The Hodge Filtration on Complements of Complex Subspace Arrangements and Integral Representations of Holomorphic Functions

Yury V. Eliyashev[*]

Institute of Mathematics and Computer Science,
Siberian Federal University,
Svobodny, 79, Krasnoyarsk, 660041
Russia

We compute the Hodge filtration on cohomology groups of complements of complex subspace arrangements. By means of this result we construct integral representations of holomorphic functions such that kernels of these representations have singularities on subspace arrangements.

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Introduction

A study of topology of coordinate subspace arrangements appears in different areas of mathematics: in toric topology and combinatorial topology [3, 4], in the theory of toric varieties, where complements to coordinate subspace arrangements play the role of homogeneous coordinate spaces [5, 6], in the theory of integral representations of holomorphic functions in several complex variables, where coordinate subspace arrangements play the role of singular sets of integral representations kernels [1, 10].

The universal combinatorial method for the computation of cohomology groups of complements to *arbitrary* subspace arrangements was developed in the book of Goresky and Macpherson [8] (see also [11]), but this method often leads to cumbersome computations. In the study of toric topology, in particular, in works of Buchstaber and Panov [3, 4], the method for the computation of the cohomology of complements to *coordinate* subspace arrangements was developed, this method is simpler than the universal method and allows to get some additional topological information.

The main purpose of this article is to compute the Hodge filtration on the cohomology rings of complements to complex coordinate subspace arrangements. We will show that the Hodge filtration is described by means of a special bigrading on the cohomology rings of complements to complex coordinate subspace arrangements, which was introduced in [3, 4], this bigrading was obtained originally from the combinatorial and topological ideas. We use these results to construct the integral representations of holomorphic functions such that the kernels of these representations have singularities on coordinate subspace arrangements.

The first section of this paper consists of different facts about topology of complements to complex coordinate subspace arrangements, in the text of this section we follow [3, 4]. Let Z be a complex coordinate subspace arrangement in $\mathbb{C}^n$. In [3, 4], from the topological reasons, the differential bigraded algebra R was introduced (R is determined by combinatorics of Z) such that the ring of cohomology $H^*(\mathbb{C}^n \setminus Z)$ is isomorphic to the ring of cohomology $H^*(R)$. Denote

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[*]eliyashev@mail.ru

by $H^{p,q}(R)$ the bigraded cohomology of the algebra R, then

$$H^s(\mathbb{C}^n \setminus Z) \simeq \bigoplus_{p+q=s} H^{p,q}(R).$$

Thus, we get a bigrading on the cohomology ring $H^*(\mathbb{C}^n \setminus Z)$.

In the second section we recall some facts and concept from differential topology and complex analysis. These facts we use in the last two sections.

In the third section the main theorem of this paper is proved. We will show that the bigrading on the cohomology of R and, consequently, the bigrading on the cohomology $H^*(\mathbb{C}^n \setminus Z)$ appear naturally from the complex structure on the manifold $\mathbb{C}^n \setminus Z$. In particular, denote by $F^k H^s(\mathbb{C}^n \setminus Z, \mathbb{C})$ a k-th term of the Hidge filtration on $H^s(\mathbb{C}^n \setminus Z, \mathbb{C})$. Then there is the following theorem.

Theorem 1.

$$F^k H^s(\mathbb{C}^n \setminus Z, \mathbb{C}) = \bigoplus_{p \geq k} H^{p,s-p}(R, \mathbb{C}).$$

In the last section we construct integral representations of holomorphic functions such that kernels of these representations have singularities on coordinate subspace arrangements.

1. General facts on topology of coordinate subspace arrangements

In this section different facts about topology of complements to coordinate subspaces arrangements are gathered. All statements of this section are taken from [4].

Let $\mathcal{K}$ be an arbitrary simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. Define a coordinate planes arrangement

$$Z_{\mathcal{K}} := \bigcup_{\sigma \notin \mathcal{K}} L_\sigma,$$

where $\sigma = \{i_1, \ldots, i_m\} \subseteq [n]$ is a subset in $[n]$ such that σ does not define a simplex in $\mathcal{K}$ and

$$L_\sigma = \{z \in \mathbb{C}^n : z_{i_1} = \cdots = z_{i_m} = 0\}.$$

Any arrangement of complex coordinate subspaces in $\mathbb{C}^n$ of codimension greater than 1 can be defined in this way.

Consider a cover $\mathcal{U}_{\mathcal{K}} = \{\mathcal{U}_\sigma\}_{\sigma \in \mathcal{K}}$ of $\mathbb{C}^n \setminus Z_{\mathcal{K}}$, where

$$\mathcal{U}_\sigma = \mathbb{C}^n \setminus \bigcup_{i \notin \sigma} \{z_i = 0\}.$$

By $D_\sigma^2 \times S_\gamma^1$ denote the following chain

$$D_\sigma^2 \times S_\gamma^1 = \{|z_i| \leqslant 1 : i \in \sigma; |z_j| = 1 : j \in \gamma, z_k = 1 : k \notin \gamma \cup \sigma\},$$

where $\sigma, \gamma \subseteq [n]$ and $\sigma \cap \gamma = \emptyset$. We define the form

$$\frac{dz_I}{z_I} = \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}, \tag{1}$$

where $I \subseteq [n], |I| = k$, $I = \{i_1, \ldots, i_k\}$, and $i_1 < \cdots < i_k$.

The orientation of the chain $D_\sigma^2 \times S_\gamma^1$ is such that the restriction of the form

$$\frac{1}{(\sqrt{-1})^{|\gamma|}} \frac{dz_\gamma}{z_\gamma} \wedge \bigwedge_{j \in \sigma} (\sqrt{-1} dz_j \wedge d\overline{z}_j)$$

on $D_\sigma^2 \times S_\gamma^1$ is positive. Then the boundary of this chain equals

$$\partial D_\sigma^2 \times S_\gamma^1 = \Sigma_{i \in \sigma}(-1)^{(i,\gamma)} D_{\sigma \setminus i}^2 \times S_{\gamma \cup i}^1,$$

where (i, γ) is the position of i in the naturally ordered set $\gamma \cup i$.

Definition 1. *The topological space*

$$\mathcal{Z}_\mathcal{K} = \bigcup_{\sigma \in \mathcal{K}} D_\sigma^2 \times S_{[n] \setminus \sigma}^1$$

is called a moment-angle complex.

