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The Solution of Adjoint Heat Problem in Spherical Area by Laplace Transform Method

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The spherically symmetric adjoint initial-boundary value problem of heat propagation in closed bounded spherical regions has been researched. The exact analytical solution of the direct and inverse nonstationary problem has been obtained using Laplace transform method. The stationary state has been found and it is shown that the nonstationary solution converges to stationary one when time tends to infinity, if such are the heat sources in media.

Keywords: initial-boundary value problem, interface, Laplace transform method, inverse problem.

1. Problem statement

Assume that the functions $u_1(r,t)$, $u_2(r,t)$ are defined in areas $\overline{\Omega}_1 = \{r|0 \leqslant r \leqslant R_1\}$, $\overline{\Omega}_2 = \{r|R_1 \leqslant r \leqslant R_2\}$, respectively, and satisfy the equations

$$u_{1t} = \chi_1 \left(u_{1rr} + \frac{2}{r} u_{1r} \right) + f_1(r,t), \quad t > 0, \quad r \in \Omega_1, \tag{1.1}$$

$$u_{2t} = \chi_2 \left(u_{2rr} + \frac{2}{r} u_{2r} \right) + f_2(r,t), \quad t > 0, \quad r \in \Omega_2. \tag{1.2}$$

The functions u_j, $j = 1,2$ are temperature fields, f_j are defined internal heat sources; χ_j are positive constants known as the thermal diffusivities.

In addition, we have initial and boundary conditions

$$u_1|_{t=0} = u_2|_{t=0} = 0; \tag{1.3}$$

$$|u_1(0,t)| < \infty, \tag{1.4}$$

$$u_1|_{r=R_1} = u_2|_{r=R_1}, \tag{1.5}$$

$$k_1 \frac{\partial u_1}{\partial r} \Big|_{r=R_1} = k_2 \frac{\partial u_2}{\partial r} \Big|_{r=R_1}, \tag{1.6}$$

$$u_2|_{r=R_2} = 0, \tag{1.7}$$

where k_j are heat conductivity coefficients. Condition (1.5) is equality of temperatures, and (1.6) is equality of heat fluxes on boundary surface $r = R_1$. It is known [1] that $\chi_j = k_j/c_j\rho_j$, where c_j are specific heats, ρ_j are densities of media.

It is necessary to find functions $u_1 \in C^2(\Omega_1) \cap C^1(\Gamma_1)$, $u_2 \in C^2(\Omega_2) \cap C^1(\Gamma_1) \cap C(\Gamma_2)$, which satisfy the equations (1.1), (1.2) and conditions (1.3)–(1.7), $\Gamma_1 = \{r|r = R_1\}$, $\Gamma_2 = \{r|r = R_2\}$.

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2. The solution of nonstationary problem by Laplace transform method

We shall find the solution of the problem (1.1)–(1.7) by Laplace transform (its using is validated in [2]). We obtain

$$U_j'' + \frac{2}{r}U_j' - \frac{p}{\chi_j}U_j = -\frac{1}{\chi_j}F_j \tag{2.1}$$

for images $U_j(r,p)$ of functions $u_j(r,t)$ by use the relation feature of original function's differentiation [2] and initial conditions (1.3). Here, $F_j = F_j(r,p)$ are images of functions $f_j(r,t)$.

Boundary conditions become such as

$$U_1|_{r=R_1} = U_2|_{r=R_1}, \tag{2.2}$$

$$k_1\frac{\partial U_1}{\partial r}\Big|_{r=R_1} = k_2\frac{\partial U_2}{\partial r}\Big|_{r=R_1}, \tag{2.3}$$

$$U_2|_{r=R_2} = 0, \tag{2.4}$$

$$|U_1(0,p)| < \infty. \tag{2.5}$$

Let us introduce change of variables $U_j = v_j/r$, then equation (2.1) take on form

$$v_j'' - \frac{p}{\chi_j}v_j = -\frac{r}{\chi_j}F_j. \tag{2.6}$$

The solution of homogeneous differential equation (2.6) has the form [3]

$$v_{j0} = C_1^j \exp\left(r\sqrt{\frac{p}{\chi_j}}\right) + C_2^j \exp\left(-r\sqrt{\frac{p}{\chi_j}}\right).$$

