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Gröbner-Shirshov Bases and PBW Theorems

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We review some applications of Gröbner-Shirshov bases, including PBW theorems, linear bases of free universal algebras, normal forms for groups and semigroups, extensions of groups and algebras, embedding of algebras.

Keywords: Gröbner-Shirshov basis, Composition-Diamond lemma, PBW theorem, normal form, group, semigroup, extension.

Introduction

Combinatorial algebra seminar at South China Normal University was organized by the authors in March 2006. Since then, there were some 30 Master Theses and 4 PhD Theses, about 40 published papers in JA, IJAC, PAMS, JPAA, Comm. Algebra, Algebra Coll., Siberian Math. J., Science in China and other Journals and Proceedings. There were organized 2 International Conferences (2007, 2009) with E. Zelmanov as Chairman of the Program Committee and several Workshops. We are going to review some of the papers.

Our main topic is Gröbner-Shirshov bases method for different varieties (categories) of linear (Ω-) algebras over a field k or a commutative algebra K over k: associative algebras (including group (semigroup) algebras), Lie algebras, dialgebras, conformal algebras, pre-Lie (Vinberg right (left) symmetric) algebras, Rota-Baxter algebras, metabelian Lie algebras, L-algebras, semiring algebras, category algebras, etc. There are some applications particularly to new proofs of some known theorems.

As it is well known, Gröbner-Shirshov (GS for short) bases method for a class of algebras based on a Composition-Diamond lemma for the class. A general form of a Composition-Diamond Lemma over a field k is as follows.

Composition-Diamond lemma for a class of algebras Let $M(X)$ be a free algebra of a category M of algebras over a field k, $(N(X), \leqslant)$ a linear basis (normal words) of $M(X)$ with an "admissible" well order and $S \subset M(X)$. Let $Id(S)$ be the ideal of $M(X)$ generated by S and $\bar{s}$ the leading term of the polynomial s. Then the following statements are equivalent.

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(i) S is a GS basis (i.e., each "composition" of polynomials from S is "trivial").

(ii) If $f \in Id(S)$, then the maximal word of f has a form $\bar{f} = (a\bar{s}b),\ s \in S,\ a, b \in N(X)$.

(iii) $Irr(S) = \{u \in N(X) | u \neq (a\bar{s}b), s \in S, a, b \in N(X)\}$ is a linear basis of $M(X|S) = M(X)/Id(S)$.

There are two kinds of compositions, a composition $(s_1, s_2)_w$ of two monic polynomials relative to $w =$ "lcm"$(\bar{s}_1, \bar{s}_2) \in N(X)$ and a composition $Com_w(s)$ of one polynomial relative to $w = \overline{(vs)}$ or $w = \overline{(sv)},\ v,\ w \in N(X)$. Namely, for monic polynomials s_1, s_2,

$$(s_1, s_2)_w = lcm(\bar{s}_1, \bar{s}_2)|_{\bar{s}_1 \to s_1} - lcm(\bar{s}_1, \bar{s}_2)|_{\bar{s}_2 \to s_2},$$

and $Com_w(s) = vs$ or sv correspondingly.

For example, for words $\overline{s_1} = u,\ \overline{s_2} = v \in X^*,\ lcm(u, v) \in \{ucv,\ c \in X^*$ (a trivial lcm); $u = avb,\ a, b \in X^*$ (an inclusion lcm); $ub = av,\ a, b \in X^*,\ deg(ub) < deg(u) + deg(v)$ (an intersection lcm)$\}$. In these cases, $(s_1, s_2)_w = s_1 c\overline{s_2} - \overline{s_1} c s_2,\ s_1 - as_2 b,\ s_1 b - as_2$ correspondingly.

For algebras over a field, a composition $(s_1, s_2)_w$ relative to a trivial lcm w is trivial $mod(s_1, s_2; w)$.

A polynomial f is trivial $mod(S, w)$ if f is a linear combination of "normal S-words" $(asb),\ s \in S,\ a, b \in N(X)$, such that $\overline{(asb)} = (a\bar{s}b) < w$.

The statement $(i) \Rightarrow (ii)$ is the main statement of a Composition-Diamond lemma since others are much easier to prove. For two known cases, conformal algebras and dialgebras, (i) and (ii) are not equivalent.

The case of Lie algebras is an exception since $\bar{s} \notin N(X)$, the maximal word of a Lie polynomial s is the maximal word of s as associative polynomial after working out all Lie brackets.