Theorem 2 ([4]). *There is a deformation retraction from $\mathbb{C}^n \setminus Z_\mathcal{K}$ to $\mathcal{Z}_\mathcal{K}$.*

Definition 2. *A Stanley–Reisner ring of a simplicial complex $\mathcal{K}$ on the vertex set $[n]$ is a ring*

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \ldots, v_n]/\mathcal{I}_\mathcal{K},$$

where $\mathcal{I}_\mathcal{K}$ is a homogeneous ideal generated by the monomials $v_\sigma = \prod_{i \in \sigma} v_i$ such that $\sigma \notin \mathcal{K}$:

$$\mathcal{I}_\mathcal{K} = (v_{i_1} \cdot \ldots \cdot v_{i_m} : \{i_1, \ldots, i_m\} \notin \mathcal{K}).$$

Consider a differential bigraded algebra $(R(\mathcal{K}), \delta_R)$:

$$R_\mathcal{K} := \Lambda[u_1, \ldots, u_n] \otimes \mathbb{Z}[\mathcal{K}]/\mathcal{J},$$

where $\Lambda[u_1, \ldots, u_n]$ is an exterior algebra, $\mathcal{J}$ is the ideal generated by monomials $v_i^2, u_i \otimes v_i, i = 1, \ldots, n$. Bidegrees of generators v_i, u_i of this algebra are equal to

$$\text{bideg } v_i = (1, 1), \text{bideg } u_i = (1, 0).$$

The differential δ_R is defined on the generators as follows

$$\delta_R u_i = v_i, \delta_R v_i = 0.$$

Remark 1. *In [4] a different bigrading on the algebra $R_\mathcal{K}$ was used, but our bigrading is equivalent to the bigrading from [4].*

We denote by $R_\mathcal{K}^{p,q}$ the homogeneous component of the algebra $R_\mathcal{K}$ of the bidegree (p, q). The differential δ_R is compatible with the bigrading, i.e., $\delta_R(R_\mathcal{K}^{p,q}) \subseteq R_\mathcal{K}^{p,q+1}$. Consider the complex

$$\cdots \xrightarrow{\delta_R} R_\mathcal{K}^{p,q-1} \xrightarrow{\delta_R} R_\mathcal{K}^{p,q} \xrightarrow{\delta_R} R_\mathcal{K}^{p,q+1} \xrightarrow{\delta_R} \cdots,$$

denote by $H^{p,q}(R_\mathcal{K})$ a cohomology group of this complex. It is clear that the cohomology of $R_\mathcal{K}$ are isomorphic to

$$H^s(R_\mathcal{K}) = \bigoplus_{p+q=s} H^{p,q}(R_\mathcal{K}).$$

Theorem 3 ([4]). *The cohomology ring $H^*(\mathbb{C}^n \setminus Z_\mathcal{K})$ is isomorphic to the ring $H^*(R_\mathcal{K})$.*

Remark 2. *The relation between Theorem 3 and the results of Goresky and Macpherson [8] on cohomology of subspace arrangements is described in [4, Ch. 8].*

Now we describe the explicit construction of the isomorphism of Theorem 3. First, we construct a cell decomposition of $\mathcal{Z}_\mathcal{K}$. Define a cell

$$E_{\sigma\gamma} = \{|z_i| < 1 : i \in \sigma; |z_j| = 1, z_j \neq 1 : j \in \gamma; z_k = 1 : k \notin \gamma \cup \sigma\},$$

where $\sigma, \gamma \subseteq [n]$ and $\sigma \cap \gamma = \emptyset$. The closure of this cell equals $\overline{E_{\sigma\gamma}} = D_\sigma^2 \times S_\gamma^1$. The orientation of $E_{\sigma\gamma}$ is defined by the orientation of $D_\sigma^2 \times S_\gamma^1$. We obtain the cell decomposition

$$\mathcal{Z}_\mathcal{K} = \bigcup_{\sigma \in \mathcal{K}, \gamma \subseteq [n] \setminus \sigma} E_{\sigma\gamma}.$$

Let $C_*(\mathcal{Z}_\mathcal{K})$ be the group of cell chains of this cell decomposition, then denote by $C^*(\mathcal{Z}_\mathcal{K})$ the group of cell cochains. Let $E'_{\sigma\gamma}$ be a cocell dual to the cell $E_{\sigma\gamma}$, i.e., $E'_{\sigma\gamma}$ is a linear functional from $C_*(\mathcal{Z}_\mathcal{K})$ such that $\langle E'_{\sigma\gamma}, E_{\sigma'\gamma'} \rangle = \delta_{\sigma'\gamma'}^{\sigma\gamma}$ (the Kronecker delta).

Denote $u_I v_J := u_{i_1} \ldots u_{i_q} \otimes v_{j_1} \ldots v_{j_p}$, where $I = \{i_1, \ldots, i_q\}$, $i_1 < \cdots < i_q$, $J = \{j_1, \ldots, j_p\}$, and $I \cap J = \emptyset, I, J \subseteq [n]$, (we suppose that $u_\emptyset v_\emptyset = 1$).

Proposition 1 ([4]). *The linear map $\phi : R_\mathcal{K} \to C^*(\mathcal{Z}_\mathcal{K})$, $\phi(v_\sigma u_\gamma) = E'_{\sigma\gamma}$ is an isomorphism of differential bigraded modules. In particular, there is an additive isomorphism $H^*(R_\mathcal{K}) \overset{\phi}{\simeq} H^*(\mathcal{Z}_\mathcal{K})$.*

From the structure of the cell decomposition of $\mathcal{Z}_\mathcal{K}$ and Theorem 2 we obtain that every cycle $\Gamma \in H_s(\mathbb{C}^n \setminus Z_\mathcal{K})$ has a representative of the form

$$\Gamma = \sum_{p+q=s} \Gamma_{p,q}, \tag{2}$$

where $\Gamma_{p,q}$ is a cycle of the form

$$\Gamma_{p,q} = \sum_{\substack{|\sigma|=q \\ |\gamma|=p-q}} C_{\sigma\gamma} \cdot D_\sigma^2 \times S_\gamma^1. \tag{3}$$

A group generated by all cycles of the form (3) is denoted by $H_{p,q}(\mathbb{C}^n \setminus Z_\mathcal{K})$. Obviously we have

$$H_s(\mathbb{C}^n \setminus Z_\mathcal{K}) = \bigoplus_{p+q=s} H_{p,q}(\mathbb{C}^n \setminus Z_\mathcal{K}).$$

It follows from Proposition 1 that $\langle \Gamma_{p,q}, \phi(\omega^{p',q'}) \rangle = 0$ for any $\Gamma_{p,q} \in H_{p,q}(\mathbb{C}^n \setminus Z_\mathcal{K})$ and $\omega^{p',q'} \in H^{p',q'}(R_\mathcal{K})$, $p' \neq p$ and $q' \neq q$. Hence, the pairing between $\Gamma_{p,q}$ and $\phi(\omega^{p',q'})$ can be nonzero only if $p' = p, q' = q$. Therefore the pairing between the vector spaces $H_{p,q}(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{R})$ and $\phi(H^{p',q'}(R_\mathcal{K} \otimes \mathbb{R}))$ is nondegenerate if $p = p'$, $q = q'$ and equals to zero otherwise.

