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Explicit Variational Formulas for Third-order Equations on Riemann Surfaces

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In this paper we deduce explicit variational formulas for the solution of an ordinary differential equation of third order and its monodromy group with respect to a variation in the space of cubic holomorphic differentials on a compact Riemann surface.

Keywords: holomorphic cubic differentials, compact Riemann surface, variational formulas, differential equation of third order, monodromy group.

Introduction

In this paper we deduce explicit variational formulas for the solution of an ordinary differential equation of third order and its monodromy group with respect to a variation in the space of cubic holomorphic differentials on a compact Riemann surface of genus $g \geqslant 2$. Since there are no explicit solutions even for second-order equations, there appeared Hejhal's approach: to understand how the generators of the monodromy group change with a small variation of the coefficients in the spaces of holomorphic differentials. Also we will find out how these objects are related to the matrix Prym differentials and to holomorphic sections of some vector bundles on compact Riemann surfaces. Variational formulas find applications in geometric theory of functions of a complex variable and in the theory of Teichmüller spaces in connection with the uniformization of compact Riemann surfaces.

In the paper, using vector and matrix notation, we present a method for deducing explicit variational formulas for the solution vector and the monodromy group with respect to a variation in the space of cubic holomorphic differentials. These theorems extend the results of D. Hejhal [1, 2] and V.V. Chueshev [3] to the case of third-order equations.

Preliminaries

Let F be a compact Riemann surface of genus $g \geqslant 2$ and D an open disk on the complex plane $\mathbb{C}$. Use Γ to denote the Fuchsian group of the first type uniformizing F on D, so F is conformally equivalent to D/Γ. Consider an ordinary linear differential equation

$$\frac{d^n v}{dt^n} + q_2(t)\frac{d^{n-2}v}{dt^{n-2}} + q_3(t)\frac{d^{n-3}v}{dt^{n-3}} + \cdots + q_n(t)v = 0,\ t \in D, \tag{1}$$

where $q_j(t)$ are meromorphic functions on D, $j = 2, ..., n$. We shall say that equation (1) is of Fuchsian type on F if it has only regular singular points and is preserved under the change of variable:

$$(t, v) \to (s, \omega), \quad s = L(t), \quad \omega = \omega(s)L'(t)^{\frac{n-1}{2}},\ L \in \Gamma. \tag{2}$$

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The solution vector is the column vector consisting of all linearly independent solutions of the equation. A holomorphic differential of order q has the form $\Phi(z)dz^q$ and is invariant under the change of coordinates on the surface, i. e. $\Phi(Lz)L'(z)^q = \Phi(z),\ z \in D,\ L \in \Gamma$. We denote by $\Omega^q(F)$ the vector space of holomorphic $q-$differentials on D/Γ, $q \in \mathbb{N}$.

Lemma 1 ([1]). *An ordinary linear differential equation* $\dfrac{d^3\upsilon(t)}{dt^3} + q_2(t)\dfrac{d\upsilon(t)}{dt} + q_3(t)\upsilon(t) = 0$ *is defined on* D/Γ *iff* $q_2(Lt)L'(t)^2 = q_2(t),\ q_3(Lt)L'(t)^3 = q_3(t),\ q_2'(t) \in \Omega^3(\Gamma),\ t \in D,\ L \in \Gamma.$

Lemma 2 ([1]). *Let the column vector* $U(t)$ *consist of* n *linearly independent solutions to equation (1) on* $F = D/\Gamma$. *Then the equality*

$$U(Lt) = \chi(L)U(t)\xi_L(t)^{n-1},\ \xi_L(t) = \sqrt{L'(t)},\ L \in \Gamma, \tag{3}$$

defines uniquely the homomorphism $\chi : \Gamma \to GL(n, \mathbb{C})$ *given by* $L \to \chi(L),\ L \in \Gamma$.

The homomorphism χ in equality (3) is called the monodromy homomorphism for the solution vector $U(t)$. The image $\chi(\Gamma)$ of the group Γ is the monodromy group for differential equation (1). It is well known that the monodromy homomorphism χ is a non-unitary irreducible representation of Γ in $GL(n, \mathbb{C})$. Moreover, the monodromy group $\chi(\Gamma)$ for $U(t)$ on the surface D/Γ of genus $g \geqslant 2$ is infinite [1].

1. Taylor expansions for the solution vector and elements of the monodromy group

Consider a perturbed equation

$$u^{(3)}(z) + Q_0(z)u^{(1)}(z) + (R_0(z) - \mu r(z))u(z) = 0, \tag{4}$$

where $Q_0(z)dz^2$, $R_0(z)dz^3$, and $r(z)dz^3$ are fixed holomorphic quadratic and cubic abelian differentials on D/Γ, respectively, and $\mu \in \mathbb{C},\ |\mu| < \varepsilon$ for a sufficiently small ε.

Denote by $U(z,\mu) = {}^t(u(z,\mu),\, v(z,\mu),\, w(z,\mu))$ the vector of linearly independent solutions of the Cauchy problem with the condition

$$U(z_0,\mu) = {}^t(1,0,0),\ U'(z_0,\mu) = {}^t(0,1,0), U''(z_0,\mu) = {}^t(0,0,1) \tag{5}$$

for every μ, where $z_0 \in D$.