For algebras over a commutative algebra K one needs to deal with a "double free" algebra $M_{k[Y]}(X)$, a free M-algebra over a polynomial algebra. In this case

$$lcm(u^Y u^X, v^Y v^X) = lcm(u^Y, v^Y) lcm(u^X, v^X)$$

and there are generally infinitely many compositions for given s_1, s_2. It is since we need to use a "trivial" $lcm(u^X, v^X) = (u^X)c^X(v^X)$. For algebras over a field, the composition corresponding to a trivial $(u^X)c^X(v^X)$ is trivial. But for algebras over a commutative algebra it is not the case if $lcm(u^Y, v^Y) \neq u^Y v^Y$.

Recently some new Composition-Diamond lemmas are given: for tensor product free algebras $k\langle X\rangle \otimes k\langle Y\rangle$ [13], for Lie algebras over commutative algebras [14], for metabelian Lie algebras [42], for semirings [24], for Rota-Baxter algebras [16], for L-algebras [17], for Vinberg-Koszul-Gerstenhaber right-symmetric (pre-Lie) algebras [18], for categories [20], for dialgebras [22], for associative algebras with multiple operations [25], for associative n-conformal algebras [27], for associative conformal algebras [29], for Lie superalgebras [31], for differential algebras [43], for λ-differential associative algebras with multiple operators [60], for commutative algebras with multiple operators and free commutative Rota-Baxter algebras [61], etc.

By using the above and the known Composition-Diamond lemmas, some applications are obtained: for embeddings of algebras [23, 52], for free inverse semigroups [28], for conformal algebras [30], for relative Gröbner-Shirshov bases of algebras and groups [36], for extensions of groups and algebras [40, 41], for some word problems [10, 44], for some Lie algebras [47], for partially commutative Lie algebras [50, 59], for braid groups [7–9, 48, 56], for PBW theorems [14, 18, 22, 35, 45, 46, 49], etc.

For development of Gröbner-Shirshov bases, one may refer to surveys: [12, 26, 32–34].

1. Gröbner-Shirshov bases for associative algebras and Lie algebras

Gröbner-Shirshov bases for Lie algebras is established by Shirshov [63,64] for the free Lie algebras (with deg-lex order) in 1962 (see also Bokut [5]). In 1976, Bokut [6] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For commutative polynomials, this is due to Buchberger [37,38].

1.1. Composition-Diamond lemma for associative algebras

Let $k\langle X\rangle$ be the free associative algebra over a field k generated by X and $(X^*,<)$ a well-ordered free monoid generated by X, $S\subset k\langle X\rangle$ such that every $s\in S$ is monic (s is monic if the coefficient of the leading word of s is 1).

Let us prove $(i)\Rightarrow(ii)$ and define a GS basis.

Let $f=\sum\limits_{i=1}^{n}\alpha_i a_i s_i b_i\in Id(S)$ where each $\alpha_i\in k,\ a_i,b_i\in X^*,\ s_i\in S$. Let $w_i=a_i\overline{s_i}b_i,\ w_1=w_2=\cdots=w_l>w_{l+1}\geqslant\cdots$.

For $l=1$, it is ok.

For $l>1,\ w_1=a_1\overline{s_1}b_1=a_2\overline{s_2}b_2$, common multiple of $\overline{s_1},\overline{s_2}$, by definition,

$$w_1=cwd,\ w=\text{“}lcm\text{”}(\overline{s_1},\overline{s_2}),\ a_is_ib_i=w|_{\overline{s_i}\mapsto s_i},\ i=1,2,$$

where $lcm(u,v)\in\{ucv,c\in X^*(a\ trivial\ lcm(u,v));\ u=avb,\ a,b\in X^*\ (an\ inclusion\ lcm(u,v));\ ub=av,\ a,b\in X^*,\ |ub|<|u|+|v|\ (an\ intersection\ lcm(u,v)\}$.

Then $a_1s_1b_1-a_2s_2b_2=c(w|_{\overline{s_1}\mapsto s_1}-w|_{\overline{s_2}\mapsto s_2})d=c(s_1,s_2)_wd$. By definition of GS basis, $(s_1,s_2)_w\equiv 0\ \ mod(S,w)$. So, $a_1s_1b_1-a_2s_2b_2\equiv 0\ \ mod(S,w_1)$. We can decrease l. By induction on l and w_1, $\bar f=a\bar sb,\ a,b\in X^*,\ s\in S$.