2. Cech cohomology, filtrations and cochains

In this section we recall some facts from differential topology and complex analysis, we mainly use a material from the books [2,9]. Let X be a complex manifold and $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ is an open, countable, locally finite cover of this manifold. Now we introduce the following notation for sheafs on X: $\mathcal{E}^s$ denotes the sheaf of C^∞-differential forms of degree s, $\mathcal{E}^{p,q}$ denotes the sheaf of C^∞-differential forms of bedegree (p,q), Ω^p denotes the sheaf of holomorphic differential forms of degree p.

Definition 3. *The decreasing filtration*

$$F^k \mathcal{E}^\bullet = \bigoplus_{p \geq k} \mathcal{E}^{p, \bullet - p},$$

on the de Rham complex $(\mathcal{E}^\bullet, d)$ is called the Hodge filtration.

The Hodge filtration induces a filtration $F^k H^s(X, \mathbb{C})$ on a de Rham cohomology, i.e.,

$$F^k H^s(X, \mathbb{C}) = \operatorname{Im}(H^s(F^k \mathcal{E}^\bullet(X), d) \to H^s(\mathcal{E}^\bullet(X), d)),$$

where $H^s(\mathcal{E}^\bullet(X), d)$ is the cohomology of the de Rham complex and $H^s(F^k \mathcal{E}^\bullet(X), d)$ is the cohomology of k-th term of the Hodge filtration. In other words, if ω lies in $F^k H^s(X, \mathbb{C})$ then there is a form $\widetilde{\omega}, [\widetilde{\omega}] = \omega$ such that

$$\widetilde{\omega} = \sum_{p \geq k} \widetilde{\omega}^{p, s-p},$$

where $\widetilde{\omega}^{p,q} \in \mathcal{E}^{p,q}(X)$.

Let $C^t(\mathcal{E}^s, \mathcal{U})$ be the Cech-de Rham double complex for the cover $\mathcal{U}$: $C^t(\mathcal{E}^s, \mathcal{U})$ with a Cech coboundary operator $\delta : C^t(\mathcal{E}^s, \mathcal{U}) \to C^{t+1}(\mathcal{E}^s, \mathcal{U})$ and a de Rham differential $d : C^t(\mathcal{E}^s, \mathcal{U}) \to C^t(\mathcal{E}^{s+1}, \mathcal{U})$ on this complex, i.e.,

$$(\delta \omega)_{i_0, \dots, i_{t+1}} = (-1)^s \sum_{j=0}^{t+1} (-1)^j \omega_{i_0, \dots, \widehat{i_j}, \dots, i_{t+1}} |_{\mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_{t+1}}},$$

$$(d\omega)_{i_0, \dots, i_t} = d(\omega)_{i_0, \dots, i_t}.$$

The associated single complex is defined by

$$K^r(\mathcal{U}, \mathcal{E}^\bullet) = \bigoplus_{s+t=r} C^t(\mathcal{E}^s, \mathcal{U})$$

the operator $D = \delta + d$ is the differential of this complex. Notice that our definition of Cech coboundary δ is different from the standard one by the factor $(-1)^s$, with this choice of sign we get $D^2 = 0$, hence $(K^\bullet(\mathcal{U}, \mathcal{E}^\bullet), D)$ is a complex. There is a natural inclusion of the de Rham complex $\varepsilon : \mathcal{E}^\bullet(X) \to C^0(\mathcal{E}^\bullet, \mathcal{U})$, $\varepsilon(\omega)_{j_0} = \omega|_{\mathcal{U}_{j_0}}$, also we denote the induced map from $\mathcal{E}^\bullet(X)$ to $K^\bullet(\mathcal{U}, \mathcal{E}^\bullet)$ by ε.

Theorem 4 ([2]). *The inclusion $\varepsilon : \mathcal{E}^\bullet(X) \to K^\bullet(\mathcal{U})$ is a quasi-isomorphism of complexes, i.e., $H^s(X, \mathbb{C}) \overset{\varepsilon}{\simeq} H^s(K^\bullet(\mathcal{U}, \mathcal{E}^\bullet), D)$.*

The Hodge filtration $F^k K^\bullet(\mathcal{U}, \mathcal{E}^\bullet)$ is defined naturally on $(K^\bullet(\mathcal{U}, \mathcal{E}^\bullet), D)$. This filtration induces a filtration on cohomology $F^k H^s(K^\bullet(\mathcal{U}, \mathcal{E}^\bullet), D)$. There is an isomorphism $F^k H^s(X, \mathbb{C}) \overset{\varepsilon}{\simeq} F^k H^s(K^\bullet(\mathcal{U}, \mathcal{E}^\bullet), D)$.

Consider a subcomplex $K^r(\mathcal{U}, \Omega^\bullet)$ of the complex $K^r(\mathcal{U}, \mathcal{E}^\bullet)$

$$K^r(\mathcal{U}, \Omega^\bullet) = \bigoplus_{s+t=r} C^t(\Omega^s, \mathcal{U}),$$

and an inclusion map $\tau : K^r(\mathcal{U}, \Omega^\bullet) \to K^r(\mathcal{U}, \mathcal{E}^\bullet)$. It is easy to get the following statement.

Theorem 5. *Suppose $\mathcal{U}$ is a $\overline{\partial}$-acyclic cover of X then the inclusion τ is a quasi-isomorphism of the complexes $K^\bullet(\mathcal{U}, \Omega^\bullet)$ and $K^\bullet(\mathcal{U}, \mathcal{E}^\bullet)$.*

Let $F^k\Omega^p$ be a stupid filtration on the de Rham complex of holomorphis forms $(\Omega^\bullet, d)$, i.e.,

$$F^k\Omega^p = \begin{cases} \Omega^p & \text{for } p \geq k, \\ 0 & \text{for } p < k. \end{cases}$$

The stupid filtration induces filtration on cohomology $F^k H^s(K^\bullet(\mathcal{U}, \Omega^\bullet), D)$. Suppose $\mathcal{U}$ is a $\bar{\partial}$-acyclic cover of X then $F^k H^s(K^\bullet(\mathcal{U}, \Omega^\bullet), D) \simeq F^k H^s(X, \mathbb{C})$.