Introduce shorter matrix notation:

$$U(z) = \begin{pmatrix} u(z) \\ \upsilon(z) \\ \omega(z) \end{pmatrix} = \begin{pmatrix} u(z) & 0 & 0 \\ 0 & \upsilon(z) & 0 \\ 0 & 0 & \omega(z) \end{pmatrix},$$

$$U_m(z) = \begin{pmatrix} u_m(z) \\ \upsilon_m(z) \\ \omega_m(z) \end{pmatrix} = \begin{pmatrix} u_m(z) & 0 & 0 \\ 0 & \upsilon_m(z) & 0 \\ 0 & 0 & \omega_m(z) \end{pmatrix}$$

for every $m \geqslant 1$. By Poincaré's small parameter method and the Cauchy-Kowalevski theorem we have a Taylor expansion in μ for the solution vector: $U(z,\mu) = U(z) + \mu\, U_1(z) + \cdots + \mu^m U_m(z) + \cdots$. By substituting this series in (4), we obtain the identity with respect to μ

$$0 \equiv U^{(3)}(z) + \mu U_1^{(3)}(z) + \mu^2 U_2^{(3)}(z) + \cdots + \mu^m U_m^{(3)}(z) + \cdots +$$

$$+Q_0(z)U^{(1)}(z) + \mu Q_0(z)U_1^{(1)}(z) + \mu^2 Q_0(z)U_2^{(1)}(z) + \cdots + \mu^m Q_0(z)U_m^{(1)}(z) + \cdots$$

$$+R_0(z)U(z) + \mu R_0(z)U_1(z) + \mu^2 R_0(z)U_2(z) + \cdots + \mu^m R_0(z)U_m(z) + \cdots$$

$$-\mu r(z)U(z) - \mu^2 r(z)U_1(z) - \mu^3 r(z)U_2(z) - \cdots - \mu^{m+1} r(z)U_m(z) + ...$$

Thus, we have an infinite system of vector equations

$$U^{(3)}(z) + Q_0(z)U^{(1)}(z) + R_0(z)U(z) = 0,$$

$$U_1^{(3)}(z) + Q_0(z)U_1^{(1)}(z) + R_0(z)U_1(z) = r(z)U(z),$$

$$U_m^{(3)}(z) + Q_0(z)U_m^{(1)}(z) + R_0(z)U_m(z) = r(z)U_{m-1}(z),$$

where $U_m(z_0) = U_m^{(n)}(z_0) = {}^t(0,0,0),\ m \geqslant 2,\ n = 1,2.$

Denote by $V(z)$ the solution to the Lagrange dual equation on D/Γ and by E the unit matrix of order three. We have the following relation (see [1]) $V(Lz) = \xi_L(z)^2 V(z)\chi(L)^{-1},\ L \in \Gamma$. Solving the second equation of the system by Lagrange's method of variation of constants, we get $U_1(z) = \Big[\int_{z_0}^{z} r(x)U(x)V(x)dx\Big]U(z)$. Denote by $B(x) = r(x)U(x)V(x),\ B_0(z) = \int_{z_0}^{z} B(x)dx$.

For $m = 1$ we have the equality $U_1(z) = B_0(z)U(z)$.

For $m > 1$ we introduce the notation $U_m(z) = B_{m-1}(z)U(z)$, where $B_{m-1}(z) = \int_{z_0}^{z} r(x)U_{m-1}(x)V(x)dx$. For $m = 2$ we have $U_2(z) = B_1(z)U(z)$. On the other hand, $U_2(z) = \int_{z_0}^{z} r(x)U_1(x)V(x)dxU(z) = \int_{z_0}^{z} B_0(x)B(x)dxU(z)$. Consequently, $B_1(z) = \int_{z_0}^{z} B_0(x)B(x)dx$. Thus, we have proved the statement for the cases $m = 1,\ m = 2$. By the induction assumption, for m we have the equality $U_m(z) = B_{m-1}(z)U(z)$, where $B_{m-1}(z) = \int_{z_0}^{z} B_{m-2}(x)B(x)dx$. We shall prove this statement for $m+1$. In our notation we have $U_{m+1}(z) = B_m(z)U(z)$. On the other hand, $U_{m+1}(z) = \Big[\int_{z_0}^{z} r(x)U_m(x)V(x)dx\Big]U(z) = \Big[\int_{z_0}^{z} B_{m-1}(x)r(x)U(x)V(x)dx\Big]U(z) = \int_{z_0}^{z} B_{m-1}(x)B(x)dxU(z)$. Consequently, the equality $B_m(z) = \int_{z_0}^{z} B_{m-1}(x)B(x)dx$ holds for $m+1$. Thus, we have proved the following theorem.