1.2. Composition-Diamond lemma for Lie algebras over a field

Let $S\subset Lie(X)\subset k\langle X\rangle$ be a nonempty set of monic Lie polynomials, $(X^*,<)$ deg-lex order, $\bar s$ means the maximal word of s as non-commutative polynomial. Then compositions are defined as follows

$$\langle s_1,s_2\rangle_w=[w|_{\overline{s_1}\mapsto s_1}]_{\overline{s_1}}-[w|_{\overline{s_2}\mapsto s_2}]_{\overline{s_2}},\quad w\in ALSW(X)$$

associative composition with the special Shirshov bracketing, where $ALSW(X)$ is the set of associative Lyndon-Shirshov words on X.

Composition-Diamond lemma for Lie algebras over a field ([11,63,64]). *The following statements are equivalent.*

(i) S is a Lie GS basis in $Lie(X)$ (any composition is trivial modulo (S,w)).

(ii) $f\in Id_{Lie}(S)\Rightarrow\bar f=a\bar sb$ for some $s\in S$ and $a,b\in X^*$.

(iii) $Irr(S)=\{[u]\in NLSW(X)\mid u\neq a\bar sb,\ s\in S,\ a,b\in X^*\}$ is a linear basis for $Lie(X|S)$.

2. PBW theorems

There are 8 PBW (Poincare-Birkhoff-Witt) theorems that are proved by using GS bases and Composition-Diamond lemmas.

2.1. Lie algebras–associative algebras (Shirshov)

Let $L = Lie_k(X|S)$ be an arbitrary Lie algebra with generators X and defining relations S and $U(L) = k\langle X|S^{(-)}\rangle$ the universal enveloping associative algebra of L. Then

(i) S is a Lie GS basis $\Leftrightarrow$ S is an associative GS basis.

(ii) In this case, a linear basis of $U(L)$ is

$$u_1u_2\cdots u_t, \quad u_1 \preceq u_2 \preceq \cdots \preceq u_t \ \text{(lex-order)}, \quad u_i \in Irr(S) \cap ALSW(X).$$

One uses Shirshov factorization theorem:

$$u \in X^*, \ \exists!\ u = u_1\cdots u_t, \quad u_1 \preceq \cdots \preceq u_t, \ u_i \in ALSW(X).$$

2.2. Lie algebras–pre-Lie algebras (Segal)

Pre-Lie algebras are defined by an identity $(x, y, z) = (x, z, y)$, where $(x, y, z) = (xy)z - x(yz)$.

Let $X = \{x_i|i \in I\}$ be a linear basis of the Lie algebra L and $[x_i, x_j] = \sum \alpha_{ij}^t x_t =: \{x_i, x_j\}$ the multiplication table of the linear basis X. Then L has a presentation

$$L = Lie(x_i, i \in I|[x_i, x_j] = \{x_i, x_j\},\ i, j \in I) = Lie(X|S)$$

and the universal enveloping pre-Lie algebra of L

$$U_{\text{pre-Lie}}(L) = \text{pre-Lie}(X|S^{(-)}).$$

Then $S^{(-)}$ is a GS basis of $U_{\text{pre-Lie}}(L)$, $L \subset U_{\text{pre-Lie}}(L)$ and $Irr(S)$ is a linear basis of $U_{\text{pre-Lie}}(L)$ by Composition-Diamond lemma for pre-Lie algebras (Bokut-Chen-Li [18]).

2.3. Leibniz algebras–dialgebras (Aymon, Grivel)

Dialgebra: $a \dashv (b \vdash c) = a \dashv b \dashv c$, $(a \dashv b) \vdash c = a \vdash b \vdash c$, $a \vdash (b \dashv c) = (a \vdash b) \dashv c$ and $\vdash, \dashv$ associative.

Leibniz identity: $[[a, b], c] = [[a, c], b] + [a, [b, c]]$.

Di-commutator: $[a, b] = a \dashv b - b \vdash a$.

$$L = Lei(x_i, i \in I|[x_i, x_j] = \{x_i, x_j\},\ i, j \in I),$$
$$U_{Dialg}(L) = D(X|S^{(-)}).$$

A GS basis for $U_{Dialg}(L)$ is given by Bokut-Chen-Liu [22] and then a linear basis for $U_{Dialg}(L)$ by Composition-Diamond lemma for dialgebras which implies $L \subset U_{Dialg}(L)$.

2.4. Akivis algebras–non-associative algebras (Shestekov)

Any nonassociative algebra is an Akivis algebra relative the commutator $[x, y] = xy - yx$ and the associator $(x, y, z) = (xy)z - x(yz)$.

