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$\mathcal{P}$-Measure in the Class of $m-wsh$ Functions

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In this work we study the $\mathcal{P}$-measure and $\mathcal{P}$-capacity in the class of $m-wsh$ functions and prove a number of their properties.

Keywords: $m-wsh$ function, $\mathcal{P}$-measure, $\mathcal{P}$-capacity, mw-regular point.

Introduction

The classical potential theory (see [1, 2]) works with classes of harmonic and subharmonic functions and involves such concepts like the condenser capacity, harmonic measures of the sets, polar sets and others. The pluripotential theory, as is known, deals with the class of *psh* functions and the Monge-Ampere operator $(dd^c u)^n = 0$ (see [3, 4]), where as usual

$$d = \partial + \overline{\partial}, d^c = \frac{\partial - \overline{\partial}}{4i}.$$

In the recent work [5] the author has studied the class of $m - wsh$ functions, introduced the concept of mw-polarity of sets and proved several of their properties. In this paper we study the $\mathcal{P}$-measure and $\mathcal{P}$-capacity in the class of $m-wsh$ functions. In section 1 we briefly give the definition of $m-wsh$ functions and some results, which we use below. In section 2 we give the definition of $\mathcal{P}$-measure and we prove some of its properties. Section 3 is dedicated to the $\mathcal{P}$-capacity and its properties.

We note that $m-sh$ and $m-wsh$ functions are related to the Hessians of function u (see [3, 6]). They can be used in different problems of multidimentional complex analysis. One of such application is shown in the work [6] (see also [7, 8]) where the characteristic functions of Nevalinna of higher order are estimated.

1. m−wsh functions

Definition 1. *A function $u(z) \in L^1_{loc}(D)$ given in a domain $D \subset \mathbb{C}^n$ is called an $m - wsh$ function (subharmonic function on $(n-m+1)$-dimensional complex surfaces) in D, $1 \leqslant m \leqslant n$, if:*

1) it is upper semicontinuous in D, i.e.

$$\overline{\lim_{z\to z^0}}\, u(z) = \lim_{\varepsilon\to 0} \sup_{B(z^0,\varepsilon)} u(z) \leqslant u(z^0);$$

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2) the current $dd^c u \wedge \beta^{n-m}$ defined on C^∞ – smooth and finite $(m-1, m-1)$ forms ω as

$$dd^c u \wedge \beta^{n-m}(\omega) = \int u\beta^{n-m} \wedge dd^c\omega \geqslant 0, \forall \omega \in F^{(m-1,m-1)},$$

is positive, i.e. $dd^c u \wedge \beta^{n-m} \geqslant 0$.

We note that a differential form

$$\omega = \left(\frac{i}{2}\right)^p (d\ell_1 \wedge d\bar{\ell}_1) \wedge ... \wedge (d\ell_p \wedge d\bar{\ell}_p)$$

is called the *main positive* form of bidegree (p,p), $0 \leqslant p \leqslant n$, where $\ell_j = a_{j_1}z_1 + ... + a_{j_n}z_n$ are linear functions in $\mathbb{C}^n$, $j = 1, 2, ..., p$.

A linear combination of such forms ω_q

$$\omega^{(p,p)} = \sum_{q=1}^{N} f_q(z)\omega_q, \ \ f_q(z) \in L^1_{loc}(D), \ \ f_q(z) \geqslant 0,$$

is called a *strongly positive* differential form of bidegree (p,p) in $D \subset \mathbb{C}^n$.

Thus, a positive differential form of bidegree $(0,0)$ or bidegree (n,n) give us a positive scalar function $\omega^{(0,0)} = f(z) \geqslant 0$ or

$$\omega^{(n,n)} = \left(\frac{i}{2}\right)^n f(z)dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n = f(z)dV, f(z) \geqslant 0,$$

where dV is the Lebesgue volume element in the space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

A differential form $\omega^{(p,p)} \in \mathcal{F}^{(p,p)}$ of bidegree (p,p) is called *weakly positive,* if $\omega^{(p,p)} \wedge \alpha \geqslant 0$ is a positive form of bidegree (n,n) for any strongly positive form $\alpha \in \mathcal{F}^{(n-p,n-p)}$. A strongly positive form is, at the same time, weakly positive, because the exterior product of two strongly positive forms is positive.

Definition 2. *A linear continuous functional $T(\omega)$ on the space of differential forms*

$$F^{(p,p)} = F^{(p,p)}(D) = \{\omega \in \mathcal{F}^{(p,p)}(D) \cap C^\infty(D) : \ supp\,\omega \subset\subset D\}$$

is called a current of bidegree $(n-p, n-p) = (q,q)$.

