1}/D_{k+1} \cap B_k$ and, at the same time, D_{k+1} are non-abelian.

Further, in view of (14), $|D_{k+1} : B_{k+1}| = |G : B_k|$. Also $|B_{k+1} : B_{k+2}| = p_{k+2}^n$. At the same time, with regard to (13), $(|D_{k+1} : B_{k+1}|, |B_{k+1} : B_{k+2}|) = 1$. Again in view of the mentioned Schur's Theorem, for some subgroup D_{k+2}/B_{k+2} of D_{k+1}/B_{k+2} ,

$$D_{k+1}/B_{k+2} = B_{k+1}/B_{k+2} \rtimes D_{k+2}/B_{k+2}. \quad (17)$$

(Of course, $D_{k+1} \supseteq D_{k+2}$.)

Then in view of (17),

$$D_{k+1} = B_{k+1}D_{k+2}. \quad (18)$$

Therefore, with regard to (15) and (18),

$$G = B_k B_{k+1} D_{k+2} = B_k D_{k+2}. \quad (19)$$

Also in view of (17),

$$B_{k+1} \cap D_{k+2} = B_{k+2}. \quad (20)$$

In particular, with regard to (19), $G/B_k \simeq D_{k+2}/D_{k+2} \cap B_k$. Consequently D_{k+2} is non-abelian. Further, with regard to (20) and (16), $B_k \cap D_{k+2} = B_k \cap (D_{k+2} \cap D_{k+1}) = (B_k \cap D_{k+1}) \cap D_{k+2} = B_{k+1} \cap D_{k+2} = B_{k+2}$. Thus

$$B_k \cap D_{k+2} = B_{k+2}. \quad (21)$$

Analogously, for some subgroup D_{k+3}/B_{k+3} of D_{k+2}/B_{k+3} , $D_{k+2}/B_{k+3} = B_{k+2}/B_{k+3} \rtimes D_{k+3}/B_{k+3}$ and $G = B_k D_{k+3}$, D_{k+3} is non-abelian, and $D_{k+2} \neq D_{k+3}, \dots$

Thus, we have the chain $D_{k+1} \supset D_{k+2} \supset D_{k+3} \supset \dots$ of non-abelian subgroups such that

$$D_j/B_{j+1} = B_j/B_{j+1} \rtimes D_{j+1}/B_{j+1}, \quad j = k+1, k+2, \dots, \quad (22)$$

$$G = B_k D_j, \quad j = k+1, k+2, \dots, \quad (23)$$

and, like to (16) and (21),

$$B_k \cap D_j = B_j, \quad j = k+1, k+2, \dots. \quad (24)$$

Then, with regard to (24), $B_k \cap (\cap_{j=k+1}^{\infty} D_j) = \cap_{j=k+1}^{\infty} (B_k \cap D_j) = \cap_{j=k+1}^{\infty} B_j = 1$. So

$$B_k \cap (\cap_{j=k+1}^{\infty} D_j) = 1. \quad (25)$$

Since G satisfies $\min - \overline{abn}$, for some $s > k$, every D_j with $j \geq s$ is normal G . Then with regard to (22), $D_j/B_{j+1} = B_j/B_{j+1} \times D_{j+1}/B_{j+1}$, $j = s, s+1, \dots$. Thus, $[B_j, D_{j+1}] \subseteq B_{j+1}$, $j = s, s+1, \dots$. Further, also $G = B_k D_{j+1}$, $j = s, s+1, \dots$, (see (23)) and $[B_k, B_j] = 1$. Hence follows:

$$B_j/B_{j+1} \subseteq Z(G/B_{j+1}), \quad j = s, s+1, \dots. \quad (26)$$

Obviously, $B/B_{j+1} = B_j/B_{j+1} \times F_j/B_{j+1}$ where $|F_j/B_{j+1}| = p_1^n \dots p_j^n$, and, with regard to (26),

$$B/F_j \subseteq Z(G/F_j), \quad j = s, s+1, \dots. \quad (27)$$

Further, B/F_j is (as B_j/B_{j+1}) an elementary abelian group of order p_{j+1}^n .

Therefore, obviously,

$$F_j = \times_{i=1}^n \langle b_i^{p_j^{j+1}} \rangle. \quad (28)$$

Taking into account (28), it is easy to see:

$$\cap_{j=s}^{\infty} F_j = 1. \quad (29)$$

In view of (27) and (29), $B \subseteq Z(G)$. Therefore, with regard to (23), $G' \subseteq \cap_{j=k+1}^{\infty} D_j$. Then in view of (25), $B_k \cap G' = 1$. Consequently, G' is finite.