Akivis identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = (x, y, z) + (z, x, y) + (y, z, x) - (x, z, y) - (y, x, z) - (z, y, x)$.

$$A = A(x_i, i \in I|[x_i, x_j] = \{x_i, x_j\},\ (x_i, x_j, x_t) = \{x_i, x_j, x_t\},\ i, j, t \in I),$$
$$U(A) = k\{X|S^{(-)}\},$$
$$S^{(-)} = \{[x_i, x_j] = \{x_i, x_j\},\ (x_i, x_j, x_t) = \{x_i, x_j, x_t\},\ i, j, t \in I\}.$$

A GS basis of $U(A)$ is given by Chen-Li [46] and then $A \subset U(A)$.

2.5. Sabinin algebras–modules (Perez-Izquierdo)

Let $(V, \langle ; \rangle)$ be a Sabinin algebra,

$$\begin{aligned} \widetilde{S}(V) &= T(V)/span\langle xaby - xbay + \sum x_{(1)}\langle x_{(2)}; a, b\rangle y | x, y \in T(V), a, b \in V\rangle \\ &\cong mod\langle X|I\rangle_{k\langle X\rangle} \text{ as } k\langle X\rangle\text{-modules} \end{aligned}$$

the universal enveloping module for V, where $I = \{xab - xba + \sum x_{(1)}\langle x_{(2)}; a, b\rangle | x \in X^*, a > b,\ a, b \in X\}$.

Then I is a GS basis (Chen-Chen-Zhong [45]) and then $V \subset \widetilde{S}(V)$.

2.6. Dendriform algebras–Rota-Baxter algebras (Chen-Mo, Kolesnikov)

Rota-Baxter identity:

$$P(x)P(y) = P(P(x)y) + P(xP(y)), \forall x, y \in A.$$

Dendriform identities: $(x \prec y) \prec z = x \prec (y \prec z + y \succ z)$, $(x \succ y) \prec z = x \succ (y \prec z)$, $(x \prec y + x \succ y) \succ z = x \succ (y \succ z)$.

$$\begin{aligned} D &= Den(X|x_i \prec x_j = \{x_i \prec x_j\},\ x_i \succ x_j = \{x_i \succ x_j\}, x_i, x_j \in X); \\ U(D) &= RB(X|x_iP(x_j) = \{x_i \prec x_j\},\ P(x_i)x_j = \{x_i \succ x_j\}, x_i, x_j \in X). \end{aligned}$$

Then $D \subset U(D)$, see Chen-Mo [49].

2.7. Shirshov's, Cartier's, Cohn's counter examples to PBW for Lie algebras over commutative algebra

Shirshov [62] 1953 and Cartier [39] 1958 give counter examples to PBW for Lie algebras over commutative algebra. Cohn [57] 1963 posts the conjecture:

$$\begin{aligned} \mathcal{L}_p &= Lie_K(x_1, x_2, x_3 \mid y_3x_3 = y_2x_2 + y_1x_1), \\ K &= k[y_1, y_2, y_3 | y_i^p = 0, i = 1, 2, 3]. \end{aligned}$$

$\mathcal{L}_p$ can not be embedded into its universal enveloping associative algebra.

Bokut-Chen-Chen [14] establish GS bases theory for Lie algebras over a commutative algebra. We prove Cohn's conjecture is correct for $p = 2, 3, 5$.

2.8. "1/2 PBW theorem" (Bokut-Fong-Ke)

Let k be a field with the characteristic 0 and C a k-algebra with operations $a(n)b,\ n \geqslant 0$, and $D(a)$. Then C is called a conformal algebra, if

(1) (the locality condition). For any $a, b \in C$, there exist $N(a, b) \geqslant 0$ such that $a(n)b = 0$ for $n \geq N(a, b)$.

(2) $D(a(n)b) = D(a)(n)b + a(n)D(b)$ and $D(a)(n)b = -na(n-1)b$.

Let C be a conformal algebra. Then C is an associative conformal algebra if

$$(As)\quad (a(n)b)(m)c = \sum_{s\geqslant 0}(-1)^s C_n{}^s a(n-s)(b(m+s)c);$$

C is a Lie conformal algebra if

$$(Lie)\ \ \text{(anti-commutativity)}\ \ (a[n]b) = \{b[n]a\},$$
$$\text{where}\ \ \{b[n]a\} = \sum_{s\geq 0}(-1)^{n+s}D^{(s)}(b[n+s]a),\ \ D^{(s)} = 1/s!D^s,$$
$$\text{(Jacoby)}\ \ ((a[n]b)[m]c) = \sum_{s\geq 0}(-1)^s C_n^s((a[n-s](b[m+s]c)) - (b[m+s](a[n-s]c))).$$

Gröbner-Shirshov bases for conformal associative algebras (n-conformal associative algebras) are established in Bokut-Fong-Ke [29] and Bokut-Chen-Zhang [27].