A current T is called strongly (weakly) positive, if $T(\omega) \geqslant 0$ for any weakly (strongly) positive form $\omega \in \mathcal{F}^{(p,p)}$. It is clear that for $q = 0, 1, n-1, n$ the weak positivity of a current is equivalent to the strong positivity.

It is known that positive currents are the measure type currents, that is differential forms with coefficients that are Borel measures. For more details about the theory of currents see [9, 10]. An important example of the currents of bidegree (p,p) in the potential theory is the currents $dd^c u \wedge \beta^{p-1}$**,** $1 \leq p \leq n$, defined as

$$dd^c u \wedge \beta^{p-1}(\omega) = \int u\beta^{p-1} \wedge dd^c\omega \ , \ \omega \in F^{(p,p)}(D), \tag{1}$$

where $u \in L^1_{loc}(D)$ is a fixed function and $\beta = dd^c|z|^2$ is the volume form in $\mathbb{C}^n$.

It is not difficult to prove that a current of the form $dd^c u \wedge \beta^{p-1}$ is strongly positive if and only if it is weakly positive.

We consider the following properties of the $m-wsh$ functions that we need further.

1) A linear combination of $m-wsh$ functions with non-negative coefficients is an $m-wsh$ function, that is

$$u_j(z) \in m-wsh(D), \quad a_j \in R_+ \quad (j=1,2,...,N) \quad \Rightarrow$$

$$a_1u_1(z)+a_2u_2(z)+...+a_Nu_N(z) \in m-wsh(D);$$

2) The limit of a monotone decreasing sequence of $m-wsh$ functions is an $m-wsh$ function, that is

$$u_j(z) \in m-wsh(D),\, u_j\,(z) \geqslant u_{j+1}\,(z)\,, (j=1,2,...) \Rightarrow$$

$$\lim_{j\to\infty} u_j(z) \in m-wsh(D);$$

3) A uniformly convergent sequence of $m-wsh$ functions converges to an $m-wsh$ function, that is if $u_j(z) \in m-wsh(D)$, $j=1,2,...$, and if $u_j(z) \rightrightarrows u(z)$, then $u(z) \in m-wsh(D)$;

4) (*The maximum principle*) Let $u(z) \in m-wsh(D)$ and at some point $z^0 \in D$ $u(z)$ attains its maximum, that is

$$u(z^0) = \sup_{z\in D} u(z), \tag{2}$$

then $u(z) \equiv const$.

5) If $u\,(z) \in m-wsh\,(D)$, then the convolution $u_j\,(z) = u * K_{1/j}\,(z-w)$ also belongs to $m-wsh\,(D)$ and $u_j\,(z) \downarrow u\,(z)$ as $j \to \infty$. Here $K_{1/j}(z) = j^n K\,(jz)$, and K is the standard infinitely smooth kernel with the support $suppK \subset B\,(0,1)$ and averaging

$$\int_{R^n} K\,(z)\,dV = \int_{B(0,1)} K\,(z)\,dV = 1.$$

Theorem 1. *An upper semicontinuous function u given in $D \subset \mathbb{C}^n$ is $m-wsh$ if and only if for any $(n-m+1)$-dimensional complex plane $\prod \subset \mathbb{C}^n$ the restriction $u|_{\prod}$ is a subharmonic function in $\Pi \bigcap D$.*

2. $\mathcal{P}$-measure

Let $E \subset D$ be an arbitrary set of $D \subset \mathbb{C}^n$, $1 \leqslant m \leqslant n$. For simplicity, we assume that D is a strictly mw-convex domain, that is $D = \{\rho(z) < 0\}$, where $\rho(z)$ is a strictly $m-wsh$ function in some neighborhood of $G \supset \overline{D}$. Recall that a twice-smooth function $\rho\,(z) \in C^2\,(D)$ is called strictly $m-wsh$ in the point $z^0 \in D$, if the operator $dd^c u \wedge \beta^{n-1}$ is strictly positive, i.e. for some positive number $\delta > 0$, $dd^c u \wedge \beta^{n-m} \geqslant \delta\beta^{n-m+1}$ in a neighborhood of z^0. It is called strictly $m-wsh$ in the domain D if this is true at each point $z^0 \in D$.

