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To Properties of Solutions to Reaction-diffusion Equation with Double Nonlinearity with Distributed Parameters

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The properties of solutions of self-similar and approximately self-similar system of the reaction-diffusion with double nonlinearity are investigated. The influence of numerical parameters to an evolution of the studied process is established. The existence of finite and quenching solutions is proved and their asymptotic behavior at the infinity is described. The condition of global solvability to the Cauchy problem, generalizing the results of other authors, is found. Knerr -Kersner type estimate for free boundary is obtained. The results of numerical experiments are enclosed.

Keywords: Reaction-diffusion equation, double nonlinearity, free boundary.

Introduction

In the domain $Q = \{(t, x): \ t > 0, \ x \in R^N\}$, let us consider the Cauchy problem for the double nonlinearity parabolic equation

$$Au \equiv -\rho(x)\frac{\partial u}{\partial t} + L(n, m, p)u + \varepsilon\gamma(t)\rho(x)u^{\beta} = 0, \tag{1}$$

$$u|_{t=0} = u_0(x) \geqslant 0, \quad x \in R^N \tag{2}$$

where $L(n, m, p)u = \nabla(|x|^n u^{m-1} |\nabla u|^{p-2} \nabla u)$, $\beta \geqslant 1$, n, p, $m-$ are the given numerical parameters, $\nabla(\cdot) -$ $\mathrm{grad}_x(\cdot)$, $0 < \gamma(t) \in C(0, \infty)$, $\varepsilon = \pm 1$, $\rho(x) = |x|^{\delta-2}$.

The equation (1) is a base for modeling many physical processes [1–7], for example the processes of reaction-diffusion, a heat conductivity, a polytrophic filtration of gases and liquids in nonlinear media with source ($\varepsilon = +1$) or absorption ($\varepsilon = -1$) having power $\rho(x)\gamma(t)u^{\beta}$.

The equation (1) is degenerated to an equation of the first order in the domain where $u = 0$ or $\nabla u = 0$. Therefore we will investigate the weak solution, because in this case there might be no solutions to (1) in the classical sense.

Before the numerical investigation and visualization of processes describing by the equation (1) it is necessary to study different qualitative properties such as finite velocity of perturbation, localization of solutions, asymptotic behavior of solutions and free boundary (fronts), depending on the numerical parameters of the equation (1).

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The properties of the solution to the problem (1), (2) depend on the value of parameters of the equation (1). The cases where $\beta \geqslant 1$, $0 < \beta < 1$, $\varepsilon = +1$, $\varepsilon = -1$, $\gamma(t) = 1$, $m = 1$ or $p = 2$ were thoroughly studied in [1–5].

From the viewpoint of physics it is reasonable to consider weak solutions that are bounded, non-negative, i.e. $0 \leqslant u(t,x)$, having the continuous source, i.e. $|x|^n u^{m-1} \left|\nabla u^{p-2}\right| \nabla u \in C(Q)$, and satisfying some integral identity [1, 6].

In the case $m + p - 3 > 0$, it is shown that there exist the solutions $u(t,x)$, which possess the property of the finite velocity of the propagation of perturbation. It means that for every $t > 0$ there exists such a continuous function $l(t)$, that $u(t,x) \equiv 0$ as $|x| \geqslant l(t)$ (in the linear case when $m = 1$, $p = 2$ it is trivial). The surface $|x| = l(t)$ is called a front of perturbation or a free boundary.

The solution of the equation (1) satisfying $l(\infty) < +\infty$, and $u(t,x) \equiv 0$ for $|x| \geqslant l(t)$ is called a localized solution.

In this paper it is studied the properties of solutions of (1) such as the finite speed of propagation of the perturbance, the localization of bounded and unbounded solutions. The main terms of the asymptotic of self-similar equations, the behavior of the front (the free boundary), depending on the parameters, the condition of the global solvability of the Cauchy problem for the equation (1) are established. We use the method of nonlinear splitting for the construction of approximately self-similar equation for nonlinear reaction-diffusion equation [1]. Based on these results, numerical calculations and visualization the process for the one-and two-dimensional cases are done.

1. Method of the nonlinear splitting and standard equations.

A method of nonlinear splitting [8–10] for construction of the self-similar, and approximately self-similar equation to the equation (1) is presented below. It facilitates the investigation of the qualitative properties of the solution to the problem (1), (2).