Let H be a non-abelian subgroup of G . Show that $H \trianglelefteq G$. Take primes q and $r \neq q$ such that $q \nmid |H'| |H : C_H(H')|$ and $r \nmid |H'| |H : C_H(H')|$. Put $Q_j = \langle g^{q^j} : g \in H \rangle$ and $R_j = \langle g^{r^j} : g \in H \rangle$, $j = 1, 2, \dots$. It is easy to see: $H' \subseteq Q_j$, R_j , and $H = Q_j R_j$, $j = 1, 2, \dots$

Let H be infinite. Then Q_1/H' and R_1/H' are finitely generated infinite abelian. Taking this into account, it is easy to see: $Q_j \neq Q_{j+1}$ and $R_j \neq R_{j+1}$, $j = 1, 2, \dots$. Thus, we have infinite strictly descending series:

$$Q_1 \supset Q_2 \supset \dots \supset Q_j \supset Q_{j+1} \supset \dots, \quad R_1 \supset R_2 \supset \dots \supset R_j \supset R_{j+1} \supset \dots \quad (30)$$

Consider the case when $H' \subseteq Z(H)$. For any $g, h \in H$, $[g^{q^j}, h^{q^j}] = [g, h]^{q^{2j}} \neq 1$ and $[g^{r^j}, h^{r^j}] = [g, h]^{r^{2j}} \neq 1$, if $[g, h] \neq 1$. Thus (30) are series of non-abelian subgroups. Since G satisfies $\min - \overline{abn}$, for some t , Q_t and R_t are normal in G . So $H = Q_t R_t \trianglelefteq G$.

Consider the case when $H' \not\subseteq Z(H)$. Since $q \nmid |H : C_H(H')|$ and $r \nmid |H : C_H(H')|$, for every $h \in H \setminus C_H(H')$, we have: $h^{q^j}, h^{r^j} \in H \setminus C_H(H')$, $j = 1, 2, \dots$. At the same time, $Q_j \not\subseteq C_H(H')$ and $R_j \not\subseteq C_H(H')$, $j = 1, 2, \dots$. Consequently, since $H' \subseteq Q_j, R_j$, Q_j and R_j are non-abelian. Then, as above, $H = Q_t R_t \trianglelefteq G$ for some t .

Now let H be finite. Then HG' is a finite normal subgroup of G . So $|G : C_G(H)| < \infty$. Consequently, $C_G(H)$ is not periodic. Let u be an element of infinite order of $C_G(H)$. Since G satisfies $\min - \overline{abn}$ and obviously $\langle u^{q^j} \rangle \times H \not\subseteq \langle u^{q^{j+1}} \rangle \times H$, for some natural d , $\langle u^{g^d} \rangle \times H \trianglelefteq G$. Since H is the set of all elements of finite order of the group $\langle u^{g^d} \rangle \times H$, $H \trianglelefteq G$.

The proof is complete. \square

Lemma 2. *Let the non-abelian group G have a local system \mathcal{M} of subgroups such that in every non-abelian $F \in \mathcal{M}$ all non-abelian subgroups are normal. Then any non-abelian subgroup H of G is normal in G .*

Proof. Let $a, b \in H$ and $[a, b] \neq 1$. Take any $g \in G$ and any $h \in H$. Now take any subgroup $F \in \mathcal{M}$, containing a, b, g and h . Then $H \cap F$ is a non-abelian subgroup of F . So $H \cap F \trianglelefteq F$. Consequently, $h^g \in H \cap F$. Thus, for any $g \in G$ and any $h \in H$, $h^g \in H$. So $H \trianglelefteq G$.

The proof is complete. \square

3. Proofs of the main results

Proof of Theorem 1. Let (i) hold. In view of Proposition 7, G is almost solvable. Let also G be non-Chernikov.

First, assume that G contains an element g of infinite order. Take $a, b \in G$ such that $[a, b] \neq 1$. Clearly all finitely generated subgroups F of G , containing g, a and b , form its local system. Since G is almost solvable, in consequence of Proposition 2, every F is almost abelian. Since F satisfies $\min - \overline{abn}$ and F is infinite almost abelian and finitely generated (and non-abelian), every non-abelian subgroup of F is normal in F (see Proposition 9). Therefore by virtue of Lemma 2, all non-abelian subgroups of G are normal in G .