Theorem 1.1. *For the solution vector of linear differential equation (4) with condition (5) on a compact Riemann surface $F = D/\Gamma$ of genus $g \geqslant 2$ the following explicit variational formulas hold: $U(z,\mu) = \Big[E + \mu B_0(z) + \mu^2 B_1(z) + \cdots + \mu^m B_{m-1}(z) + \cdots\Big]U(z)$, where $B(x) = r(x)U(x)V(x)$, $B_0(z) = \int_{z_0}^{z} B(x)dx$, $B_m(z) = \int_{z_0}^{z} B_{m-1}(x)B(x)dx$ for every $m \geqslant 1$ and $|\mu| < \varepsilon$.*

In Theorem 1.1 we obtain variations of every order, i.e., the whole variational Taylor series in parameter μ.

Now, using variational formulas for the solution $U(z,\mu)$, we shall find variational formulas for the monodromy group of equation (4).

Write down the basic relation

$$\chi(L;\mu)U(z,\mu)\xi_L(z)^2 = U(Lz,\mu) = U_0(Lz) + \mu U_1(Lz) + \mu^2 U_2(Lz) + \ldots + \mu^m U_m(Lz) + \cdots.$$

Express $U_m(Lz)$ in terms of $U(z)$ and the coefficients of the equation. We have

$$U_1(Lz) = \Big[\int_{z_0}^{Lz} r(x)U(x)V(x)dx\Big]U(Lz) =$$

$$= \Big[\int_{z_0}^{Lz_0} r(x)U(x)V(x)dx + \int_{Lz_0}^{Lz} r(x)U(x)V(x)dx\Big]U(Lz) =$$

$$= B_0(Lz_0)\chi(L)U(z)\xi_L(z)^2 + \chi(L)B_0(z)U(z)\xi_L(z)^2 =$$

$$= B_0(Lz_0)\chi(L)U(z)\xi_L(z)^2 + \chi(L)U_1(z)\xi_L(z)^2.$$

Indeed,

$$\Big(\int_{Lz_0}^{Lz} r(x)U(x)V(x)dx\Big)\chi(L)U(z)\xi_L(z)^2 =< x = L(t) >=$$

$$=\Big(\int_{z_0}^{z} r(Lt)U(Lt)V(Lt)d(Lt)\Big)\chi(L)U(z)\xi_L(z)^2 =$$

$$=\Big(\int_{z_0}^{z} r(t)L'(t)^{-3}\chi(L)U(t)\xi_L(t)^2\xi_L(t)^2V(t)\chi(L^{-1})L'(t)dt\Big)\chi(L)U(z)\xi_L(z)^2 =$$

$$=\Big(\int_{z_0}^{z} r(t)\chi(L)U(t)V(t)dt\Big)U(z)\xi_L(z)^2 = \chi(L)U_1(z)\xi_L(z)^2.$$

Next, $U_2(Lz) =\Big(\int_{z_0}^{Lz} r(x)U_1(x)V(x)dx\Big)U(Lz) =$

$$=\Big(\int_{z_0}^{Lz_0} r(x)U_1(x)V(x)dx + \int_{Lz_0}^{Lz} r(x)U_1(x)V(x)dx\Big)U(Lz) =$$

$$= B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2+\Big(\int_{z_0}^{z} r(Lt)U_1(Lt)V(Lt)dL(t)\Big)U(Lz) =$$

$$= B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2+\Big[\int_{z_0}^{z} r(t)L'(t)^{-3}\Big(B_0(Lz_0)\chi(L)U(t)\xi_L(t)^2+$$

$$+\chi(L)U_1(t)\xi_L(t)^2\Big)\xi_L(t)^2V(t)\chi(L^{-1})L'(t)dt\Big]U(Lz) =$$

$$= B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2+\Big(\int_{z_0}^{z} r(t)B_0(Lz_0)\chi(L)U(t)V(t)\chi(L^{-1})dt\Big)U(Lz)+$$

$$+\Big(\int_{z_0}^{z} r(t)\chi(L)U_1(t)V(t)\chi(L^{-1})dt\Big)U(Lz) = B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2+$$

$$+B_0(Lz_0)\chi(L)\Big[\int_{z_0}^{z} r(t)U(t)V(t)dt\Big]\xi_L(z)^2U(z)+$$

$$+\chi(L)\Big(\int_{z_0}^{z} r(t)U_1(t)V(t)dt\Big)\xi_L(z)^2U(z) = B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2+$$

$$+B_0(Lz_0)\chi(L)B_0(z)\xi_L(z)^2U(z) + \chi(L)B_1(z)\xi_L(z)^2U(z) =$$

$$= B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2 + B_0(Lz_0)\chi(L)U_1(z)\xi_L(z)^2 + \chi(L)U_2(z)\xi_L(z)^2.$$

Assume that the statement holds for m, i.e.,

$$U_m(Lz) = \sum_{j=0}^{m-1} B_{m-1-j}(Lz_0)\chi(L)U_j(z)\xi_L(z)^2 + \chi(L)U_m(z)\xi_L(z)^2.$$