The self-similar equation associated with the partial differential equation is an ordinary differential equation obtained from the initial one by the transformation depending on an unknown function of one variable with the argument being a combination of the independent variables.

As shown in [1, 8, 11], self-similar equations play an important role in the study of qualitative properties of solutions to nonlinear partial differential equations, because the use of self-similar analysis of solutions can detect the characteristic properties of new nonlinear phenomena [1]. Below we propose a method for constructing self-similar equation associated to (1), based on the splitting of the original partial differential equation [8].

With this purpose, first we solve the simple equation

$$\frac{d\bar{u}}{dt} = \varepsilon\gamma(t)\bar{u}^{\beta} \tag{3}$$

Now we are looking for solutions to the equation (1) in the form

$$u(t,x) = \bar{u}(t)w(\tau(t), \varphi(x)), \tag{4}$$

where the function

$$\overline{u}(t) = [(T + \varepsilon(\beta - 1)\int_0^t \gamma(\eta)d\eta)]^{-1/(\beta-1)}$$

is a solution to the equation (3) and $w(\tau, x)$ is a solution to the equation (1) without lower order term and in the radially symmetric form:

$$Aw \equiv -\frac{\partial w}{\partial \tau} + \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m-1}\left|\frac{\partial w}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) = 0, \tag{5}$$

$$\tau(t) = \begin{cases} \displaystyle\int_0^t [\bar{u}(x)]^{p+m-3}dx, & \text{if } p+m-3 \neq 0; \\ (T+t), \quad \text{if } \ p+m-3=0, \ T>0 & - \text{const.} \end{cases} \tag{6}$$

$\varphi(x) = \dfrac{1}{p_1}|x|^{p_1}$, $\quad p_1 = 1 + \dfrac{\delta-(n+2)}{p}$, $\quad s = p(N+\delta-2)/(p+\delta-(n+2))$.

Substituting (4) to the equation (1) reduces the consideration to the following approximately self-similar equation:

$$\frac{\partial w}{\partial \tau} = \varphi^{1-s}\frac{\partial}{\partial \varphi}\left(\varphi^{s-1}w^{m-1}\left|\frac{\partial w}{\partial \varphi}\right|^{p-2}\frac{\partial w}{\partial \varphi}\right) + \varepsilon\gamma(t)\bar{u}^{\beta-(p+m-2)}(-w+w^{\beta}) \tag{7}$$

It is easy to establish, that equation (5) has the self-similar solution of the type:

$$w(\tau, x) = f(\eta). \tag{8}$$

Here $\eta = \dfrac{\varphi(x)}{[\tau(t)]^{\frac{1}{p}}}$ and $f(\eta)$ satisfies to the ordinary differential equation

$$L_1(s,m,p)f + \varepsilon\gamma(t)\tau(t)\left[\bar{u}(t)\right]^{\beta-(p+m-2)}\left(-f+f^{\beta}\right) = 0, \tag{9}$$

where

$$L_1(s,m,p)f = \eta^{1-s}\frac{d}{d\eta}\left(\eta^{s-1}f^{m-1}\left|\frac{df}{d\eta}\right|^{p-2}\frac{df}{d\eta}\right) + \frac{\eta}{p}\frac{df}{d\eta}.$$

Note that, if

$$\gamma(t)\tau(t)[\bar{u}(t)]^{\beta-(p+m-2)} = \frac{N}{p}, \quad t>0,$$

equation (9) become self-similar.

It is easy to establish, that, when $\gamma(t) = (T+t)^{\sigma}$ with a constant σ, equation (9) has the self-similar solution of the kind

$$L_1(s,p,m)f + \frac{\varepsilon(\sigma+1)}{\beta-1-(\sigma+1)(p+m-3)}(-f+f^{\beta}) = 0 \tag{10}$$

In the case $s=1$, $p=2$, different properties of the weak solutions to equation (10) with boundary condition

$$f(0) = c \geqslant 0, f(\infty) = 0, \tag{11}$$

$$f(0) = c \geqslant 0, \quad f(b) = 0, \ \ b < \infty$$

were discussed in [1, 2, 5]. The properties of the different solutions, the existence of thermal structures and numerical solutions were studied in [1–4].