There is a 1/2 PBW-theorem for Lie conformal algebras: Let

$$L = LieCon(\{a_i,\ i\in I\}, N|a_i[n]a_j = \sum \alpha_{ij}^k a_k,\ i\geqslant j,\ n<N)$$

be a Lie conformal algebra with the linear basis $\{a_i,\ i\in I\}$ over $k[D]$ and the locality N. Let

$$U(L) = AsCon(\{a_i,\ i\in I\}, N|S = \{s_{ij}^n = a_i(n)a_j - \{a_j(n)a_i\} - a_i[n]a_j,\ i\geqslant j,\ n<N\})$$

be the universal enveloping associative conformal algebra for L.

Then any composition $(s_{ij}^n, s_{jk}^m)_w = s_{ij}^n(m)a_k - a_i(n)s_{jk}^m,\ i>j>k,\ n,m<N$ is trivial $mod(S,w),\ w = a_i(n)a_j(m)a_k$.

There is also a 1/2 PBW theorem between n-conformal Lie algebras and n-conformal associative algebras, see [27].

3. Linear bases of free universal algebras

–Bases of free Lie algebras

M. Hall and A.I. Shirshov use constructions and check axioms.

One may use anti-commutative Gröbner-Shirshov bases for a free Lie algebra.

Hall basis (Bokut-Chen-Li [19]): $Lie(X) = AC(X|S_1)$, S_1 is a anti-commutative GS basis, $Irr(S_1)$ =Hall basis in X.

Lyndon-Shirshov basis (Bokut-Chen-Li [21]): $Lie(X) = AC(X|S_2)$, S_2 is a anti-commutative GS basis, $Irr(S_2)$ =Lyndon-Shirshov basis in X.

–Loday basis of a free dialgebra

$D(X) = L(X|S)$, L-identity: $(a\vdash b)\dashv c = a\vdash(b\dashv c)$, S an L-GS basis (GS basis as L-algebra), $Irr(S)$ =Loday basis in X (Bokut-Chen-Huang [17]).

–Bases of a free dentriform algebra

$Den(X) = L(X|S)$, $Irr(S)$=a linear basis of $Den(X)$ (Bokut-Chen-Huang [17]).

–Bases of a free Rota-Baxter algebra (Rota, Cartier)

Via GS method for Ω-algebras (Bokut-Chen-Qiu [25]).

–Free inverse semigroup (Polyakova-Schain)

An associative GS basis is given by (Bokut-Chen-Zhao [28]), $Irr(S)$ is a normal form of free inverse semigroup.

–Free idempotent semigroup (Chen-Yang [53]).

4. Normal forms for groups and semigroups

–Braid groups
in Artin-Burau generators (Bokut-Chanikov-Shum [9]);
in Artin-Garside generators (Bokut [7]);
in Birman-Ko-Lee generators (Bokut [8]);
in Adyan-Thurston generators (Chen-Zhong [56]).

–Chinese monoid (Chen-Qiu [51])

–Plactic monoid (Bokut-Chen-Chen-Li [15]).

–HNN extension Britton Lemma and Lyndon-Schupp normal form lemma for an HNN-extension of a group was proved using an associative Composition-Diamond lemma relative to a "S-partially" monomial order of words (Chen-Zhong [54]).

–One-relator groups In (Chen-Zhong [55]), some one-relator groups were studying by means of groups with the standard normal forms (the standard GS bases) in the sense (Bokut [3,4]). It is known that any one-relator group can be effectively embedded into 2-generator one-relator group $G = gp\langle x, y | x^{i_1} y^{j_1} \dots x^{i_k} y^{j_k} \rangle,\ k \geqslant 1$ is the depth. It is proved that a group G of depth $\leqslant 3$ is computably embeddable into a Magnus-Moldavanskii tower of HNN-extensions with the standard normal form of elements. There are quite a lot of examples that support an old conjecture that the result is valid for any depth.