We will show that it holds for $m+1$. We have

$$U_{m+1}(Lz)=\Big[\int_{z_0}^{Lz} r(x)U_m(x)V(x)dx\Big]U(Lz)=$$
$$=\Big[\int_{z_0}^{Lz_0} r(x)U_m(x)V(x)dx+\int_{Lz_0}^{Lz} r(x)U_m(x)V(x)dx\Big]U(Lz)=$$
$$=B_m(Lz_0)\chi(L)U(z)\xi_L(z)^2+\Big[\int_{z_0}^{z} r(Lt)U_m(Lt)V(Lt)dL(t)\Big]U(Lz)=$$
$$=B_m(Lz_0)\chi(L)U(z)\xi_L(z)^2+\sum_{j=0}^{m-1} B_{m-1-j}(Lz_0)\chi(L)U_{j+1}(z)\xi_L(z)^2+$$
$$+\chi(L)U_{m+1}(z)\xi_L(z)^2=\sum_{j=0}^{m} B_{m-j}(Lz_0)\chi(L)U_j(z)\xi_L(z)^2+\chi(L)U_{m+1}(z)\xi_L(z)^2.$$

Since

$$\Big[\int_{z_0}^{z} r(Lt)U_m(Lt)V(Lt)d(L(t))\Big]U(Lz)=\Big[\int_{z_0}^{z} r(t)L'(t)^{-3}\Big(\sum_{j=0}^{m-1} B_{m-1-j}(Lz_0)+\chi(L)U_j(t)\xi_L(t)^2+$$
$$+\chi(L)U_m(t)\xi_L(t)^2\Big)\xi_L(t)^2V(t)\chi(L^{-1})L'(t)\Big]dt\chi(L)U(z)\xi_L(z)^2=$$
$$=\Big[\sum_{j=0}^{m-1}\int_{z_0}^{z} r(t)B_{m-1-j}(Lz_0)\chi(L)U_j(t)+\int_{z_0}^{z} r(t)\chi(L)U_m(t)\Big]V(t)dtU(z)\xi_L(z)^2=$$
$$=\Big[\sum_{j=0}^{m-1} B_{m-1-j}(Lz_0)\chi(L)\int_{z_0}^{z} r(t)U_j(t)V(t)dt\Big]U(z)\xi_L(z)^2+$$
$$+\chi(L)\Big[\int_{z_0}^{z} r(t)U_m(t)V(t)dt\Big]U(z)\xi_L(z)^2=$$
$$=\sum_{j=0}^{m-1} B_{m-1-j}(Lz_0)\chi(L)U_{j+1}(z)\xi_L(z)^2+\chi(L)U_{m+1}(z)\xi_L(z)^2.$$

Using the expression for $U_m(Lz)$ proved by induction, we obtain an explicit variational formula for the elements of the monodromy group. We have

$$\chi(L;\mu)U(z,\mu)\xi_L(z)^2=U(Lz,\mu)=U(Lz)+\mu U_1(Lz)+$$
$$+\mu^2U_2(Lz)+\mu^3U_3(Lz)+\cdots+\mu^mU_m(Lz)+\cdots=$$
$$=E\chi(L)U(z)\xi_L(z)^2+\mu B_0(Lz_0)\chi(L)U(z)\xi_L(z)^2+\mu\chi(L)U_1(z)\xi_L(z)^2+$$
$$+\mu^2B_1(Lz_0)\chi(L)U(z)\xi_L(z)^2+\mu^2B_0(Lz_0)\chi(L)U_1(z)\xi_L(z)^2+\mu^2\chi(L)U_2(z)\xi_L(z)^2+$$
$$+\mu^3B_2(Lz_0)\chi(L)U(z)\xi_L(z)^2+\mu^3B_1(Lz_0)\chi(L)U_1(z)\xi_L(z)^2+$$
$$+\mu^3B_0(Lz_0)\chi(L)U_2(z)\xi_L(z)^2+\mu^3\chi(L)U_3(z)\xi_L(z)^2+\cdots+$$
$$+\mu^m\sum_{j=0}^{m-1} B_{m-1-j}(Lz_0)\chi(L)U_j(z)\xi_L(z)^2+\mu^m\chi(L)U_m(z)\xi_L(z)^2+\cdots=$$
$$=E\chi(L)\Big[U(z)+\mu U_1(z)+\mu^2U_2(z)+\mu^3U_3(z)+\cdots+$$
$$+\mu^mU_m(z)+\cdots\Big]\xi_L(z)^2+\mu B_0(Lz_0)\chi(L)\Big[U(z)+\mu U_1(z)+\mu^2U_2(z)+$$
$$+\mu^3U_3(z)+\cdots+\mu^{m-1}U_{m-1}(z)+\cdots\Big]\xi_L(z)^2+\mu^2B_1(Lz_0)\chi(L)(U(z)+$$

$$+\mu U_1(z)+\mu^2U_2(z)+\mu^3U_3(z)+\cdots+\mu^{m-2}U_{m-2}(z)+\cdots)\xi_L(z)^2+\cdots+$$
$$+\mu^m B_{m-1}(Lz_0)\chi(L)(U(z)+\mu U_1(z)+\cdots+\mu^k U_k(z)+\cdots)\xi_L(z)^2+\cdots$$

Thus we have proved

Theorem 1.2. *For the elements of the monodromy group of equation (4) on a compact Riemann surface F of genus $g\geqslant 2$ there hold the following explicit variational formulas:*

$$\chi(L;\mu)=\Big[E+\mu B_0(Lz_0)+\mu^2B_1(Lz_0)+\cdots+\mu^m B_{m-1}(Lz_0)+\cdots\Big]\chi(L),\ L\in\Gamma,\ |\mu|<\varepsilon.$$