2. Conditions for localization of Cauchy problem.

Now we consider the Cauchy problem (1), (2) in the case, where $\beta = 1$, $\gamma(t) \neq 0$, $u_0(x) = q\delta(x)$, q is the power of instantaneous source, and $\delta(x)$ is Dirac's function. We will find the condition of localization of solution to problem (1), (2).

In this case the problem (1), (2) has the following exact solution

$$u(t,x) = \tau(t)^{-\frac{s}{p+(p+m-3)s}} \overline{f}(\eta), \tag{12}$$

$$\overline{f}(\eta) = \begin{cases} \left(a - |\eta|^{p/(p-1)}\right)_+^{(p-1)/(p+m-3)}, & \text{if } p+m-3 \neq 0, \\ \exp\left(-\left(\frac{\eta}{p}\right)\right)^p, & \text{if } p+m-3 = 0, \end{cases} \tag{13}$$

where $b = (p+m-3)\left(\dfrac{1}{p}\right)^{\frac{p}{p-1}}$, $\eta = |\varphi(x)|/\tau^{\frac{1}{p+(p+m-3)s}}$ if $p+m-3 > 0$ and $\eta = \dfrac{\varphi(x)}{(T+t)^{1/p}}$ if $p+m-3 = 0$, $T > 0$ — constant. Here the notation $(a)_+ = \max(0,a)$ was used.

The constant $a > 0$ in (13) can be found with the use of the condition

$$\int_{R^N} u(t,x)dx = q. \tag{14}$$

In the case, where $m = 1$, $\delta = 2$, $n = 0$ or $p = 2$, $\delta = 2$, $a = 1$, and $u_0(x) \neq q\delta(x)$ it is easy to see, that the solution to the equation (1) is the function $u(t,x) = (T+\tau(t))^{-\frac{s}{p+(p+m-3)s}} \overline{f}(\eta)$, with

$$\eta = \begin{cases} \dfrac{\varphi(x)}{(T+\tau(t))^{1/(p+(p+m-3)s)}}, & \text{if } p+m-3 > 0, \\ \dfrac{\varphi(x)}{(T+\tau(t))^{1/p}}, & \text{if } m+p-3 = 0, \end{cases}$$

$T > 0$ is a constant, and the function $\varphi(x)$ is defined above.

To obtain an exact solution to the equation (1), with $\beta = 1$, we make the change of variables in (1):

$$u(t,x) = \bar{u}(t) w(\tau(t), \varphi(x)),$$

where $\tau(t) = \displaystyle\int_0^t [\overline{u}(t)]^{p+m-3} dt$, $\overline{u}(t) = \exp(\varepsilon \displaystyle\int_0^t \gamma(\eta) d\eta)$.

Let $\bar{u}(t) < +\infty$, $\forall t > 0$ and $\tau(\infty) < +\infty$. Then there exist the localization of the problem (1), (2), if $u_0(x) \leqslant \bar{u}(0)\bar{f}(\xi)|_{t=0}$ and the equation (1) takes the form (5), which has the solution (12).

Theorem 1. *Let $u(t,x)$ be a weak solution of the problem (1)–(2) and $u_0(x) \leqslant u_+(0,x)$, where $u_+(t,x) = \overline{f}(\eta)$, $\overline{f}(\eta)$ being defined as in (13). Then for weak solution $u(t,x)$ of the problem (1)–(2) the estimate*

$$u(t,x) \leqslant u_+(t,x) \quad in \quad Q,$$

and the estimate

$$|x| \leqslant \left(\frac{a}{b}\right)^{\frac{p-1}{p}} [\tau(t)]^{1/(p+\delta-(n+2))}, \quad x \in R^N, \ t > 0$$

for the free boundary are valid.

Proof. The proof of the Theorem 1 is based on the comparison theorem for solutions (see [1]). In our case, the comparable function defined above is taken as $w_1(\tau,\varphi) = \bar{f}(\eta)$. It is easy to see that

$$Aw_1 \leqslant 0$$

at $D_1 = \{(t,x) : t > 0, |x| < l_1(t)\}$, $l_1(t) = (a/b)^{(p-1)/p}[T + \tau(t)]^{1/p}$, $T > 0$, where $w_1(\tau,\varphi) = \bar{f}(\eta)$, $\eta = \varphi(x)/[\tau(t)]^{1/p}$.