5. Schreier extensions of groups and algebras

In (Chen [40]), it is dealing with a Schreier extension

$$1 \to A \to C \to B \to 1$$

of a group A by B. M. Hall [58] mentioned that for any B it is difficult to find an analogous conditions. Actually this problem was solved in [40] using the GS bases technique. As applications there were given above conditions for cyclic and free abelian cases, as well as for the case of HNN-extensions.

The same kind of result was established for Schreier extensions of associative algebras (Chen [41]).

6. Embedding of algebras

In Bokut-Chen-Mo [23], we were dealing with the problem of embedding of countably generated associative and Lie algebras, groups, semigroups, Ω-algebras into (simple) 2-generated ones. We proved some known results (of Higman-Neuman-Neuman, Evance, Malcev, Shirshov) and some new ones using GS bases technique. For example

Theorem 1. *Every countable Lie algebra is embeddable into simple 2-generated Lie algebra.*

Theorem 2. *Every countable differential algebra is embeddable into a simple 2-generated differential algebra.*

G. Bergman (Private communication, 2013 [2]) gave an idea how to avoid the restriction on cardinality of the ground field. Now Qiuhui Mo proved that the Bergman's idea works.

References

[1] G.M.Bergman, The diamond lemma for ring theory, *Adv. in Math.*, **29**(1978), 178–218.

[2] G.M.Bergman, Privite communication, 2013.

[3] L.A.Bokut, On one property of the Boone group, *Algebra Logika*, **5**(1966), 5–23 (in Russian).

[4] L.A.Bokut, On the Novikov groups, *Algebra Logika*, **6**(1967), 25–38 (in Russian).

[5] L.A.Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.*, **36**(1972), 1173–1219 (in Russian).

[6] L.A.Bokut, Imbeddings into simple associative algebras, *Algebra Logika*, **15**(1976), 117–142 (in Russian).

[7] L.A.Bokut, Gröbner-Shirshov bases for braid groups in Artin-Garside generators, *J. Symbolic Computation*, **43**(2008), 397–405.

[8] L.A.Bokut, Gröbner-Shirshov bases for the braid group in the Birman-Ko-Lee generators, *J. Algebra*, **321**(2009), 361–379.

[9] L.A.Bokut, V.V.Chainikov, K.P.Shum, Markov and Artin normal form theorem for braid groups, *Commun. Algebra*, **35**(2007), 2105–2115

[10] L.A.Bokut, V.Chainikov, Gröbner-Shirshov bases of Adjan extension of the Novikov group, *Discrete Mathematics*, 2008.

[11] L.A.Bokut, Y.Q.Chen, Gröbner-Shirshov bases for Lie algebras: after A.I. Shirshov, *Southeast Asian Bull. Math.*, **31**(2007), 1057–1076

[12] L.A.Bokut, Y.Q.Chen, Gröbner-Shirshov bases: some new results, Advance in Algebra and Combinatorics. Proceedings of the Second International Congress in Algebra and Combinatorics, Eds. K. P. Shum, E. Zelmanov, Jiping Zhang, Li Shangzhi, World Scientific, 2008, 35–56.

[13] L.A.Bokut, Y.Q.Chen, Y.S.Chen, Composition-Diamond lemma for tensor product of free algebras, *J. Algebra*, **323**(2010), 2520–2537.

[14] L.A.Bokut, Y.Q.Chen, Y.S.Chen, Gröbner-Shirshov bases for Lie algebras over a commutative algebra, *J. Algebra*, **337**(2011), 82–102.

[15] L.A.Bokut, Y.Q.Chen, W.P.Chen, J.Li, Gröbner-Shirshov bases for plactic monoids, Preprint, 2012.

[16] L.A.Bokut, Y.Q.Chen, X.M.Deng, Gröbner-Shirshov bases for Rota-Baxter algebras, *Siberian Math. J.*, **51** (2010), 978–988.

[17] L.A.Bokut, Y.Q.Chen, J.P.Huang, Gröbner-Shirshov bases for L-algebras, *Internat. J. Algebra Comput.*, **23**(2013), 547–571.

[18] L.A.Bokut, Y.Q.Chen, Y.Li, Gröbner-Shirshov bases for Vinberg-Koszul-Gerstenhaber right-symmetric algebras, *J. Math. Sci.*, **166**(2010), 603–612.

[19] L.A.Bokut, Y.Q.Chen, Y.Li, Anti-commutative Gröbner-Shirshov basis of a free Lie algebra, *Science in China Series A: Mathematics*, **52**(2009), 244–253.