2. Elements of the monodromy group for the variation with respect to the base of cubic differentials

Consider the perturbed differential equation

$$u^{(3)}(z)+Q_0(z)u^{(1)}(z)+(R_0(z)-\sum_{j=1}^{d}\mu_j r_j)u(z)=0 \tag{6}$$

on the surface $F=D/\Gamma$, where $r_1,...,r_d$ is the basis for cubic holomorphic differentials in the space $\Omega^3(F)$, $d=5g-5$. Put $\mu=(\mu_1,...,\mu_d)$. As before, we denote by $U(z,\mu)={}^t(u(z,\mu),v(z,\mu),w(z,\mu))$ the vector of three linearly independent solutions to the Cauchy problem with the conditions at the point z_0

$$U(z_0,\mu)={}^t(1,0,0);\quad U^{(1)}(z_0,\mu)={}^t(0,1,0);\quad U^{(2)}(z_0,\mu)={}^t(0,0,1), \tag{7}$$

for every μ.

Rewrite scalar equation (6) in the vector form

$$U^{(3)}(z,\mu)+Q_0(z)U^{(1)}(z,\mu)+(R_0(z)-\sum_{j=1}^{d}\mu_j r_j)U(z,\mu)=0. \tag{8}$$

By Poincaré's small parameter method and the Cauchy-Kowalevski theorem we have a Taylor series for the solution of equation (8)

$$U(z,\mu)=U(z)+\sum_{|k|=1}^{\infty}U_{|k|;(k_1,...,k_d)}(z)\mu_1^{k_1}\cdots\mu_d^{k_d}=U(z)+\sum_{|k|=1}^{\infty}U_{|k|;k}(z)\mu^k \tag{9}$$

for sufficiently small $\mu\in\mathbb{C}^d$, $\|\mu\|<\varepsilon$, where $\|\mu\|=\max\limits_{1\leqslant j\leqslant d}|\mu_j|$, $k=(k_1,...,k_d)$ is a vector with integer nonnegative coordinates, $|k|=k_1+\cdots+k_d$, and $U(z)=U_{|0|;(0,...,0)}(z)$. By substituting series (9) in equation (8), we obtain the vector equality

$$\sum_{|k|=0}^{\infty}U^{(3)}_{|k|;k}(z)\mu^k+Q_0(z)\sum_{|k|=0}^{\infty}U^{(1)}_{|k|;k}(z)\mu^k+(R_0(z)-\sum_{j=1}^{d}\mu_j r_j(z))\sum_{|k|=0}^{\infty}U_{|k|;k}(z)\mu^k=0.$$

Note that for every $|k|\geqslant 1$ the following conditions hold

$$U_{|k|;k}(z_0)=U^{(1)}_{|k|;k}(z_0)=U^{(2)}_{|k|;k}(z_0)=0. \tag{10}$$

Hence, we have an infinite system of vector linear differential equations of the form:

$$U^{(3)}(z)+Q_0(z)U^{(1)}(z)+R_0(z)U(z)=0;$$

$$U^{(3)}_{1;k}(z) + Q_0(z)U^{(1)}_{1;k}(z) + R_0(z)U_{1;k}(z) = r_j(z)U(z)$$

for $k = e_j$ (the unit is on jth place, while others equal zero), $j = 1, ..., d$;

$$U^{(3)}_{|k|;k}(z) + Q_0(z)U^{(1)}_{|k|;k} + R_0(z)U_{|k|;k}(z) = \sum_{j=1,\ k_j\neq 0}^{d} r_j(z)U_{|k|-1;\ k-e_j}(z)$$

for $k = (k_1, ..., k_d),\ |k| \geqslant 2$.

Solving the second equation by Lagrange's method of variation of constants, we get $U_{1;k}(z) = \int_{z_0}^{z} r_j(t)U(t)V(t)dtU(z)$ for $k = e_j, j = 1, ..., d$. Let $B_j(z) = r_j(z)U(z)V(z),\ j = 1, ..., d,\ B_{0;e_j} = \int_{z_0}^{z} B_j(t)dt$. Then, $U_{1;e_j}(z) = B_{0;e_j}(z)U(z)$ for every $k = e_j, j = 1, ..., d$.

For $k = (k_1, ..., k_d),\ |k| = 2$, we have

$$U_{2;k}(z) = \sum_{j=1,\ k_j\neq 0}^{d} \int_{z_0}^{z} B_{0;k-e_j}(t)B_j(t)dtU(z) = B_{1;k}(z)U(z).$$

For $k = (k_1, ..., k_d),\ |k| \geqslant 3$, we obtain the equality

$$U_{|k|;k}(z) = \sum_{(j:\ k_j\neq 0)} \int_{z_0}^{z} B_{|k|-2;k-e_j}(t)B_j(t)dtU(z) = B_{|k|-1;k}(z)U(z).$$

Thus, we have proved the following theorem.