From now until the end of this section we will follow the paper [7].

Definition 4. *A $\mathcal{U}$-chain of degree t and of dimension s on the manifold X is an alternating function Γ from the set of indexes $\mathcal{A}^{t+1}$ to the group of singular chains in X of dimension s such that Γ is nonzero on a finite number of points from $\mathcal{A}^{t+1}$ and*

$$\operatorname{supp}(\Gamma_{i_0,\dots,i_t}) \subset \mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_t},$$

for every $(i_0, \dots, i_t) \in \mathcal{A}^{t+1}$, where $\operatorname{supp}(\Gamma_{i_0,\dots,i_t})$ is the support of the chain $\Gamma_{i_0,\dots,i_t}$.

Let $C_{t,s}(\mathcal{U})$ be an additive group of $\mathcal{U}$-chains of degree t and of dimension s on the manifold X. Define maps $\delta' : C_{t,s}(\mathcal{U}) \to C_{t-1,s}(\mathcal{U})$

$$(\delta'\Gamma)_{i_0,\dots,i_{t-1}} = (-1)^s \sum_{i \in \mathcal{A}} \Gamma_{i,i_0,\dots,i_{t-1}},$$

and $\partial : C_{t,s}(\mathcal{U}) \to C_{t,s-1}(\mathcal{U})$

$$(\partial\Gamma)_{i_0,\dots,i_t} = \partial(\Gamma)_{i_0,\dots,i_t},$$

i.e., the operator ∂ is a boundary operator on each chain $\Gamma_{i_0,\dots,i_t}$. The groups $C_{t,s}(\mathcal{U}), t, s \geqslant 0$ together with the differentials δ', ∂ form a double complex. Define a map $\varepsilon' : C_{0,s}(\mathcal{U}) \to C_s(X)$ in the following way

$$\varepsilon'(\Gamma) = \sum_{i \in \mathcal{A}} \Gamma_i.$$

Now we will construct a pairing between elements of $C_{t,s}(\mathcal{U})$ and $C^t(\mathcal{E}^s, \mathcal{U})$. Suppose $\Gamma \in C_{t,s}(\mathcal{U})$ and $\omega \in C^t(\mathcal{E}^s, \mathcal{U})$, then

$$\langle \omega, \Gamma \rangle = \frac{1}{(t+1)!} \sum_{(i_0,\dots,i_t) \in \mathcal{A}^{t+1}} \int_{\Gamma_{i_0,\dots,i_t}} \omega_{i_0,\dots,i_t}.$$

There are the following relations for the pairing:

$$\langle \omega^{t,s}, \partial\Gamma_{t,s+1} \rangle = \langle d\omega^{t,s}, \Gamma_{t,s+1} \rangle,$$

$$\langle \delta\omega^{t,s}, \Gamma_{t+1,s} \rangle = \langle \omega^{t,s}, \delta'\Gamma_{t+1,s} \rangle,$$

$$\int_{\varepsilon'(\Gamma_{0,s})} \omega^s = \langle \varepsilon\omega^s, \Gamma_{0,s} \rangle,$$

where $\omega^{t,s} \in C^t(\mathcal{E}^s, \mathcal{U})$, $\omega^s \in \mathcal{E}^s(X)$, and $\Gamma_{t,s} \in C_{t,s}(\mathcal{U})$.

Definition 5. *Let Γ be a singular cycle of dimension s on X, then a $\mathcal{U}$-resolvent of length k of the cycle Γ is a collection of $\mathcal{U}$-chains $\Gamma^i \in C_{i,s-i}(\mathcal{U}), 0 \leqslant i \leqslant k$ such that $\Gamma = \varepsilon'\Gamma^0$ and $\partial\Gamma^i = -\delta'\Gamma^{i+1}$.*

Proposition 2. *Given an s-dimensional cycle Γ, a closed differential form ω of degree s, a $\mathcal{U}$-resolvent $\Gamma^0, \dots, \Gamma^k$ of the cycle Γ and a cocycle $\widetilde{\omega} \in K^s(\mathcal{U})$ such that $\widetilde{\omega} = \sum_{i \leqslant k} \widetilde{\omega}^{i,s-i}, \widetilde{\omega}^{i,s-i} \in C^i(\mathcal{E}^{s-i}, \mathcal{U})$ and the cocycle $\varepsilon\omega$ is cohomologous to $\widetilde{\omega}$ in $H^s(K^\bullet(\mathcal{U}, \mathcal{E}^\bullet), D)$, then*

$$\int_\Gamma \omega = \sum_{i \leqslant k} \langle \widetilde{\omega}^{i,s-i}, \Gamma^i \rangle.$$

This proposition follows directly from the properties of the pairing.

3. The Hodge filtration of cohomology of complements to coordinate subspace arrangements

In this section we compute the Hodge filtration of the cohomology ring $H^*(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C})$. It follows from Theorem 3 and Proposition 1 that there is the isomorphism $H^*(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C}) \overset{\phi}{\simeq} H^*(R_\mathcal{K} \otimes \mathbb{C})$.

Theorem 1. *Let $H^{p,q}(R_\mathcal{K} \otimes \mathbb{C})$ be the bigraded cohomology group of the complex $R_\mathcal{K}^{p,q} \otimes \mathbb{C}$, then there is an isomorphism*

$$F^k H^s(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C}) \overset{\phi}{\simeq} \bigoplus_{p \geq k} H^{p,s-p}(R_\mathcal{K} \otimes \mathbb{C}).$$

Proof. First, we will prove the lemma.

Lemma 1. *Let*

$$\Gamma_{p,q} = \sum_{\substack{|\sigma|=q \\ |\gamma|=p-q}} C_{\sigma\gamma} \cdot D_\sigma^2 \times S_\gamma^1$$

be a cycle in $\mathbb{C}^n \setminus Z_\mathcal{K}$. Then there is a $\mathcal{U}_\mathcal{K}$-resolvent of the cycle $\Gamma_{p,q}$ of length q:

$$\Gamma_{p,q}^0, \ldots, \Gamma_{p,q}^q,$$

where $\Gamma_{p,q}^k$ is a $\mathcal{U}_\mathcal{K}$-chain of dimension $q + p - k$ and of degree k of the form

$$(\Gamma_{p,q}^k)_{\alpha_0,\ldots,\alpha_k} = \sum_{\substack{|\sigma|=q-k \\ |\gamma|=p-q+k}} C_{\sigma\gamma,\alpha_0\ldots\alpha_k} \cdot D_\sigma^2 \times S_\gamma^1.$$

Proof. We will use the induction on the length k of the resolvent. We going to construct the resolvent of the special form

$$(\Gamma_{p,q}^k)_{\sigma_k,\sigma_{k-1},\ldots,\sigma_0} = \sum_{|\gamma|=p-q+k} C_{\sigma_k\gamma,\sigma_k\ldots\sigma_0} \cdot D_{\sigma_k}^2 \times S_\gamma^1,$$

for $|\sigma_j| = q - j$, $\sigma_j \subset \sigma_t$, $j > t$ $j = 0, \ldots, k$ (in other words, $\{\sigma_j\}$ is a chain of subsets $\sigma_k \subset \cdots \subset \sigma_0$, and $|\sigma_j| = q - j$); and $(\Gamma_{p,q}^k)_{\alpha_0,\ldots,\alpha_k} = 0$ for any other indexes $\alpha_0, \ldots, \alpha_k$.