Then by the hypothesis of Theorem 1 and the comparison theorem for solutions (see [1]), we have $u(t,x) \leqslant w_1(\tau,\varphi)$ in the domain Q. □

3. Asymptotic behavior of the self-similar solutions

We note that, if $\gamma(t) = \gamma > 0$ in (1) then the equation (7) becomes self-similar:

$$\xi^{1-s}\frac{d}{d\xi}\left(\xi^{s-1}f^{m-2}\left|\frac{df}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \frac{\xi}{p}\cdot\frac{df}{d\xi} + \frac{\varepsilon}{\beta-(p+m-2)}\left(-f+f^{\beta}\right) = 0 \qquad (15)$$

We consider this equation with the following boundary condition

$$f(0) = c > 0, \quad f(d) = 0, \quad d < \infty. \qquad (16)$$

Set

$$\bar{f}(\xi) = \left(a - \xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p+m-3}}.$$

The existence of a self - similar weak solution to the problem (15)– 16), where $n = 0$, $N = 1$, $p = 2$, $\varepsilon = -1$ in (1), was studied in [1,5].

Theorem 2. *Let $\gamma(t) = 1$, $p+m-3 > 0$, $\beta > 1$, $\beta > p+m-2$. Then the solution of the problem (15)–(16) in a neighborhood of the free boundary has the following asymptotic behavior*

$$f(\xi) = c_1\bar{f}(\xi)(1+o(1)), \quad a = c^{\frac{p+m-3}{p-1}}$$

while $\eta \to \infty \left(\eta = -ln\left(a - \xi^{\frac{p}{p-1}}\right)\right)$.

Proof. We make the following change of variables in (15)

$$f(\xi) = \bar{f}(\xi)\,W(\eta), \quad \eta = -\ln\left(a - b\xi^{\frac{p}{p-1}}\right).$$

This change reduces equation (19) to an equation with nearly constant coefficients with $(p+m-3 \neq 0,\ \beta \neq p+m-2)$. Really, after this transformation the equation (15) reduced to the form

$$\frac{d}{d\eta}\left(W^{m-1}\left|\frac{dW}{d\eta} - \frac{p-1}{p+m-3}W\right|^{p-2}\left(\frac{dW}{d\eta} - \frac{p-1}{p+m-3}W\right)\right) +$$
$$+\left(s\frac{p-1}{p}\frac{1}{a-e^{-\eta}}e^{-\eta} - \frac{bp}{p+m-3}\right)W^{m-1}\left|\frac{dW}{d\eta} - \frac{p-1}{p+m-3}W\right|^{p-2}\times$$
$$\times\left(\frac{dW}{d\eta} - \frac{p-1}{p+m-3}W\right) + \frac{1}{p}\left(\frac{p-1}{pb}\right)^{p-1}\left(\frac{dW}{d\eta} - \frac{p-1}{p+m-3}W\right) +$$
$$+\frac{1}{\beta-(p+m-2)}\left(\frac{p-1}{pb}\right)^{p}\left(\frac{b}{a-e^{-\eta}}e^{-\eta}W \pm \frac{b}{a-e^{-\eta}}e^{-\left(\frac{(p-1)(\beta-1)}{p+m-3}+1\right)\eta}W^{\beta}\right) = 0$$

This equation has such a solution W that $W \to p^{\frac{p-1}{p+m-3}}$, as $\eta \to \infty$, which was to be proved. □

Remark. Applying this theorem for the case $T = 0,\ m = p = \delta = 2,\ \ N = 1,\ \ \gamma(t) \equiv 0$ and the problem

$$\frac{d^2 f^2}{d\xi} + \beta\xi\frac{df}{d\xi} + (1-2\beta)f = 0 \tag{17}$$

$$f(1) = 0, \quad f'(1) = -\frac{\beta}{2} \tag{18}$$

we obtain the following asymptotic of the solution to (17), (18):

$$f(\xi) \cong \frac{\beta}{4}(1-\xi^2)$$

near neighborhood the point $\xi = 1$, that was proved in [2]. If $\beta = 1/3$ then the application of the Lemma yields the exact solution $f(\xi) = \left(1 - \frac{1}{3}\xi^2\right)_+$ of the problem (17)–(18).