[20] L.A.Bokut, Y.Q.Chen, Y.Li, Gröbner-Shirshov bases for categories, *Nankai Series in Pure, Applied Mathematics and Theoretical Physical, Operads and Universal Algebra*, **9**(2012), 1–23.

[21] L.A.Bokut, Y.Q.Chen, Y.Li, Lyndon-Shirshov words and anti-commutative algebras, *J. Algebra*, **378**(2013), 173–183.

[22] L.A.Bokut, Y.Q.Chen, C.H.Liu, Gröbner-Shirshov bases for dialgebras, *Internat. J. Algebra Comput.*, **20**(2010), 391–415.

[23] L.A.Bokut, Y.Q.Chen, Q.H.Mo, Gröbner-Shirshov bases and embeddings of algebras, *Internat. J. Algebra Comput.*, **20**(2010), 875–900.

[24] L.A.Bokut, Y.Q.Chen, Q.H.Mo, Gröbner-Shirshov bases for semirings, *J. Algebra*, **378** (2013), 47–63.

[25] L.A.Bokut, Y.Q.Chen, J.J.Qiu, Gröbner-Shirshov bases for associative algebras with multiple operations and free Rota-Baxter algebras, *J. Pure Applied Algebra*, **214**(2010), 89–100.

[26] L.A.Bokut, Y.Q.Chen, K.P.Shum, Some new results on Gröbner-Shirshov bases, Proceedings of International Conference on Algebra 2010, Advances in Algebraic Structures, 2012, 53–102.

[27] L.A.Bokut, Y.Q.Chen, G.L.Zhang, Composition-Diamond lemma for associative n-conformal algebras, arXiv:0903.0892.

[28] L.A.Bokut, Y.Q.Chen, X.G.Zhao, Gröbner-Shirshov beses for free inverse semigroups, *Internat. J. Algebra Comput*, **19**(2009), 129–143.

[29] L.A.Bokut, Y.Fong, W.-F.Ke, Composition-Diamond lemma for associative conformal algebras, *J. Algebra*, **272**(2004), 739–774.

[30] L.A.Bokut, Y.Fong, W.-F.Ke, P.S.Kolesnikov, Gröbner and Gröbner-Shirshov bases in algebra and conformal algebras, *Fundamental and Applied Mathematics*, **6**(2000), 669–706.

[31] L.A.Bokut, S.-J.Kang, K.-H.Lee, P.Malcolmson, Gröbner-Shirshov bases for Lie superalgebras and their universal enveloping algebras, *J. Algebra*, **217**(1999), 461–495.

[32] L.A.Bokut, P.S.Kolesnikov, Gröbner-Shirshov bases: from their incipiency to the present, *J. Math. Sci.*, **116**(2003), 2894–2916.

[33] L.A.Bokut, P.S.Kolesnikov, Gröbner-Shirshov bases, conformal algebras and pseudo-algebras, *J. Math. Sci.*, **131**(2005), 5962–6003.

[34] L.A.Bokut, P.Malcolmson, Gröbner-Shirshov bases for Lie and associative algebras, Collection of Abstracts, ICAC,97, Hong Kong, 1997, 139–142.

[35] L.A.Bokut, P.Malcolmson, Gröbner-Shirshov bases for relations of a Lie algebra and its enveloping algebra. Shum, Kar-Ping (ed.) et al., Algebras and combinatorics. Papers from the international congress, ICAC'97, Hong Kong, August 1997, Singapore, Springer, 1999, 47–54.

[36] L.A.Bokut, K.P.Shum, Relative Gröbner-Shirshov bases for algebras and groups, *St. Petersbg. Math. J.*, **19**(2008), 867–881.

[37] B.Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal, Ph.D. thesis, University of Innsbruck, Austria, 1965.

[38] B.Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, *Aequationes Math.*, **4**(1970), 374–383.

[39] P.Cartier, Remarques sur le théorème de Birkhoff-Witt, *Annali della Scuola Norm. Sup. di Pisa,* série III, **12**(1958), 1–4.

[40] Y.Q.Chen, Gröbner-Shirshov basis for Schreier extensions of groups, *Commun. Algebra*, **36**(2008), 1609–1625

[41] Y.Q.Chen, Gröbner-Shirshov basis for extensions of algebras, *Algebra Colloq.*, **16**(2009) 283–292.

[42] Y.S.Chen, Y.Q.Chen, Gröbner-Shirshov bases for matabelian Lie algebras, *J. Algebra*, **358**(2012), 143–161.