The base of induction: define $(\Gamma_{p,q}^0)_{\sigma_0} = \sum_{|\gamma|=p-q} C_{\sigma_0\gamma} \cdot D_{\sigma_0}^2 \times S_\gamma^1$ with $|\sigma_0| = q$ and $(\Gamma_{p,q}^0)_\alpha = 0$ for any other indexes α. We get

$$\Gamma_{p,q} = \sum_{\substack{|\sigma_0|=q \\ |\gamma|=p-q}} C_{\sigma_0\gamma} \cdot D_{\sigma_0}^2 \times S_\gamma^1 = \sum_{\sigma \in \mathcal{K}} (\Gamma_{p,q}^0)_\sigma = \varepsilon' \Gamma_{p,q}^0,$$

therefore $\Gamma_{p,q}^0$ is the resolvent of length 0.

Suppose that the resolvent $\Gamma_{p,q}^0, \ldots, \Gamma_{p,q}^k$ of length k is already constructed. Recall that (i, γ) is the position of i in the naturally ordered set $\gamma \cup i$. Define

$$(\Gamma_{p,q}^{k+1})_{\sigma_k \setminus i, \sigma_k\ldots\sigma_0} = (-1)^{p+q-k} \sum_{|\gamma|=p-q} (-1)^{(i,\gamma)} C_{\sigma_k\gamma,\sigma_k\ldots\sigma_0} D_{\sigma_k \setminus i}^2 \times S_{\gamma \cup i}^1,$$

for $i \in \sigma_k$, $|\sigma_j| = q - j$, $\sigma_{j+1} \subset \sigma_j$, and

$$(\Gamma_{p,q}^{k+1})_{\alpha_0,\ldots,\alpha_{k+1}} = 0,$$

for any other indexes $\alpha_0, \ldots, \alpha_{k+1}$. Let us show that $\Gamma_{p,q}^0, \ldots, \Gamma_{p,q}^{k+1}$ is a resolvent of length $k+1$:

$$- (\delta' \Gamma_{p,q}^{k+1})_{\sigma_k \ldots \sigma_0} = (-1)^{p+q-k} \sum_{i \in \sigma_k} (\Gamma_{p,q}^{k+1})_{\sigma_k \backslash i, \sigma_k \ldots \sigma_0} =$$

$$= \sum_{\substack{i \in \sigma_k \\ |\gamma| = p-q}} (-1)^{(i,\gamma)} C_{\sigma_k \gamma, \sigma_k \ldots \sigma_0} D_{\sigma_k \backslash i}^2 \times S_{\gamma \cup i}^1 = \sum_{|\gamma| = p-q} C_{\sigma_k \gamma, \sigma_k \ldots \sigma_0} \partial D_{\sigma_k}^2 \times S_\gamma^1 = (\partial \Gamma_{p,q}^k)_{\sigma_k \ldots \sigma_0}.$$

For any indexes $\alpha_0, \ldots, \alpha_k$ different from $\sigma_k, \ldots, \sigma_{m+1}, \sigma_{m-1}, \ldots, \sigma_0$, $0 \le m \le k$, directly from definition of $\Gamma_{p,q}^k, \Gamma_{p,q}^{k+1}$, we get

$$(\partial \Gamma_{p,q}^k)_{\alpha_0, \ldots, \alpha_k} = -(\delta' \Gamma_{p,q}^{k+1})_{\alpha_0, \ldots, \alpha_k} = 0.$$

Consider the last case $\sigma_k \backslash i, \sigma_k, \ldots, \sigma_{m+1}, \sigma_{m-1}, \ldots, \sigma_0$, for $0 \le m \le k$. Since by the induction hypothesis $\Gamma_{p,q}^0 \ldots \Gamma_{p,q}^k$ is a resolvent, $-\delta' \Gamma_{p,q}^k = \partial \Gamma_{p,q}^{k-1}$, hence we have $\delta' \partial \Gamma_{p,q}^k = 0$, and

$$(\delta' \partial \Gamma_{p,q}^k)_{\sigma_k \ldots \sigma_{m+1} \sigma_{m-1} \ldots \sigma_0} =$$
$$= (-1)^{p+q-k+1} \sum_{\substack{\sigma_{m+1} \subset \sigma_m \subset \sigma_{m-1} \\ |\sigma_m| = q-m}} \sum_{|\gamma| = p-q+k} \sum_{i \in \sigma_k} (-1)^{(i,\gamma)} C_{\sigma_k \gamma, \sigma_k \ldots \sigma_0} \cdot D_{\sigma_k \backslash i}^2 \times S_{\gamma \cup i}^1 = 0.$$

Therefore, for a fixed $i \in \sigma_k$, we get

$$\sum_{\substack{\sigma_{m+1} \subset \sigma_m \subset \sigma_{m-1} \\ |\sigma_m| = q-m}} \sum_{|\gamma| = p-q+k} (-1)^{(i,\gamma)} C_{\sigma_k \gamma, \sigma_k \ldots \sigma_0} \cdot D_{\sigma_k \backslash i}^2 \times S_{\gamma \cup i}^1 = 0.$$

On the other side,

$$(\delta' \Gamma_{p,q}^{k+1})_{\sigma_k \backslash i, \sigma_k \ldots \sigma_{m+1} \sigma_{m-1} \ldots \sigma_0} =$$
$$= (-1)^{p+q-k+1} \sum_{\substack{\sigma_{m+1} \subset \sigma_m \subset \sigma_{m-1} \\ |\sigma_m| = q-m}} \sum_{|\gamma| = p-q+k} (-1)^{(i,\gamma)} C_{\sigma_k \gamma, \sigma_k \ldots \sigma_0} \cdot D_{\sigma_k \backslash i}^2 \times S_{\gamma \cup i}^1,$$

hence $(\delta' \Gamma_{p,q}^{k+1})_{\sigma_k \backslash i, \sigma_k \ldots \sigma_{m+1} \sigma_{m-1} \ldots \sigma_0} = 0$. We have shown that $\partial \Gamma_{p,q}^k = -\delta' \Gamma_{p,q}^{k+1}$. $\qquad \square$