Consider the class of functions

$$\mathcal{U}(E,D) = \{u \in m-wsh(D) : \; u\,|_D \leqslant 0, \; u\,|_E \leqslant -1\}$$

and put

$$\omega(w,E,D) = \sup\{u(w) : u \in \mathcal{U}(E,D)\}.$$

Definition 3. *The regularization $\omega^*(z,E,D) = \overline{\lim_{w\to z}}\,\omega\,(w,E,D)$ is called $\mathcal{P}$-measure ($m-wsh$ measure) of the set E with respect to D.*

Here there are some simple properties of $\mathcal{P}$-measure.

1) *(monotony)* If $E_1 \subset E_2$, then $\omega^*(z,E_1,D) \geqslant \omega^*(z,E_2,D)$. If $E \subset D_1 \subset D_2$, then $\omega^*(z,E,D_1) \geqslant \omega^*(z,E,D_2)$;

2) $\omega^*(z,U,D) \in \mathcal{U}(U,D)$ for open sets $U \subset D$ and therefore $\omega^*(z,U,D) \equiv \omega(z,U,D)$. In fact, it is easy to prove that $\omega^*(z,E,D) = -1$ if z is an interior point of the set E, $z \in E^0$. Hence $\omega^*(z,U,D) = -1$ for any $z \in U$ and $\omega^*(z,U,D) \in \mathcal{U}(U,D)$. It follows that $\omega^*(z,U,D) \leqslant \omega(z,U,D)$ and, therefore $\omega^*(z,U,D) \equiv \omega(z,U,D)$.

3) If $U \subset D$ is an open set and $U = \bigcup_{j=1}^{\infty} K_j$, where $K_j \subset \underset{j+1}{K}$, then $\omega^*(z,K_j,D) \downarrow \omega(z,U,D)$ (*It follows easily from property 2*).

4) If $E \subset D$ is an arbitrary subset, then there is a sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$ $(j = 1,2,...)$, such that $\omega^*(z,E,D) = [\lim\limits_{j\to\infty} \omega(z,U_j,D)]^*$. Indeed, by Choquet's Lemma (see [1]) there exists a countable family $\mathcal{U}' \subset \mathcal{U}$ such that $\left\{\sup\limits_{u\in\mathcal{U}'} u\left(z\right)\right\}^* = \left\{\sup\limits_{u\in\mathcal{U}} u\left(z\right)\right\}^* = \omega^*\left(z,E,D\right)$.

Hence if $\mathcal{U}' = \{u_1,u_2,...\}$ and $v_j\left(z\right) = \max\left\{u_1\left(z\right),...,u_j\left(z\right)\right\}$, then $v_j(z) \uparrow v(z)$ and $v^*\left(z\right) \equiv \omega^*\left(z,E,D\right)$. Now, if we put $U_j = \left\{v_j < -1 + \dfrac{1}{j}\right\}$, then U_j is open and $E \subset U_j \subset D$. Therefore, $\omega^*\left(z,E,D\right) \geqslant \omega^*\left(z,U_j,D\right)$, $j = 1,2,....$ On the other side $v_j - \dfrac{1}{j} \in \mathcal{U}\left(U_j,D\right)$ and $v_j - \dfrac{1}{j} \leqslant \omega\left(z,U_j,D\right)$. Consequently, $\lim\limits_{j\to\infty} v_j\left(z\right) \leqslant \lim\limits_{j\to\infty} \omega\left(z,U_j,D\right)$ and $v^*\left(z\right) = \omega^*\left(z,E,D\right) \leqslant \left[\lim\limits_{j\to\infty} \omega\left(z,U_j,D\right)\right]^*$. □

5) The $\mathcal{P}$-measure $\omega^*(z,E,D)$ is nowhere equal to zero or identically zero. $\omega^*(z,E,D) \equiv 0$ if and only if E is mw-polar in D (it is proved analogously to the corresponding property of the potential theory)

6) *(theorem of two constants)* If in the domain $D \subset \mathbb{C}^n$ a function $u(z) \in m-wsh$ and $u\,|_D \leqslant R$, $u\,|_E \leqslant r$, $(E \subset D)$, then for all $z \in D$ we have the inequality

$$u(z) \leqslant R(1 + \omega^*(z,E,D)) - r\omega^*(z,E,D).$$

This follows from the fact that the function

$$\frac{u\left(z\right) - R}{R - r} \in \mathcal{U}\left(E,D\right).$$

7) If the set E compactly lies in a strictly mw- convex domain $D = \left\{\rho\left(z\right) < 0\right\}$, $E \subset\subset D$, then the $\mathcal{P}$-measure $\omega^*(z,E,D)$ $m-wsh$ continues to G, $G \supset \overline{D}$.