[43] Y.Q.Chen, Y.S.Chen, Y.Li, Composition-Diamond lemma for differential algebras, *The Arabian Journal for Science and Engineering*, **34**(2009), 135–145.

[44] Y.Q.Chen, W.S.Chen, R.I.Luo, Word problem for Novikov's and Boone's group via Gröbner-Shirshov bases, *Southeast Asian Bull. Math.*, **32**(2008), 863–877.

[45] Y.Q.Chen, Y.S.Chen, C.Y.Zhong, Composition-Diamond lemma for modules, *Czechoslovak Math. J.*, **60**(2010), 59–76.

[46] Y.Q.Chen, Y.Li, Some remarks for the Akivis algebras and the Pre-Lie algebras, *Czechoslovak Math. J.*, **61**(136)(2011), 707–720.

[47] Y.Q.Chen, Y.Li, Q.Y.Tang, Gröbner-Shirshov bases for some Lie algebras, Preprint, 2012.

[48] Chen, Y.Q., Mo, Q.H.: Artin-Markov normal form for braid group, *Southeast Asian Bull. Math.*, **33**(2009), 403–419.

[49] Y.Q.Chen, Q.H.Mo, Embedding dendriform algebra into its universal enveloping Rota-Baxter algebra, *Proc. Am. Math. Soc.*, **139**(2011), 4207–4216.

[50] Y.Q.Chen, Q.H.Mo, Gröbner-Shirshov bases for free partially commutative Lie algebras, *Commun. Algebra*, **41**(2013), 3753–3761.

[51] Y.Q.Chen, J.J.Qiu, Gröbner-Shirshov basis for the Chinese monoid, *Journal of Algebra and its Applications*, **7**(2008), 623–628.

[52] Y.Q.Chen, H.S.Shao, K.P.Shum, On Rosso-Yamane theorem on PBW basis of $U_q(A_N)$, *CUBO A Mathematical Journal*, **10**(2008), 171–194.

[53] Y.Q.Chen, M.M.Yang, A Gröbner-Shirshov basis for free idempoten semigroup, Preprint, 2012.

[54] Y.Q.Chen, C.Y.Zhong, Gröbner-Shirshov basis for HNN extensions of groups and for the alternative group, *Commun. Algebra*, **36**(2008), 94–10.

[55] Y.Q.Chen, C.Y.Zhong, Gröbner-Shirshov basis for some one-relator groups, *Algebra Colloq*, **19**(2011), 99–116.

[56] Y.Q.Chen, C.Y.Zhong, Gröbner-Shirshov bases for braid groups in Adyan-Thurston generators, *Algebra Colloq*, **20**(2013), 309–318.

[57] Cohn, P.M. A remark on the Birkhoff-Witt theorem, *Journal London Math. Soc.*, **38**(1963), 197-203.

[58] Jr.Marshall Hall, The Theory of Groups, The Macmillan Company, 1959.

[59] E.N.Poroshenko, Bases for partially commutative Lie algebras, *Algebra Logika*, **50**(2011), 405–417 (in Russian).

[60] J.J.Qiu, Y.QChen, Composition-Diamond lemma for λ-differential associative algebras with multiple operators, *Journal of Algebra and its Applications*, **9**(2010), 223–239 .

[61] J.J.Qiu, Gröbner-Shirshov bases for commutative algebras with multiple operators and free commutative Rota-Baxter algebras, *Asian-European Jour. Math.*, to appear.

[62] A.I.Shirshov, On the representation of Lie rings in associative rings, *Uspekhi Mat. Nauk N. S.*, **8**(1953), (5)(57), 173–175 (in Russian).

[63] A.I.Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Zh.*, **3**(1962), no. 2, 292–296; English translation in *SIGSAM Bull*, **33**(1999), 3–6 .

[64] Selected works of A.I. Shirshov, Eds. Bokut, L.A., Latyshev, V., Shestakov, I., Zelmanov, E., Bremner, Trs.M., Kochetov, M. Birkhäuse, Basel, Boston, Berlin, 2009.

Базисы Грёбнера–Ширшова и PBW-теоремы

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Статья является обзором некоторых приложений Грёбнера–Ширшова базисов, включая PBW теоремы, линейные базисы свободных универсальных алгебр, нормальных форм для групп и полугрупп, расширения групп и алгебр, вложения алгебр.

Ключевые слова: базис Грёбнева–Ширшова, лемма о композиции-диаманте, PBW-теорема, нормальная форма, группа, полугруппа, расширение.