It follows from Theorem 2 and the construction of the cell decomposition of the moment-angle complex $\mathcal{Z}_\mathcal{K}$ that any cycle $\Gamma_s \in H_s(\mathbb{C}^n \backslash Z_\mathcal{K})$ can be represented as a sum of the cycles $\Gamma_{p,q}$:

$$\Gamma^s = \sum_{\substack{p+q=s \\ p \ge q}} \Gamma_{p,q},$$

where $\Gamma_{p,q}$ is the cycles of the form (3). From Lemma 1 we have the construction of the resolvent $\Gamma_{p,q}^0, \ldots, \Gamma_{p,q}^q$ of the cycle $\Gamma_{p,q}$.

The cover $\mathcal{U}_\mathcal{K}$ is $\overline{\partial}$-acyclic. Indeed, all elements of the cover and their intersections are isomorphic to $\mathbb{C}^{n-k} \times (\mathbb{C}^*)^k$ for appropriate choice of k and consequently are a Stein manifolds.

From Theorem 4 and Theorem 5 we obtain that $H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet), D)$ is isomorphic to the de Rham cohomology group $H^s(\mathbb{C}^n \backslash Z_\mathcal{K}, \mathbb{C})$. Recall that we use the following notation for the inclusions of complexes

$$\varepsilon : \mathcal{E}^\bullet(\mathbb{C}^n \backslash Z_\mathcal{K}) \to K^\bullet(\mathcal{U}_\mathcal{K}, \mathcal{E}^\bullet),$$
$$\tau : K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet) \to K^\bullet(\mathcal{U}_\mathcal{K}, \mathcal{E}^\bullet).$$

We will use the same notation for the induced isomorphisms on the cohomology groups:

$$H^s(\mathbb{C}^n \backslash Z_\mathcal{K}, \mathbb{C}) \overset{\varepsilon}{\simeq} H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \mathcal{E}^\bullet), D) \overset{\tau}{\simeq} H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet), D).$$

Lemma 2. *Let* $\omega \in H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet), D)$, *then there is a cocycle* $\widetilde{\omega}$ *such that* $\widetilde{\omega} = \omega$ *in* $H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet), D)$, $\widetilde{\omega} = \sum_{p+q=s} \widetilde{\omega}^{p,q}$, $\widetilde{\omega}^{p,q} \in C^q(\mathcal{U}_\mathcal{K}, \Omega^p)$, *and*

$$(\widetilde{\omega}^{p,q})_{\sigma_0,\ldots,\sigma_q} = \sum_{|I|=p} C_{I,\sigma_0,\ldots,\sigma_q} \frac{dz_I}{z_I},$$

where $D\widetilde{\omega}^{p,q} = 0$ *for any* p *and* q.

Proof. Consider an arbitrary element ω of $H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet), D)$, this element is representable by cocycle $\omega = \sum_{p+q=s} \omega^{p,q}$, where $\omega^{p,q} \in C^q(\mathcal{U}_\mathcal{K}, \Omega^p)$ and $\delta\omega^{p,q} = -d\omega^{p-1,q+1}$. The cocycle ω has a unique decomposition $\omega = \widetilde{\omega} + \psi$, where $(\widetilde{\omega}^{p,q})_{\sigma_0,\ldots,\sigma_q}$ is the following form

$$(\widetilde{\omega}^{p,q})_{\sigma_0,\ldots,\sigma_q} = \sum_{|I|=p} C_{I,\sigma_0,\ldots,\sigma_q} \frac{dz_I}{z_I}, \tag{4}$$

and the Laurent expansion of $(\psi^{p,q})_{\sigma_0,\ldots,\sigma_q}$ does not contain summands $\dfrac{dz_I}{z_I}$.

Let us show that $\widetilde{\omega}$ and ψ are cocycles. Since ω is a cocycle, we have $\delta\widetilde{\omega}^{p,q} + \delta\psi^{p,q} = -d\widetilde{\omega}^{p-1,q+1} - d\psi^{p-1,q+1}$. The forms $\dfrac{dz_I}{z_I}$ are closed, hence $d\widetilde{\omega}^{p-1,q+1} = 0$. Since the Laurent expansions of the components of the cochain $\delta\psi^{p,q}$ do not contain summand $\dfrac{dz_I}{z_I}$ and the cochain $\delta\widetilde{\omega}^{p,q}$ can be exact if and only if $\delta\widetilde{\omega}^{p,q} = 0$ (because nonzero linear combinations of the forms $\dfrac{dz_I}{z_I}$ are nonexact on any elements of the cover $\mathcal{U}_\mathcal{K}$), $\delta\widetilde{\omega}^{p,q} = 0$. We get that $\delta\widetilde{\omega}^{p,q} = d\widetilde{\omega}^{p,q} = 0$, consequently, $\widetilde{\omega}$ is a cocycle. The cochain $\psi = \omega - \widetilde{\omega}$ is a difference of two cocycles, hence ψ is a cocycle.

Now we going to show that ψ is a coboundary.

Lemma 3. *Let* $\Gamma \in H_s(\mathbb{C} \setminus Z_\mathcal{K})$, *then* $\int_\Gamma \varepsilon^{-1} \circ \tau(\psi) = 0$.