Indeed, from the condition $E \subset\subset D$ it follows that there exists a constant $M > 0$ such that $M \cdot \rho\left(z\right) \leqslant -1$, $z \in E$. Hence, $\omega^*(z,E,D) \geqslant M \cdot \rho\left(z\right)$, $z \in D$. Therefore, the function

$$w\left(z\right) = \begin{cases} \omega^*(z,E,D)\ , & if\ z \in D \\ M \cdot \rho\left(z\right)\ , & if\ z \notin D \end{cases}$$

is $m-wsh$ in G which gives continuation of the $\mathcal{P}$-measure in G.

Definition 4. *$z^0 \in K$ is called an mw-regular point of the compact K, if $\omega^*(z^0,K,D) = -1$. $K \subset D$ is called an mw-regular compact if every its point $z^0 \in K$ is mw-regular.*

Note that the regular compacts are always mw-regular. It is well known that if the boundary of a bounded open set G consists of a finite number of twice smooth surfaces, $\overline{G}$ is a regular compact.

This implies that for any compact $K \subset U \subset D$, where U is an open set, there is always an mw-regular compact $E: K \subset E \subset U \subset D$.

The next theorem is very important in the study of $m - wsh$ functions.

Theorem 2. *If $K \subset D$ is an mw-regular compact, then the $\mathcal{P}$–measure $\omega^*(z, K, D) \equiv \omega(z, K, D)$ and is a continuous function in D.*

(It can be proved similarly to the continuity of $\mathcal{P}$-measures in the pluripotential theory).

3. $\mathcal{P}$–capacity

Let $E \subset D$ and $\omega^*(z, E, D)$ be its $\mathcal{P}$-measure. The value

$$\mathcal{P}(E, D) = -\int_D \omega^*(z, E, D)dV$$

is called the $\mathcal{P}$-capacity of the set E with respect to D.

Thus, the $\mathcal{P}$-capacity expresses the capacitive value of the pair (E, D). Such a pair is usually called a condenser in $\mathbb{C}^n$. For $m = n$ the $\mathcal{P}$-capacity was introduced by Sadullayev (see [8]). It has been used in the estimates of volumes of analytic sets, in the descriptions of the family of defective divisor of a holomorphic mapping, etc. (see [8]).

The $\mathcal{P}$-capacity has the following properties:

1) $\mathcal{P}(E, D) = -\int_D \omega(z, E, D)dV$ (this follows from the fact that the set $\{\omega(z, E, D) < \omega^*(z, E, D)\}$ is a polar set. Consequently, it has zero Lebesgue measure);

2) $\mathcal{P}(E, D) \geqslant 0$ and $\mathcal{P}(E, D) = 0$ if and only if E is an mw-polar set in D *(the proof follows from the definition of the $\mathcal{P}$-capacity and the relevant property of $\mathcal{P}$-measure).*

Measurability of the $\mathcal{P}$-capacity $\mathcal{P}(E, D)$ is based on the following theorem.

Theorem 3. *The value $\mathcal{P}(E, D)$ is an increasing and countable sub-additive function of the set: $\mathcal{P}(E_1, D) \leqslant P(E_2, D)$ for $E_1 \subset E_2$ and*

$$P\left(\bigcup_{j=1}^{\infty} E_j, D\right) \leqslant \sum_{j=1}^{\infty} P(E_j, D). \tag{3}$$

Moreover, $\mathcal{P}(E, D)$ is right-continuous, that is for any set $E \subset D$ and for any $\varepsilon > 0$ there exists an open set $U \supset E$ such that $\mathcal{P}(U, D) - \mathcal{P}(E, D) < \varepsilon$

Proof. Monotony of $\mathcal{P}(E, D)$ clearly follows from the monotonicity of the $\mathcal{P}$-measure. The proof of (3) follows from a similar inequality

$$-\omega\left(z, \bigcup_{j=1}^{\infty} E_j, D\right) \leqslant -\sum_{j=1}^{\infty} \omega(z, E_j, D)$$

for $\mathcal{P}$-measure: for any set of $u_j(z) \in U(E_j, D)$ the sum $\sum\limits_{j=1}^{\infty} u_j(z)$ is an $m-wsh$ function in the wide sense (that is, it can also be equal to $-\infty$). Moreover, $\sum\limits_{j=1}^{\infty} u_j(z) \in \mathcal{U}\left(\bigcup_{j=1}^{\infty} E_j, D\right)$ and hence $\sum\limits_{j=1}^{\infty} u_j(z) \leqslant \omega\left(z, \bigcup_{j=1}^{\infty} E_j, D\right)$.