The situation, where $m + p < 3$, corresponds to the so-called quick diffusion. In this case we consider the equation (15) with boundary condition

$$f\,(0) = c > 0, \quad f\,(\infty) = 0. \tag{19}$$

The properties of the solution of this problem with $p = \delta = 2, \quad n = 0, \quad \varepsilon = -1, \quad N = 1$ were studied in [10].

Below we investigate the asymptotic behavior of solutions to the problem (17), (19) with $\varepsilon = -1$, $\eta \to +\infty$.

Let's consider the following function

$$\tilde{f}\,(\xi) = \xi^{-\frac{p}{\beta-(p+m-2)}}.$$

Theorem 3. *Let $\beta > (p+m-2)$. Then the solutions to the problem (17), (19) have the following asymptotic behavior while as $\xi \to \infty$:*

$$f\,(\xi) = \left((s(\beta-(m+p-2)) - p\beta)\left(\frac{p}{\beta-(m+p-2)}\right)^{p-1}\right)^{\frac{1}{\beta-(p+m-2)}} \bar{f}\,(\xi)\,,$$

if $s > \dfrac{p\beta}{\beta-(m+p-2)}$,

$$f\,(\xi) = \left(-(s(\beta-(m+p-2)) - p\beta)\left(\frac{p}{\beta-(m+p-2)}\right)^{p-1}\right)^{\frac{1}{\beta-(p+m-2)}} \bar{f}\,(\xi)\,,$$

if $s < \dfrac{p\beta}{\beta-(m+p-2)}$.

Proof. The change of the variables $f\,(\xi) = \tilde{f}\,(\xi)\,W\,(\eta)\,,\ \eta = \ln\xi$ reduces the equation (17) to the form

$$\frac{d}{d\eta}\left(W^{m-1}\left|\frac{dW}{d\eta} - \frac{p}{\beta-(p+m-2)}W\right|^{p-2}\left(\frac{dW}{d\eta} - \frac{p}{\beta-(p+m-2)}W\right)\right) +$$
$$+\left(s - \frac{bp}{\beta-(p+m-2)}\right)W^{m-1}\left|\frac{dW}{d\eta} - \frac{p}{\beta-(p+m-2)}W\right|^{p-2} \times$$
$$\times\left(\frac{dW}{d\eta} - \frac{p}{\beta-(p+m-2)}W\right) + \frac{1}{p}e^{\frac{p(\beta-1)}{\beta-(p+m-2)}\eta}\frac{dW}{d\eta} \pm \frac{1}{\beta-(p+m-2)}W^{\beta} = 0.$$

From here it follows, that

$$W \to \left(\pm \left(s \left(\beta - (m+p-2) \right) - p\beta \right) \left(\frac{p}{\beta - (m+p-2)} \right)^{p-1} \right)^{\frac{1}{\beta-(m+p-2)}}, \quad \text{at} \quad \eta \to \infty.$$

Let's consider the following function

$$\breve{f}(\xi) = \left(a + b\xi^{\frac{p}{p-1}} \right)^{\frac{p-1}{p+m-3}}, \tag{20}$$

$b = (3-p-m)\left(\frac{1}{p}\right)^{\frac{p}{p-1}}$, if $p+m-3<0$. □

Theorem 4. *Let* $p+m-3<0$, $s > -\dfrac{p}{p+m-3}$, $\beta > p+m-2$. *Then the asymptotic behavior of solutions to the problem (17), (19), while* $\xi \to \infty$, *have the following form*

$$f(\xi) = \left((3-(p+m)) \frac{\beta - 1}{\beta - (p+m-2)} \frac{1}{s(p+m-3)+p} \right)^{\frac{1}{p+m-3}} \bar{f}(\xi), \quad a = c^{\frac{p+m-3}{p-1}}.$$