Now let G be periodic. In view of Proposition 8, for some Chernikov subgroup $N \triangleleft G$, G/N is abelian. In consequence of O.Yu. Schmidt's Theorem (see, for instance, [25]), G is locally finite. Let U be an arbitrary non-abelian subgroup of G ; $u, v \in U$ and $[u, v] \neq 1$; g be an arbitrary element of U and $L = \langle u, v, g \rangle$. Then L is finite non-abelian, $LN \trianglelefteq G$ and LN is Chernikov. In view of R. Baer-Polovickii-S.N. Chernikov Theorem [27], [28], [29], $G/C_G(LN)$ is Chernikov. Then with regard to (S.N. Chernikov's) Theorem 1.4 [3], $C_G(LN)$ is non-Chernikov. Since it is periodic almost solvable (as the group G), by virtue of S.N. Chernikov's Theorem (see, for instance, [3, Theorem 4.3]), it contains some abelian non-Chernikov subgroup A . Obviously, there is some descending series $A_1 \supset A_2 \supset \dots \supset A_j \supset A_{j+1} \supset \dots \supset 1$ of A with $A_1 \cap L = 1$ and $\bigcap_{j=1}^{\infty} A_j = 1$. Since G satisfies $\min - \overline{abn}$ and $A_j \times L \not\subseteq A_{j+1} \times L$, $j = 1, 2, \dots$, for some natural

$k, A_j \times L \trianglelefteq G, j = k, k+1, \dots$. Then $L = \bigcap_{j=k}^{\infty} (A_j \times L) \triangleleft G$. At the same time, for any $b \in G$, $g^b \in L \subseteq U$. Consequently, $U \trianglelefteq G$.

Thus, G is an almost solvable group with normal non-abelian subgroups.

Let $R \trianglelefteq G$ be a solvable subgroup with $|G : R| < \infty$. If G/R is non-abelian, take its non-abelian subgroup K/R with abelian proper subgroups. In view of Miller-Moreno Theorem [30], K/R is solvable. Since K is non-abelian, obviously, G/K is Dedekind. Therefore G/K is obviously solvable. Consequently, G/R and, at the same time, G are solvable.

Clearly, (ii) implies (i).

The proof is complete. \square

Proof of Theorem 2. Indeed, any Shunkov group is primitive graded. Therefore the assertion (i) of Theorem 2 is a consequence of Theorem 1 [9].

Further, any periodic Shunkov group is weakly graded. Therefore the assertion (ii) of Theorem 2 is a consequence of Theorem 1.

The proof is complete. \square

Proof of Theorem 3. First, remind: an arbitrary class of groups is closed with respect to forming subcartesian products, if it is closed with respect to forming subgroups and series (see [31, Lemma 1.37]). Further, let \mathfrak{H} be from Corollary 9. Let \mathfrak{M} be the intersection of all local classes of groups that are closed with respect to forming series and contain \mathfrak{G} . In view of Corollary 9, \mathfrak{H} is local and closed with respect to forming series. In view of Theorem 1, also $\mathfrak{G} \subseteq \mathfrak{H}$. Consequently, $\mathfrak{M} \subseteq \mathfrak{H}$. Further, \mathfrak{M} is obviously the union of the classes \mathfrak{M}_α where $\mathfrak{M}_0 = \mathfrak{G}$ and for ordinals $\alpha > 0$ by induction: if $\alpha = \beta + 1$ for some ordinal β , then \mathfrak{M}_α is the class of all groups with a local system of subgroups having a series with \mathfrak{M}_β -factors; if there is no such β , $\mathfrak{M}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{M}_\beta$. Let us assume that all \mathfrak{M}_β with $\beta < \alpha$ are closed with respect to forming subgroups. Then it is easy to see: \mathfrak{M}_α is also closed with respect to forming subgroups. Thus \mathfrak{M} is closed with respect to forming subgroups.

Therefore clearly $\mathfrak{M} = \mathfrak{L}$. So $\mathfrak{L} \subseteq \mathfrak{H}$.

Let G satisfy $\min - \overline{abn}$. Since $G \in \mathfrak{L} \subseteq \mathfrak{H}$, G is a Chernikov group or a solvable group with normal non-abelian subgroups.

If G is a Chernikov group or a solvable group with normal non-abelian subgroups, then it satisfies $\min - \overline{abn}$.

The proof is complete. \square

Also the following assertion holds.

Assertion. Let R be a commutative and associative ring with 1, M be a finitely generated unital module over R , n be a natural number. The group $G \leq \text{Aut}_R(M)$ or $G \leq \mathbf{GL}_n(R)$ satisfies $\min - \overline{abn}$ iff it is Chernikov or abelian, or a solvable non-abelian group with normal non-abelian subgroups.

Proof. Since $\text{Aut}_R(M)$, $\mathbf{GL}_n(R)$ are locally graded (see [32, P. 428–429]), by virtue of Corollary 8, the present assertion is true. \square

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Группы, удовлетворяющие условию минимальности для неабелевых ненормальных подгрупп

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В настоящей статье устанавливается, что в очень многих больших и экстремально-больших классах групп неабелевы группы, удовлетворяющие указанному условию, — это в точности неабелевы черниковские группы и неабелевы разрешимые группы с нормальными неабелевыми подгруппами.

Ключевые слова: условия минимальности, ненормальные подгруппы, неабелевы, черниковские, артиновы, дедекиндовы, шунковские, разрешимые, периодические группы, слабо, бинарно, примитивно, локально ступенчатые группы.