Theorem 2.1. *For the solution vector of equation* (6) *with normalization* (7) *and with a perturbation with respect to the base of* $r_j,\ j = 1, ..., d = 5g - 5$, *one has the following explicit variational formulas:*

$$U(z, \mu) = \Big[E + \sum_{|k|=1} B_{0;k}(z)\mu^k + \sum_{|k|=2} B_{1;k}(z)\mu^k + \cdots + \sum_{|k|=n} B_{n-1;k}(z)\mu^k + \cdots\Big]U(z),$$

where $B_{0;k}(z) = \int_{z_0}^{z} B_j(t)dt$ *for* $k = e_j,\ B_j(t) = r_j(t)U(t)V(t),\ j = 1, ..., d,\ B_{|k|;(k_1,...,k_d)}(z) = \int_{z_0}^{z} \sum_{(j:\ k_j\neq 0)} B_{|k|-1;k-e_j}(t)B_j(t)dt$ *for* $|k| \geqslant 2$, $\|\mu\| < \varepsilon$.

To deduce variational formulas for the monodromy group, we need some relations associated with the group Γ. First, for $k = (k_1, ..., k_d) = e_j,\ |k| = 1, j = 1, ..., d$, we find that

$$\begin{aligned} U_{1;k}(Lz) = \int_{z_0}^{Lz} B_j(t)dtU(Lz) &= \xi_L(z)^2\int_{z_0}^{Lz_0} B_j(t)\,\chi(L)dtU(z) + \xi_L(z)^2\int_{Lz_0}^{Lz} B_j(t)\chi(L)dtU(z) = \\ &= \xi_L(z)^2 B_{0;k}(Lz_0)\chi(L)U(z) + \xi_L(z)^2\chi(L)\int_{z_0}^{z} B_j(t)dtU(z) = \\ &= \xi_L(z)^2 B_{0;k}(Lz_0)\,\chi(L)U(z) + \xi_L(z)^2\chi(L)U_{1;k}(z), \end{aligned}$$

since

$$\begin{aligned} \int_{Lz_0}^{Lz} B_j(x)\,\chi(L)dx = < x = L(t) > &= \int_{z_0}^{z} B_j(Lt)\,\chi(L)d(Lt) = \int_{z_0}^{z} r_j(Lt)U(Lt)V(Lt)\,\chi(L)L'(t)dt = \\ &= \int_{z_0}^{z} r_j(t)L'(t)^{-3}L'(t)\,\chi(L)U(t)L'(t)V(t)\chi(L^{-1})\,\chi(L)L'(t)dt = \\ &= \chi(L)\int_{z_0}^{z} r_j(t)U(t)V(t)dt = \chi(L)B_{0;e_j}(z). \end{aligned}$$

For $k=(k_1,...,k_d),\ |k|=2$, we find that

$$U_{2;k}(Lz)=\xi_L(z)^2B_{1;k}(Lz)\,\chi(L)U(z)=\xi_L(z)^2B_{1;k}(Lz_0)\,\chi(L)U(z)+$$

$$+\xi_L(z)^2\int_{Lz_0}^{Lz}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(x)B_j(x)dx\,\chi(L)U(z).$$

Next,

$$\int_{Lz_0}^{Lz}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(x)B_j(x)dx\,\chi(L)=<x=L(t)>=\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lt)B_j(Lt)d(Lt)\chi(L)=$$

$$=\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lt)r_j(Lt)U(Lt)V(Lt)L'(t)dt\,\chi(L)=$$

$$=\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lt)r_j(t)L'(t)^{-3}L'(t)\,\chi(L)U(t)L'(t)V(t)L'(t)dt=$$

$$=\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lt)r_j(t)\,\chi(L)U(t)V(t)dt=$$

$$=\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lt)\chi(L)B_j(t)dt=\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lz_0)\,\chi(L)B_j(t)dt+$$

$$+\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}\sum_{(i:\ (k-e_j)_i\neq 0)}\int_{Lz_0}^{Lt}B_i(s)\,\chi(L)dsB_j(t)dt=$$

$$=\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lz_0)\chi(L)B_{0;e_j}(z)+\chi(L)\int_{z_0}^{z}\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(t)B_j(t)dt.$$

Thus,

$$U_{2;k}(Lz)=\xi_L(z)^2B_{1;k}(Lz_0)\,\chi(L)U(z)+\xi_L(z)^2\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lz_0)\chi(L)B_{0;e_j}(z)U(z)+$$

$$+\xi_L(z)^2\chi(L)\sum_{(j:\ k_j\neq 0)}\int_{z_0}^{z}B_{0;k-e_j}(t)B_j(t)dtU(z)=\xi_L(z)^2B_{1;k}(Lz_0)\,\chi(L)U(z)+$$

$$+\xi_L(z)^2\sum_{(j:\ k_j\neq 0)}B_{0;k-e_j}(Lz_0)\,\chi(L)U_{1;e_j}(z)+\xi_L(z)^2\,\chi(L)U_{2;k}(z).$$