Proof. The change of the variables

$$f(\xi) = \overset{\frown}{f}(\xi)\, W(\eta), \quad \eta = \ln\left(a + b\xi^{\frac{p}{p-1}} \right)$$

in (15) reduces the equation (17) to the form

$$\begin{aligned}
&\frac{d}{d\eta}\left(W^{m-1} \left| \frac{dW}{d\eta} + \frac{p-1}{p+m-3} W \right|^{p-2} \left(\frac{dW}{d\eta} + \frac{p-1}{p+m-3} W \right) \right) + \\
&- \left(s\frac{p-1}{pb} \frac{b}{e^\eta - a} e^\eta - \frac{p-1}{p+m-3} \right) W^{m-1} \left| \frac{dW}{d\eta} + \frac{p-1}{p+m-3} W \right|^{p-2} \times \\
&\times \left(\frac{dW}{d\eta} + \frac{p-1}{p+m-3} W \right) + \frac{1}{p} \left(\frac{p-1}{pb} \right)^{p-1} \left(\frac{dW}{d\eta} + \frac{p-1}{p+m-3} W \right) + \\
&+ \frac{1}{\beta - (p+m-2)} \left(\frac{p-1}{pb} \right)^{p} \left(\frac{b}{e^\eta - a} e^\eta W \pm \frac{b}{e^\eta - a} e^{\left(\frac{(p-1)(\beta-1)}{p+m-3} + 1 \right)\eta} W^\beta \right) = 0
\end{aligned}$$

Therefore $W \to \left(-(p+m-3) \dfrac{\beta-1}{\beta-(p+m-2)} \cdot \dfrac{1}{s(p+m-3)+p} \right)^{\frac{1}{p+m-3}}$ as $\eta \to \infty$. □

4. The critical case

The described above method of the nonlinear splitting yields explanation of the so-called critical case:

$$\gamma(t)\tau(t)[\overline{u}(t)]^{\beta-(p+m-2)} = s/p, \quad t > 0.$$

In this case the equation (1) becomes a self-similar equation:

$$L(m,p)f + (s/p)(f + \varepsilon f^\beta) = 0$$

The double critical case corresponds to the following condition:

$$\gamma(t)(T+t)[\overline{u}(t)]^{\beta-1} = s/p.$$

For example, if $\gamma(t) = (T+t)^\sigma, \sigma > -1$, then the critical value is $\beta = \beta_* = 1 + (\sigma+1)(p+m-3+p/s)$ and the double critical case corresponds to $\beta = \beta_* = 1 + (\sigma+1)(p/s)$.

It is easy to check that in this case the function $\bar{f}(\xi)$ defined above will be the sup-solution when $\varepsilon = -1$ and the sub-solution in the case where $\varepsilon = +1$.

It was proved [1, 6, 12] that in the critical case for large time the asymptotic behavior of solutions to the problem (1), (2) with $\gamma(t) = 1, \quad \delta = 2, \quad n = 0$, is different. In the case, where $\varepsilon = +1$, it was proved in [1], that every solution to the problem (1), (2) (in the presence of the source in the equation (1)) is blowing up.

Theorem 5. *Let $\varepsilon = -1, \quad p+m-3 = 0$ in (7). Then the asymptotic behavior of solutions to the problem (1)–(2), while $t \to +\infty$, is the following:*

$$u(t,x) \approx \tilde{u}(t)\bar{f}(\eta),$$

where $\tilde{u}(t) = [(T+t)\ln(T+t)]^{\frac{1}{1-\beta_*}}, \ \eta = \dfrac{\varphi(x)}{(T+\tau(t))^{1/p}}$ *and the function $\bar{f}(\eta)$ is defined above.*

This result was established earlier by Galaktionov V.A., Vaskes H.L. in [6, 12] for particular value of parameters: $m = 1$ or $p = 2$ and $\gamma(t) = 1, \quad n = \delta = 2, \quad \varepsilon = -1$.

5. Knerr-Kershner type estimation

Knerr [13] has established the following estimate for the free boundary:

$$x(t) \geqslant a\,(ln(t))^{\frac{1}{2}}\,, \quad t > 1$$

with a constant $a > 0$ in the case where $n = 0,\ p = 2,\ \gamma(t) = 1,\ N = 1,\ \beta = m$ in equation (1).

Another proof of this result was given by Kershner [7].

Below we prove the following

Theorem 6. *Let $\beta = p+m-2$, $\varepsilon = -1$. Then, for the sufficient large values of t, the following estimate holds*

$$x(t) \geqslant \left(\frac{a}{b}\right)^{\frac{p-1}{p}} \left(\int_0^t \left(T + (p+m-3)\int_0^\eta \gamma(t)dt\right)^{-1} d\eta\right)^{\frac{1}{p+\delta-(n+2)}}$$

for the free boundary of the problem (1)–(2).