Proof. For the cycle Γ we have the expansion to the sum $\Gamma = \sum_{p+q=s} \Gamma_{p,q}$. Lemma 1 gives the explicit construction of the resolvent $\Gamma^0_{p,q}, \ldots, \Gamma^q_{p,q}$ of the cycle $\Gamma_{p,q}$. Let us construct a cocycle $\widetilde{\psi}_k = \sum_{p+q=s} \widetilde{\psi}^{p,q}_k$ cohomologous to $\tau(\psi)$ of $K^\bullet(\mathcal{U}_\mathcal{K}, \mathcal{E}^\bullet)$

$$\widetilde{\psi}^{p,q}_k = \begin{cases} \psi^{p,q} & \text{for } q<k, \\ \psi^{p,q} - d\delta^{-1}(\psi^{p-1,q+1} - d\delta^{-1}(\psi^{p-2,q+2} - d\delta^{-1}(\cdots - d\delta^{-1}(\psi^{0,p+q})))) & \text{for } q=k, \\ 0 & \text{for } q>k. \end{cases}$$

From Proposition 2 we obtain

$$\int_\Gamma \varepsilon^{-1} \circ \tau(\psi) = \sum_{p+q=s} \sum_{k \leqslant q} \langle \Gamma^k_{p,q}, \widetilde{\psi}^{s-k,k}_q \rangle.$$

Let $k < q$ and $\omega \in \Omega^{p+q-k}(\mathcal{U}_{\sigma'})$, it is easy to see, that $\omega|_{D^2_\sigma \times S^1_\gamma} = 0$ for $|\sigma| = q - k > 0, |\gamma| = p - q + k, \sigma \subseteq \sigma'$. The forms $(\widetilde{\psi}^{s-k,k}_q)_{\alpha_0,\ldots,\alpha_k}$ are holomorphic on $\mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_k}$, indeed, $(\widetilde{\psi}^{s-k,k}_q)_{\alpha_0,\ldots,\alpha_k} = (\psi^{s-k,k})_{\alpha_0,\ldots,\alpha_k}$, on the other side, $(\Gamma^k_{p,q})_{\alpha_0,\ldots,\alpha_k}$ is a linear combination of the chains $D^2_\sigma \times S^1_\gamma$, $|\sigma| = q - k > 0, |\gamma| = p - q + k$, combining these two facts we get $\langle \Gamma^k_{p,q}, \widetilde{\psi}^{s-k,k}_q \rangle = 0$.

Consider the case $k = q$. From the definition of $\widetilde{\psi}$ it follows that $\widetilde{\psi}^{p,q}_q = \psi^{p,q} + d\varphi$ for some $\varphi \in C^q(\mathcal{E}^{p-1}, \mathcal{U})$. Since $(\Gamma^q_{p,q})_{\alpha_0,\ldots,\alpha_k}$ is a linear combination of the cycles $S^1_\gamma, |\gamma| = p$,

$$S^1_\gamma = \{|z_j| = 1 : j \in \gamma, z_k = 1 : k \notin \gamma\},$$

$\langle \Gamma^q_{p,q}, \widetilde{\psi}^{s-q,q}_q \rangle = \langle \Gamma^q_{p,q}, \psi^{s-q,q} \rangle$. Indeed, by the Stokes formula $\int_{S^1_\gamma} d\varphi = 0$, hence $\langle \Gamma^q_{p,q}, d\varphi \rangle = 0$.

Expand the from $(\psi^{s-q,q})_{\alpha_0,\dots,\alpha_q}$ to the Laurent series,

$$(\psi^{s-q,q})_{\alpha_0,\dots,\alpha_q} = \sum_{a=(a_1,\dots,a_n)\in\mathbb{Z}^n} \sum_{|I|=p} C_{a,I,\alpha_0\dots\alpha_q} z_1^{a_1}\dots z_n^{a_n} \frac{dz_I}{z_I}.$$

The integral $\displaystyle\int_{S^1_\gamma} z_1^{a_1}\dots z_n^{a_n} \frac{dz_I}{z_I}$ is nonzero only if $I = \gamma$ and $a = 0$, i.e., for the forms $\frac{dz_\gamma}{z_\gamma}$, but by the construction of $\psi^{s-q,q}$ the Laurent expansion of $(\psi^{s-q,q})_{\alpha_0,\dots,\alpha_q}$ does not contain the summands $\dfrac{dz_\gamma}{z_\gamma}$. Consequently, $\langle \Gamma^q_{p,q}, \psi^{s-q,q} \rangle = 0$.

We have shown that $\int_\Gamma \varepsilon^{-1}\circ\tau(\psi) = 0$. Lemma 3 is proved. □

By the de Rham Theorem any closed form ω of degree s on $\mathbb{C}^n \setminus Z_\mathcal{K}$ is exact if and only if $\int_\Gamma \omega = 0$ for any cycle $\Gamma \in H_s(\mathbb{C}^n \setminus Z_\mathcal{K})$. It follows from Lemma 3 that $\varepsilon^{-1}\circ\tau(\psi)$ is cohomologous to zero, hence ψ is a coboundary . Lemma 2 is proved. □

By Lemma 2 any cocycle $\omega \in H^s(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D)$ is cohomologous to $\widetilde{\omega} = \sum_{p+q=s} \widetilde{\omega}^{p,q}$, where $\widetilde{\omega}^{p,q}$ is of the form (4). Moreover, $\widetilde{\omega}^{p,q} \in C^q(\mathcal{U}_\mathcal{K},\Omega^p)$ is a cocycle, i.e., $D\widetilde{\omega}^{p,q} = 0$. We denote by $H^{p,q}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D)$ a subspace of $H^s(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D)$ generated by cocycles $\widetilde{\omega}^{p,q}$. We obtain

$$H^s(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D) = \bigoplus_{p+q=s} H^{p,q}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D).$$

Then the filtration $F^k H^s(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D)$ equals

$$F^k H^s(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D) = \bigoplus_{p\geq k} H^{p,s-p}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D).$$

Hence,

$$F^k H^s(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C}) \overset{\varepsilon^{-1}\circ\tau}{\simeq} \bigoplus_{p\geq k} H^{p,s-p}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D).$$

By the same argument as in Lemma 3 we obtain that for every cycle $\Gamma_{p,q} \in H_{p,q}(\mathbb{C}^n \setminus Z_\mathcal{K})$, and every cocycle $\widetilde{\omega}^{p',q'} \in H^{p',q'}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D)$, the following equality holds

$$\int_{\Gamma_{p,q}} \varepsilon^{-1}\circ\tau(\widetilde{\omega}^{p',q'}) = 0,$$

for $p \neq p'$, $q \neq q'$. It follows from nondegeneracy of the pairing between cohomology and homology that the pairing between elements of $H_{p,q}(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C})$ and $\varepsilon^{-1}\circ\tau(H^{p',q'}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D))$ is nondegenerate if $p = p'$, $q = q'$ and equals to zero otherwise. Thus, $\varepsilon^{-1}\circ\tau(H^{p',q'}(K^\bullet(\mathcal{U}_\mathcal{K},\Omega^\bullet), D)) = \phi(H^{p',q'}(R_\mathcal{K}\otimes\mathbb{C}))$. □

4. Integral representations of holomorphic functions

In the last section we study integral representations of holomorphic functions such that kernels of these integral representations have singularities on coordinate subspace arrangements in $\mathbb{C}^n$. The examples of such integral representations are the multidimensional Cauchy integral representation, whose kernel has singularity on $(\{z_1 = 0\} \cup \dots \cup \{z_n = 0\})$, and the Bochner–Martinelli integral representation, whose kernel has singularity on $\{0\}$. In [10] a family of new integral representations of this kind was obtained, the kernels of these integral representations have singularities on the subspace arrangements defined by simple polytopes.