By induction, for every $n>1$ we obtain the equality:

$$U_{n+1;k}(Lz)=B_{n;k}(Lz)U(Lz)=\sum_{(j:\ k_j\neq 0)}\int_{z_0}^{Lz}r_j(s)U_{n;k-e_j}(s)V(s)dsU(Lz)=$$

$$=\xi_L(z)^2B_{n;k}(Lz_0)\chi(L)U(z)+\xi_L(z)^2\sum_{(j:\ k_j\neq 0)}\int_{Lz_0}^{Lz}r_j(x)U_{n;k-e_j}(x)V(x)dx\chi(L)U(z)=$$

$$=\xi_L(z)^2B_{n;k}(Lz_0)\,\chi(L)U(z)+\xi_L(z)^2\sum_{(j:\ k_j\neq 0)}\int_{z_0}^{z}r_j(s)L'(s)^{-1}U_{n;k-e_j}(Ls)V(s)dsU(z)=$$

$$=\xi_L(z)^2B_{n;k}(Lz_0)\chi(L)U(z)+\xi_L(z)^2\sum_{(j:\ k_j\neq 0)}\int_{z_0}^{z}r_j(s)\Big[B_{n-1;k-e_j}(Lz_0)\,\chi(L)U(s)+$$

$$+\sum_{(j_1\colon (k-e_j)_{j_1}\neq 0)} B_{n-2;k-e_j-e_{j_1}}(Lz_0)\chi(L)U_{1;e_{j_1}}(s)+$$

$$+\sum_{(j_1\colon (k-e_j)_{j_1}\neq 0)}\ \sum_{(j_2\colon\ (k-e_j-e_{j_1})_{j_2}\neq 0)} B_{n-3;k-e_j-e_{j_1}-e_{j_2}}(Lz_0)\,\chi(L)U_{2;e_{j_1}+e_{j_2}}(s)+\cdots$$

$$+\sum_{(j_1\colon (k-e_j)_{j_1}\neq 0)}\ \sum_{(j_2\colon (k-e_j-e_{j_1})_{j_2}\neq 0)}\cdots$$

$$\sum_{(j_{n-1}\colon (k-e_j-e_{j_1}-\ldots-e_{j_{n-2}})_{j_{n-1}}\neq 0)} B_{0;k-e_j-e_{j_1}-e_{j_2}-\ldots-e_{j_{n-1}}}(Lz_0)\,\chi(L)U_{n-1;e_{j_1}+e_{j_2}+\ldots+e_{j_{n-1}}}(s)+$$

$$+\chi(L)U_{n;k-e_j}(s)\Big]V(s)dsU(z)=$$

$$=\xi_L(z)^2B_{n;k}(Lz_0)\chi(L)U(z)+\xi_L(z)^2\Big[\sum_{(j\colon k_j\neq 0)} B_{n-1;k-e_j}(Lz_0)\chi(L)U_{1;e_j}(z)+$$

$$+\sum_{(j\colon k_j\neq 0)}\ \sum_{(j_1\colon (k-e_j)_{j_1}\neq 0)} B_{n-2;k-e_j-e_{j_1}}(Lz_0)\chi(L)U_{2;e_j+e_{j_1}}(z)+$$

$$+\sum_{(j\colon k_j\neq 0)}\ \sum_{(j_1\colon (k-e_j)_{j_1}\neq 0)}\ \sum_{(j_2\colon (k-e_j-e_{j_1})_{j_2}\neq 0)} B_{n-3;k-e_j-e_{j_1}-e_{j_2}}(Lz_0)\chi(L)U_{3;e_j+e_{j_1}+e_{j_2}}(z)+$$

$$\cdots+\sum_{(j\colon k_j\neq 0)}\ \sum_{(j_1\colon (k-e_j)_{j_1}\neq 0)}\ \sum_{(j_2\colon (k-e_j-e_{j_1})_{j_2}\neq 0)}\cdots$$

$$\sum_{(j_{n-1}\colon (k-e_j-e_{j_1}-\ldots-e_{j_{n-2}})_{j_{n-1}}\neq 0)} B_{0;k-e_j-e_{j_1}-e_{j_2}-\ldots-e_{j_{n-1}}}(Lz_0)\,\chi(L)U_{n;e_j+e_{j_1}+e_{j_2}+\ldots+e_{j_{n-1}}}(z)+$$

$$+\chi(L)U_{n+1;k}(z)\Big].$$

Taking into account the previous equalities we obtain the following

$$\xi_L(z)^2\chi(L;\mu)U(z,\mu)=U(Lz,\mu)=U(Lz)+\sum_{|k|=1}\mu^kU_{1;k}(Lz)+$$

$$+\sum_{|k|=2}\mu^kU_{2;k}(Lz)+\cdots+\sum_{|k|=n}\mu^kU_{n;k}(Lz)+\cdots=\xi_L(z)^2\Big[\chi(L)U(z)+$$

$$+\sum_{|k|=1}\mu^kB_{0;k}(Lz_0)\chi(L)U(z)+\sum_{|k|=1}\mu^k\chi(L)U_{1;k}(z)+\sum_{|k|=2}\mu^kB_{1;k}(Lz_0)\chi(L)U(z)+$$

$$+\sum_{|k|=2}\mu^k\sum_{(j\colon\ k_j\neq 0)} B_{0;k-e_j}(Lz_0)\chi(L)U_{1;e_j}(z)+\sum_{|k|=2}\mu^k\chi(L)U_{2;k}(z)+$$