Proof. Indeed, the following change of variables in (7)

$$w(\tau, \varphi(x)) = f(\eta), \quad \eta = \varphi(x)/\tau^{1/p}, \quad \tau(t) = \int_0^t [\overline{u}(\eta)]^{p+m-3} d\eta$$

yields the approximately self-similar equation

$$L_2(f) = \eta^{1-s}\frac{d}{d\eta}\left(\eta^{s-1}\left|\frac{df}{d\eta}\right|^{n-1}\frac{df}{d\eta}\right) + \frac{\eta}{2}\cdot\frac{df}{d\eta} + \tau(t)\gamma(t)\left(f - f^\beta\right) = 0.$$

For the function $\bar{f}(\eta)$ defined above we have

$$L_2(\bar{f}) = -(1/2)\bar{f} + \tau(t)\gamma(t)\left(\bar{f} - \bar{f}^\beta\right) = \left[\left(-\frac{1}{2} + \tau(t)\gamma(t)\right) - \tau(t)\gamma(t)\bar{f}^{\beta-1}\right]\bar{f}$$

and $L_2(\bar{f}) \geqslant 0$ for some value of $t > t_0 > 0$.

Therefore applying the comparison theorem [1] we obtain the desired estimate for the free boundary. □

6. Results of the numerical experiments and a visualization

As for the nonlinear self-similar problems, the most essential difficulty is the absence of the uniqueness of the solution, which distinguishes them from the classical problems with a unique solution. The following problems naturally arise:

– to find a "good" approximation to each type of the solutions;

– to construct an iterative method, which always converges to the desired solution (corresponding to the initial approximation), converges quickly and provides a sufficient accuracy;

– to achieve an automatization of the calculations so that one can quickly find a proper method for the different solutions corresponding to the given parameters of the problem.

For the numerical solution of the problem we used the approximate equation on a grid under the implicit circuit of variable directions (for a multidimensional case) in a combination to the method of balance. The iterative processes were constructed on the base of the Picard Method, Newton Newton and a special method. Results of computational experiments show, that all the listed iterative methods are effective for the solution of nonlinear problems and lead to the nonlinear effects if we will use as initial approximation the solutions of self-similar equations constructed by the method of nonlinear splitting and by the method of standard equation [4]. As it was expect, in order achieve an identical accuracy the method of Newton demands smaller quantity of iterations than the methods of Picard and the special method because of a successful choice of an initial approximation. We observe that in each of the considered cases Newton's method has the best results.

On the other side, for applications it is important to produce the the numerical investigations and visualizations of processes describing by equation (1). To achieve this goal, one needs to investigate the different qualitative properties such as finite velocity of the perturbation, the localization of the solutions, asymptotic behavior of the solutions and the free boundary (fronts), depending on numerical parameters related to the equation (1). The direct linearization of the non-linear equation leads to the disappearance of the typical properties (effects) of the studying processes that are specific to the non-linear case only.

Numerical results of the problem with $n = 1,\ m = 2,\ p = 3,\ \beta = 2,\ \varepsilon = +1$ are presented below as a graph (Fig. 1).

In the two-dimensional case for approximation of the problem we apply the method of variable directions. Below we show the ensemble of the numerical experiments (for $0 < \beta < 1$) for final time in animation form: comparison of the cases $n = 2,\ m = 2,\ p = 2.5,\ \beta = 0.5,\ \varepsilon = -1$ and $n = 2,\ m = 2,\ p = 2.5,\ \beta = 0.8,\ \varepsilon = +1$ (Fig. 2).