Now

$$\sup\left\{\sum_{j=1}^{\infty}u_j(z):u_j(z)\in\mathcal{U}(E_j,D)\right\}=\sum_{j=1}^{\infty}\sup\left\{u_j(z):u_j(z)\in\mathcal{U}(E_j,D)\right\}=\sum_{j=1}^{\infty}\omega(z,E_j,D),$$

and

$$\sum_{j=1}^{\infty}\omega(z,E_j,D)\leqslant\omega\left(z,\bigcup_{j=1}^{\infty}E_j,D\right).$$

Integrating this inequality and using Levi's theorem, we get

$$-\int\omega\left(z,\bigcup_{j=1}^{\infty}E_j,D\right)dV\leqslant-\sum_{j=1}^{\infty}\int\omega(z,E_j,D)\,dV,$$

$$\mathcal{P}\left(\bigcup_{j=1}^{\infty}E_j,D\right)\leqslant\sum_{j=1}^{\infty}\mathcal{P}(E_j,D).$$

To show the right-continuity of the set-function $\mathcal{P}(E,D)$, we fix the set $E\subset D$ and according to property of $\mathcal{P}$-measure we construct open sets $U_j\supset E$, $U_j\supset U_{j+1}$: $\left[\lim\limits_{j\to\infty}\omega(z,U_j,D)\right]^*\equiv\omega^*(z,E,D)$.

As $\omega(z,U_j,D)$ is increasing, then again by Levi's theorem

$$\lim_{j\to\infty}\mathcal{P}(U_j,D)=-\lim_{j\to\infty}\int\omega(z,U_j,D)\,dV=-\int\lim_{j\to\infty}\omega(z,U_j,D)=$$

$$=-\int\left[\lim_{j\to\infty}\omega(z,U_j,D)\right]^*dV=\mathcal{P}(E,D).$$

Hence, for any $\varepsilon>0$ there is such j_0, when $j\geqslant j_0$ we have the inequality $\mathcal{P}(U_j,D)-\mathcal{P}(E,D)<\varepsilon$. The theorem is proved. □

Corollary 1. *For any decreasing sequence of compacts $K_1\supset K_2\supset\ldots$ we have the equality*

$$\mathcal{P}\left(\bigcup_{j=1}^{\infty}K_j,D\right)=\lim_{j\to\infty}\mathcal{P}(K_j,D)$$

If $G_1\subset G_2\subset\ldots$are the sequences of open sets and $G=\bigcup_{j=1}^{\infty}G_j$ then

$$\mathcal{P}(G,D)=\lim_{j\to\infty}\mathcal{P}(G_j,D).\tag{4}$$

Corollary 2. *The set function $\mathcal{P}(E,D)$ has all the properties of Choquet measurability (see [1, 2]) and therefore any Borel sets are measurable with respect to the capacity $\mathcal{P}$.*

Thus, if $E\subset D$ is a Borel set, then its interior and exterior capacities coincide: ${}_*\mathcal{P}(E,D)=\mathcal{P}^*(E,D)=\mathcal{P}(E,D)$, where $\mathcal{P}^*(E,D)=\inf\{\mathcal{P}(U,D):\ U\supset E\text{ is open}\}$ and

$${}_*\mathcal{P}(E,D)=\sup\{\mathcal{P}(K,D):K\subset E\text{ is compact}\}.$$

Remark. In the classical potential theory and in the pluripotential theory (4) is proved for any arbitrary sequence of sets $E_1\subset E_2\subset\ldots$. It is based on the following problem, which solution is unknown to us: let $\{u_j(z)\}$ be a locally uniformly bounded, monotonically increasing sequence of $m-wsh$ functions and $u(z)=\lim\limits_{j\to\infty}u_j(z)$. Then will there be the set $\{u(z)<u^*(z)\}$ mw-polar in D?

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$\mathcal{P}$-мера в классе $m-wsh$ функций

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В данной работе изучена $\mathcal{P}$-мера и $\mathcal{P}$-емкость в классе $m-wsh$ функций, также доказаны их некоторые свойства.

Ключевые слова: $m-wsh$ функции, $\mathcal{P}$-мера, mw-регулярная точка, $\mathcal{P}$-емкость.