We define fundamental system of solutions $\varphi_1^j = \exp\left(r\sqrt{p/\chi_j}\right)$, $\varphi_2^j = \exp\left(-r\sqrt{p/\chi_j}\right)$ for finding of particular solution. Then Wronsky's determinant $W(r) = \varphi_1^j\varphi_2^{j\prime} - \varphi_2^j\varphi_1^{j\prime} = -2\sqrt{p/\chi_j}$ and solution of equations (2.1) can be represented by formulae (subject to boundary conditions (2.2)–(2.5))

$$U_1 = \frac{\operatorname{sh}\left(r\sqrt{\frac{p}{\chi_1}}\right)\exp\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}{r\sqrt{p\chi_2}\operatorname{sh}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)}\int_{R_1}^{R_2}\xi F_2 \operatorname{sh}\left((R_2-\xi)\sqrt{\frac{p}{\chi_2}}\right)d\xi +$$
$$+\frac{1}{r\sqrt{p\chi_1}}\left(\frac{\operatorname{sh}\left(r\sqrt{\frac{p}{\chi_1}}\right)}{\operatorname{sh}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)}\int_0^{R_1}\xi F_1\operatorname{sh}\left((R_1-\xi)\sqrt{\frac{p}{\chi_1}}\right)d\xi - \int_0^r \xi F_1 \operatorname{sh}\left((r-\xi)\sqrt{\frac{p}{\chi_1}}\right)d\xi\right)+$$
$$+\frac{C\operatorname{sh}\left(r\sqrt{\frac{p}{\chi_1}}\right)}{r\operatorname{sh}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)}\left(\exp\left(R_1\sqrt{\frac{p}{\chi_2}}\right)-\exp\left((2R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)\right), \tag{2.7}$$

$$U_2 = \frac{C}{r}\left(\exp\left(r\sqrt{\frac{p}{\chi_2}}\right)-\exp\left((2R_2-r)\sqrt{\frac{p}{\chi_2}}\right)\right) - \frac{1}{r\sqrt{p\chi_2}}\int_{R_1}^r \xi F_2\operatorname{sh}\left((r-\xi)\sqrt{\frac{p}{\chi_2}}\right)d\xi +$$
$$+\frac{1}{r\sqrt{p\chi_2}}\exp\left((R_2-r)\sqrt{\frac{p}{\chi_2}}\right)\int_{R_1}^{R_2}\xi F_2\operatorname{sh}\left((R_2-\xi)\sqrt{\frac{p}{\chi_2}}\right)d\xi, \tag{2.8}$$

where

$$C = \frac{\frac{C_1 \exp\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}{\sqrt{p\chi_2}} \int_{R_1}^{R_2} \xi F_2 \operatorname{sh}\left((R_2-\xi)\sqrt{\frac{p}{\chi_2}}\right) d\xi - \frac{k}{\chi_1 \operatorname{sh}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)} \int_0^{R_1} \xi F_1 \operatorname{sh}\left(\xi\sqrt{\frac{p}{\chi_1}}\right) d\xi}{C_2 \exp\left(R_1\sqrt{\frac{p}{\chi_2}}\right) + C_1 \exp\left((2R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}, \tag{2.9}$$

$$C_1 = \sqrt{\frac{p}{\chi_2}} + \frac{1-k}{R_1} + k\sqrt{\frac{p}{\chi_1}} \operatorname{cth}\left(R_1\sqrt{\frac{p}{\chi_1}}\right), \quad C_2 = \sqrt{\frac{p}{\chi_2}} - \frac{1-k}{R_1} - k\sqrt{\frac{p}{\chi_1}} \operatorname{cth}\left(R_1\sqrt{\frac{p}{\chi_1}}\right). \tag{2.10}$$

3. The solution of stationary problem

Assume that $f_j(r,t) \to f_j^0(r)$, when $t \to \infty$. Then the question arise whether solution of nonstationary problem will converge to solution of stationary one. The steady-state condition of heat conduction satisfies the equations

$$u_j^{0\prime\prime}(r) + \frac{2}{r} u_j^{0\prime}(r) = -\frac{1}{\chi_j} f_j^0(r) \tag{3.1}$$

with boundary conditions

$$u_1^0(R_1) = u_2^0(R_1), \tag{3.2}$$

$$k_1 u_1^{0\prime}(R_1) = k_2 u_2^{0\prime}(R_1), \tag{3.3}$$

$$u_2^0(R_2) = 0, \tag{3.4}$$

$$\left|u_1^0(0)\right| < \infty. \tag{3.5}$$

It is easy to see that the problem (3.1)–(3.5) has the following solution:

$$u_1^0 = \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \frac{k}{\chi_1} \int_0^{R_1} \xi^2 f_1^0(\xi)\, d\xi - \frac{1}{\chi_1} \int_0^{R_1} \left(\frac{\xi^2}{R_1} - \xi\right) f_1^0(\xi)\, d\xi - \\ - \frac{1}{\chi_2} \int_{R_1}^{R_2} \left(\frac{\xi^2}{R_2} - \xi\right) f_2^0(\xi)\, d\xi + \frac{1}{\chi_1} \int_0^{r} \left(\frac{\xi^2}{r} - \xi\right) f_1^0(\xi)\, d\xi, \tag{3.6}$$

$$u_2^0 = \left(\frac{1}{r} - \frac{1}{R_2}\right) \frac{k}{\chi_1} \int_0^{R_1} \xi^2 f_1^0(\xi)\, d\xi - \frac{1}{\chi_2} \int_{R_1}^{R_2} \left(\frac{\xi^2}{R_2} - \xi\right) f_2^0(\xi)\, d\xi + \\ + \frac{1}{\chi_2} \int_{R_1}^{r} \left(\frac{\xi^2}{r} - \xi\right) f_2^0(\xi)\, d\xi. \tag{3.7}$$

We use the well-known limiting relations $\lim\limits_{p\to 0} pF(p) = f(\infty)$ [2], $\operatorname{sh} x \sim x$, $\operatorname{ch} x \sim 1$, $\exp x \sim 1 + x$, when $x \to 0$, and formulae (2.7)–(2.10) for finding of limits

$$\lim_{p\to 0} pU_1(r,p) = \frac{1}{\chi_1}\lim_{p\to 0}\left(\frac{1}{R_1}\int_0^{R_1}\xi(R_1-\xi)f_1^0\,d\xi + \frac{1}{r}\int_0^r \xi(\xi-r)f_1^0\,d\xi\right)+$$

$$+\frac{2(R_1-R_2)}{R_1}\lim_{p\to 0}\frac{\sqrt{\frac{p}{\chi_2}}\left[\frac{1+(R_2-R_1)\sqrt{\frac{p}{\chi_2}}}{\chi_2}\left(\sqrt{\frac{p}{\chi_2}}+\frac{1}{R_1}\right)\int_{R_1}^{R_2}\xi(R_2-\xi)f_2^0\,d\xi - \frac{k}{\chi_1 R_1}\int_0^{R_1}\xi^2 f_1^0\,d\xi\right]}{\left(\sqrt{\frac{p}{\chi_2}}-\frac{1}{R_1}\right)\left(1+R_1\sqrt{\frac{p}{\chi_2}}\right)+\left(\sqrt{\frac{p}{\chi_2}}+\frac{1}{R_1}\right)\left(1+(2R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}+$$

$$+\lim_{p\to 0}\frac{1+(R_2-R_1)\sqrt{\frac{p}{\chi_2}}}{R_1\chi_2}\int_{R_1}^{R_2}\xi(R_2-\xi)f_2^0\,d\xi = u_1^0(r),$$

$$\lim_{p\to 0} pU_2(r,p) = \frac{1}{r\chi_2}\lim_{p\to 0}\left(\int\limits_{R_1}^{r}\xi(\xi-r)f_2^0\,d\xi + \left(1+(R_2-r)\sqrt{\frac{p}{\chi_2}}\right)\int_{R_1}^{R_2}\xi(R_2-\xi)f_2^0\,d\xi\right)+$$

$$+\lim_{p\to 0}\frac{\frac{2(r-R_2)}{r}\sqrt{\frac{p}{\chi_2}}\left[\frac{1+(R_2-R_1)\sqrt{\frac{p}{\chi_2}}}{\chi_2}\left(\sqrt{\frac{p}{\chi_2}}+\frac{1}{R_1}\right)\int_{R_1}^{R_2}\xi(R_2-\xi)f_2^0\,d\xi - \frac{k}{\chi_1 R_1}\int_0^{R_1}\xi^2 f_1^0\,d\xi\right]}{\left(\sqrt{\frac{p}{\chi_2}}-\frac{1}{R_1}\right)\left(1+R_1\sqrt{\frac{p}{\chi_2}}\right)+\left(\sqrt{\frac{p}{\chi_2}}+\frac{1}{R_1}\right)\left(1+(2R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)} = u_2^0(r)$$

with functions u_1^0, u_2^0 from (3.6), (3.7). In other words, we have proved that the nonstationary solution converges to stationary one when time tends to infinity, if such are the heat sources in media.