Denote by U the unit polydisc in $\mathbb{C}^n$:

$$U = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}.$$

Notice that the moment-angle complex $\mathcal{Z}_\mathcal{K}$ is lying on the boundary ∂U of the polydisc.

Theorem 6. *Given a nontrivial element ω' from $F^n H^s(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C})$. Then there exists a closed $(n, s-n)$-form ω, $[\omega] = \omega'$ and an s-dimensional cycle Γ in $\mathbb{C}^n \setminus Z_\mathcal{K}$ with support in $\mathcal{Z}_\mathcal{K}$, such that for any function f holomorphic in some neighborhood of U the following integral representation holds*

$$f(\zeta) = \int_\Gamma f(z)\omega(z - \zeta)$$

for $\zeta \in U$.

Proof. Since $\omega' \in F^n H^s(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C})$, by Theorem 1 there is a cycle $\Gamma \in H_s(\mathbb{C}^n \setminus Z_\mathcal{K}, \mathbb{C})$,

$$\Gamma = \sum_{\substack{|\sigma|=s-n \\ |\gamma|=2n-s}} C_{\sigma\gamma} \cdot D_\sigma^2 \times S_\gamma^1,$$

such that $\langle \Gamma, \omega' \rangle = 1$. It follows from Lemma 2 that there exists a cocycle $\omega^{n,s-n} \in C^{s-n}(\mathcal{U}_\mathcal{K}, \Omega^n)$,

$$(\omega^{n,s-n})_{\alpha_0,\ldots,\alpha_{s-n}} = B_{\alpha_0,\ldots,\alpha_{s-n}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

that is cohomologous to $\tau^{-1} \circ \varepsilon(\omega')$ in $H^s(K^\bullet(\mathcal{U}_\mathcal{K}, \Omega^\bullet), D)$. The form $\omega = \varepsilon^{-1} \circ (-\overline{\partial}\delta^{-1})^{s-n}\omega^{n,s-n}$ is cohomologous to ω', so $\int_\Gamma \omega = 1$.

Let us show that ω and γ define an integral representation. Consider the integral $\int_\Gamma \omega(z-\zeta)$, where $\zeta \in U$, here the notation $\omega(z - \zeta)$ stands for the form ω after the change of coordinates $z \to z-\zeta$. Notice that the form $\omega(z-\zeta)$ is closed in U, thus the integral of this form depends only on the homological class of the integration cycle. Let us make a change of coordinates $\int_{\Gamma-\zeta} \omega(z)$ where $\Gamma-\zeta$ is a cycle Γ shifted by the vector $-\zeta$. In the sequel we will use the subindex $-\zeta$ to denote chains, cycles, and sets in $\mathbb{C}^n$ shifted by the vector $-\zeta$.

Let us show that $\Gamma - \zeta$ is homologous to Γ. Notice that $(\mathcal{Z}_\mathcal{K} - \zeta) \cap Z_\mathcal{K} = \varnothing$ for any $\zeta \in U$. Indeed,

$$\mathcal{Z}_\mathcal{K} - \zeta = \bigcup_{\sigma \in \mathcal{K}} (D_\sigma^2 \times S_{[n]\setminus\sigma}^1 - \zeta), \qquad Z_\mathcal{K} = \bigcup_{\sigma \notin \mathcal{K}} L_\sigma,$$

we see that $(D_\sigma^2 \times S_{[n]\setminus\sigma}^1 - \zeta) \cap L_{\sigma'} = \emptyset$ for any $\sigma \in K, \sigma' \notin K$ and $\zeta \in U$. Consider the chain $\widetilde{\Gamma_{-\zeta}} = \{y : y = x - t\zeta, x \in \Gamma, t \in [0,1]\}$, the support of the chain $\widetilde{\Gamma_{-\zeta}}$ is a subset of $\bigcup_{t \in [0,1]}(\mathcal{Z}_\mathcal{K} - t\zeta)$, therefore $\widetilde{\Gamma_{-\zeta}}$ is a subset of $\mathbb{C}^n \setminus Z_\mathcal{K}$. Its boundary equals $\partial\widetilde{\Gamma_{-\zeta}} = (\Gamma - \zeta) - \Gamma$, i.e., $(\Gamma - \zeta)$ and Γ are homologous. So we have returned to the case $\int_\Gamma \omega(z)$, which was already considered. We get

$$\int_\Gamma \omega(z - \zeta) = \int_{\Gamma-\zeta} \omega(z) = 1.$$

By dentition $\omega(z - \zeta)$ is an $(n, s - n)$-form. Let $f(z)$ be a function holomorphic in some neighborhood of unit polydisc U. Since the operators $\overline{\partial}$ and δ are interchangeable with the multiplication by a holomorphic function, we get $f(z) \cdot \omega(z - \zeta) = \varepsilon^{-1} \circ (-\overline{\partial}\delta^{-1})^{s-n} f(z) \cdot \omega^{n,s-n}(z - \zeta)$. By Lemma 1 there is a resolvent $\Gamma^0, \ldots, \Gamma^{s-n}$ of the cycle Γ such that

$$\Gamma_{\alpha_0,\ldots,\alpha_{s-n}}^{s-n} = C'_{\alpha_0,\ldots,\alpha_{s-n}} \cdot S_{[n]}^1, \quad S_{[n]}^1 = \{|z_1| = \cdots = |z_n| = 1\}.$$

Since

$$\int_{\Gamma} \omega(z - \zeta) = \langle \Gamma^{s-n}, \omega^{n,s-n}(z - \zeta) \rangle = 1,$$

from the Cauchy integral representation formula we get

$$\int_{\Gamma} f(z)\omega(z - \zeta) = \langle \Gamma^{s-n}, f(z) \cdot \omega^{n,s-n}(z - \zeta) \rangle = f(\zeta).$$

$\square$

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Фильтрация Ходжа на дополнениях к наборам комплексных координатных подпространств и интегральные представления голоморфных функций

Юрий В. Элияшев

В статье вычисляется фильтрация Ходжа на когомологиях дополнений к наборам комплексных координатных подпространств. Эти результаты используются для нахождения интегральных представлений голоморфных функций, в которых ядра имеют сингулярности на наборах координатных подпространств.

Ключевые слова: фильтрация Ходжа, конфигурации плоскостей, интегральные представления, торическая топология.