$$+\sum_{|k|=3}\mu^kB_{2;k}(Lz_0)\chi(L)U(z)+\sum_{|k|=3}\mu^k\sum_{(j\colon\ k_j\neq 0)} B_{1;k-e_j}(Lz_0)\,\chi(L)U_{1;e_j}(z)+$$

$$+\sum_{|k|=3}\mu^k\sum_{(j\colon\ k_j\neq 0)}\ \sum_{(j_1\colon\ (k-e_j)_{j_1}\neq 0)} B_{0;k-e_j-e_{j_1}}(Lz_0)\chi(L)U_{2;e_j+e_{j_1}}(z)+$$

$$+\sum_{|k|=3}\mu^k\chi(L)U_{3;k}(z)+\cdots+\sum_{|k|=n}\mu^kB_{n-1;k}(Lz_0)\,\chi(L)U(z)+$$

$$+\sum_{|k|=n}\mu^k\sum_{(j:\ k_j\neq 0)}B_{n-2;k-e_j}(Lz_0)\,\chi(L)U_{1;e_j}(z)+$$

$$+\sum_{|k|=n}\mu^k\sum_{(j:\ k_j\neq 0)}\sum_{(j_1:\,(k-e_j)_{j_1}\neq 0)}B_{n-3;k-e_j-e_{j_1}}(Lz_0)\chi(L)U_{2;e_j+e_{j_1}}(z)+\cdots+$$

$$+\sum_{|k|=n}\mu^k\sum_{(j:\ k_j\neq 0)}\sum_{(j_1:\,(k-e_j)_{j_1}\neq 0)}\cdots\times$$

$$\times\sum_{(j_{n-2}:\,(k-e_j-e_{j_1}-\ldots-e_{j_{n-3}})_{j_{n-2}}\neq 0)}B_{0;k-e_j-e_{j_1}-\ldots-e_{j_{n-2}}}(Lz_0)\,\chi(L)U_{n-1;e_j+e_{j_1}+\ldots+e_{j_{n-2}}}(z)+$$

$$+\sum_{|k|=n}\mu^k\chi(L)U_{n;k}(z)+\cdots=\xi_L(z)^2\Big[\chi(L)U(z,\mu)+\sum_{|k|=1}\mu^kB_{0;k}(Lz_0)\,\chi(L)U(z,\mu)+$$

$$+\sum_{|k|=2}\mu^kB_{1;k}(Lz_0)\chi(L)U(z,\mu)+\sum_{|k|=3}\mu^kB_{2;k}(Lz_0)\,\chi(L)U(z,\mu)+\cdots+$$

$$+\sum_{|k|=n}\mu^kB_{n-1;k}(Lz_0)\chi(L)U(z,\mu)+\cdots\Big].$$

Thus, we have proved the following theorem.

Theorem 2.2. *For the elements of the monodromy group of equation (6) with normalization (7) on compact Riemann surface $F=D/\Gamma$ of genus $g\geqslant 2$ the following explicit variational formulas hold*

$$\chi(L;\mu)=\Big[E+\sum_{|k|=1}B_{0;k}(Lz_0)\mu^k+\sum_{|k|=2}B_{1;k}(Lz_0)\mu^k+\cdots\sum_{|k|=n}B_{n-1;k}(Lz_0)\mu^k+\cdots\Big]\chi(L),$$

where $L\in\Gamma$, for $\|\mu\|<\varepsilon$.

Remark 2.1. Variational formulas show how the monodromy group and the solution to the equation of third order depend on the parameters $(\mu_1,...,\mu_d)$ for a variation with respect to the base of cubic holomorphic differentials on F. In particular, they give explicit variational formulas for the generators $\chi(A_1),\ldots,\chi(A_g),\chi(B_1),\ldots,\chi(B_g)$ of the monodromy group.

Remark 2.2. It follows from the relation $dB_0(Lz)=\chi(L)dB_0(z)\chi(L)^{-1}$ that the matrix differential form $dB_0(z)$ can be considered as a kind of Prym differentials, in Gunning's sense, with respect to the character that is equal to the monodromy homomorphism of equation (6) for a fixed group Γ with values in $GL(3,\mathbb{C})$ [4].

Remark 2.3. The equality $U(Lz)(L'(z))^{-1}=\chi(L)U(z),\ L\in\Gamma$ means that the solution vector $U(z)$ for the Cauchy problem at the point z_0 is a kind of vector Prym 1-differentials of third order on $F=D/\Gamma$ with respect to the matrix character χ of the group Γ with values in $GL(3,\mathbb{C})$. More precisely, $U(z)$ is a holomorphic section of the vector bundle $\chi\otimes K^{-1}$, where K is the canonical bundle on $F=D/\Gamma$ [4].

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Точные вариационные формулы для уравнения третьего порядка на римановой поверхности

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В статье выводятся точные вариационные формулы для решения и его группы монодромии обыкновенного дифференциального уравнения третьего порядка при вариации в пространстве голоморфных кубических дифференциалов на компактной римановой поверхности.

Ключевые слова: голоморфные кубические дифференциалы, компактная риманова поверхность, вариационные формулы, группа монодромии, дифференциальное уравнение третьего порядка.