Let us now put $f_j = f_j(t)$. In this case formulae (2.7)–(2.9) are simplified up to

$$U_1 = \frac{\operatorname{sh}\left(r\sqrt{\frac{p}{\chi_1}}\right)}{rp\operatorname{sh}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)}\left(C_3F_2\exp\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)-R_1F_1\right)+$$

$$+\frac{C_4\operatorname{sh}\left(r\sqrt{\frac{p}{\chi_1}}\right)}{r\operatorname{sh}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)}\left[\exp\left(R_1\sqrt{\frac{p}{\chi_2}}\right)-\exp\left((2R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)\right]+\frac{F_1}{p}, \quad (3.8)$$

$$U_2 = \frac{C_4}{r}\left(\exp\left(r\sqrt{\frac{p}{\chi_2}}\right)-\exp\left((2R_2-r)\sqrt{\frac{p}{\chi_2}}\right)\right)+\frac{F_2}{p}+\frac{C_3F_2}{rp}\exp\left((R_2-r)\sqrt{\frac{p}{\chi_2}}\right)-$$

$$-\frac{F_2}{rp}\left[\sqrt{\frac{\chi_2}{p}}\operatorname{sh}\left((r-R_1)\sqrt{\frac{p}{\chi_2}}\right)+R_1\operatorname{ch}\left((r-R_1)\sqrt{\frac{p}{\chi_2}}\right)\right], \quad (3.9)$$

where

$$C_3 = \sqrt{\frac{\chi_2}{p}}\operatorname{sh}\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)+R_1\operatorname{ch}\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)-R_2, \quad (3.10)$$

$$C_4 = \frac{\frac{kF_1}{\sqrt{p\chi_1}}\left(\sqrt{\frac{\chi_1}{p}}-R_1\operatorname{cth}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)\right)+\frac{C_1C_3F_2}{p}\exp\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}{C_2\exp\left(R_1\sqrt{\frac{p}{\chi_2}}\right)+C_1\exp\left((2R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}, \quad (3.11)$$

and constants C_1, C_2 are determined in (2.10).

4. On the evaluation of internal heat sources

Consider the problem (1.1)–(1.7) when $f_2 = 0$ (there is no heat sources in second medium), add condition of overdetermination (we specify heat flux on surface) and regard temperature on sphere surface as non-nil constant. Then problem take on form

$$u_{1t} = \chi_1 \left(u_{1rr} + \frac{2}{r} u_{1r} \right) + f_1(r,t), \;\; t > 0, \;\; r \in \Omega_1, \tag{4.1}$$

$$u_{2t} = \chi_2 \left(u_{2rr} + \frac{2}{r} u_{2r} \right), \;\; t > 0, \;\; r \in \Omega_2; \tag{4.2}$$

$$u_1|_{t=0} = u_2|_{t=0} = 0; \tag{4.3}$$

$$|u_1(0,t)| < \infty, \tag{4.4}$$

$$u_1|_{r=R_1} = u_2|_{r=R_1}, \tag{4.5}$$

$$k_1 u_{1r}|_{r=R_1} = k_2 u_{2r}|_{r=R_1}, \tag{4.6}$$

$$u_2|_{r=R_2} = T = \text{const}, \tag{4.7}$$

$$u_{2r}|_{r=R_2} = Q = \text{const}. \tag{4.8}$$

Let us introduce change of variables $\widetilde{u_j} = u_j - T$ and apply Laplace transform to the problem (4.1)–(4.8). Thus we get following problem for images U_j of functions $\widetilde{u_j}$ (F_1 is image of function f_1):

$$U_1'' + \frac{2}{r} U_1' - \frac{p}{\chi_1} U_1 = -\frac{1}{\chi_1}(F_1 - T), \tag{4.9}$$

$$U_2'' + \frac{2}{r} U_2' - \frac{p}{\chi_2} U_2 = \frac{T}{\chi_2}; \tag{4.10}$$

$$|U_1(0,p)| < \infty, \tag{4.11}$$

$$U_1|_{r=R_1} = U_2|_{r=R_1}, \tag{4.12}$$

$$k_1 U_{1r}|_{r=R_1} = k_2 U_{2r}|_{r=R_1}, \tag{4.13}$$

$$U_2|_{r=R_2} = 0, \tag{4.14}$$

$$U_{2r}|_{r=R_2} = \frac{Q}{p}. \tag{4.15}$$

By using of the task (2.1)–(2.5) solution (see formulae (2.7)–(2.10)) and condition of overdetermination (4.15) we derive integral equation of the first kind for $F_1(r,p)$:

$$\int_0^{R_1} \xi F_1 \operatorname{sh}\left(\xi \sqrt{\frac{p}{\chi_1}} \right) d\xi = \frac{\chi_1 T}{p} \left(\frac{R_1 \sqrt{p} \operatorname{ch} a}{\sqrt{\chi_1}} - \operatorname{sh} a - \frac{C_1 C_3 e^b \operatorname{sh} a}{k} \right) +$$
$$+ \frac{\chi_1 R_2 \operatorname{sh} a \left(C_1 e^b + C_2 e^{-b} \right)}{2k\sqrt{p\chi_2}} \left(\frac{T}{R_2} \left[\sqrt{\frac{\chi_2}{p}} \Big((R_1 + 1) \operatorname{ch} b - R_2 - 1 \Big) + \left(\frac{\chi_2}{p} + R_1 \right) \operatorname{sh} b \right] - \frac{Q\chi_2}{p} \right), \tag{4.16}$$

where $a = R_1 \sqrt{p/\chi_1}$, $b = (R_2 - R_1)\sqrt{p/\chi_2}$. Assume that $f_1 = f_1(t)$, then $F_1(p)$ can be found from (3.9) and (4.15), or (4.16), as

$$F_1(p) = T + \left((QR_2 + T)\sqrt{\frac{\chi_2}{p}} - T \left[\left(\sqrt{\frac{\chi_2}{p}} + R_1 \right) \exp\left((R_2 - R_1)\sqrt{\frac{p}{\chi_2}} \right) - R_2 \right] \right) \times$$

$$\times\ \frac{C_2\exp\left((R_1-R_2)\sqrt{\frac{p}{\chi_2}}\right)+C_1\exp\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}{2k\left(1-R_1\sqrt{\frac{p}{\chi_1}}\,\mathrm{cth}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)\right)}+\frac{C_1C_3T\exp\left((R_2-R_1)\sqrt{\frac{p}{\chi_2}}\right)}{k\left(1-R_1\sqrt{\frac{p}{\chi_1}}\,\mathrm{cth}\left(R_1\sqrt{\frac{p}{\chi_1}}\right)\right)}. \tag{4.17}$$

The constants C_1, C_2, C_3 are specified by formulae (2.10), (3.10). So the heat source in sphere $0\leqslant r\leqslant R_1$ (that is function $f_1(t)$) can be obtained by using of inverse Laplace transform. Here we evaluate only $\lim\limits_{t\to\infty} f_1(t)$.

Since $e^x\approx 1+x+x^2/2$ when $x\to 0$, we derive from (4.17)

$$\lim_{p\to 0} pF_1(p)=-\frac{\chi_1}{kR_1^2}\lim_{p\to 0}\left(\sqrt{\frac{p}{\chi_2}}-\frac{1}{R_1}-\frac{kpR_1}{2\chi_1}\right)\left(1+(R_1-R_2)\sqrt{\frac{p}{\chi_2}}+\frac{p(R_1-R_2)^2}{2\chi_2}\right)\times$$
$$\times\left(QR_2\sqrt{\frac{\chi_2}{p}}-T\left[\frac{R_2^2-R_1^2}{2}\sqrt{\frac{p}{\chi_2}}+\frac{pR_1(R_2-R_1)^2}{2\chi_2}\right]\right)-$$
$$-\frac{\chi_1}{kR_1^2}\lim_{p\to 0}\left(\sqrt{\frac{p}{\chi_2}}+\frac{1}{R_1}+\frac{kpR_1}{2\chi_1}\right)\left(1-(R_1-R_2)\sqrt{\frac{p}{\chi_2}}+\frac{p(R_1-R_2)^2}{2\chi_2}\right)\times$$
$$\times\left(QR_2\sqrt{\frac{\chi_2}{p}}-T\left[\frac{R_2^2-R_1^2}{2}\sqrt{\frac{p}{\chi_2}}-\frac{pR_1(R_2-R_1)^2}{2\chi_2}\right]\right)+\lim_{p\to 0}Tp=-\frac{2\chi_1QR_2^2}{kR_1^3}\equiv f_1(\infty).$$

Thus the boundedness of internal heat sources at infinity is proved.

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Решение сопряженной тепловой задачи в шаровой области методом преобразования Лапласа

Илона А. Резникова

Исследована сферически симметричная сопряжённая начально-краевая задача распространения тепла в замкнутых ограниченных шаровых областях. Точное аналитическое решение прямой и обратной нестационарной задачи получено методом преобразования Лапласа. Найдено стационарное состояние и показано, что оно является предельным при больших временах, если таковыми служат источники тепла в средах.

Ключевые слова: начально-краевая задача, поверхность раздела, преобразование Лапласа, обратная задача.