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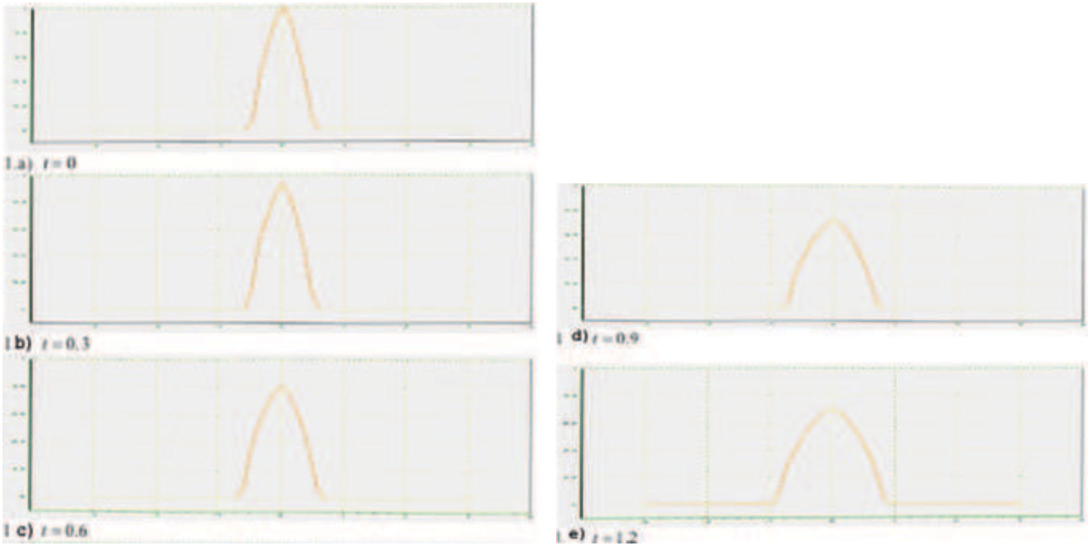

Fig. 1

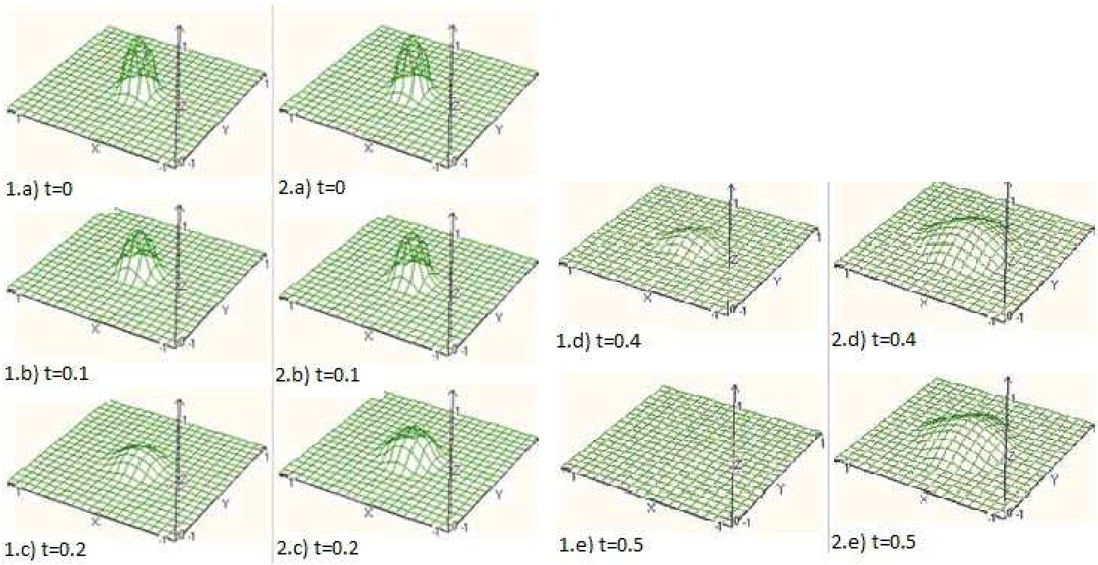

Fig. 2

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О свойствах решений уравнения реакции диффузии с двойной нелинейностью с распределенными параметрами

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Исследуются свойства решений автомодельных и приближенно-автомодельных уравнений для уравнения реакции-диффузии с двойной нелинейностью. Исследовано влияние параметров системы реакции-диффузии к эволюции процесса. Доказано существование финитных решений и решений, исчезающих на бесконечности, и их асимптотика. Найдено условие глобальной разрешимости задачи Коши, обобщающее ранее известные результаты, и получена оценка типа Кнерра–Кершинера для свободной границы. Приводятся результаты численных экспериментов.

Ключевые слова: реакция-диффузия, двойная нелинейность, свободная граница.