Journal of Siberian Federal University. Mathematics & Physics 2013, 6(1), 18–27

УДК 517.55

On Asymptotic Expansion of the Conormal Symbol of the Singular Bochner-Martinelli Operator on the Surfaces with Conical Wedges

Davlatboi Kh. Dzhumabaev*
National University of Uzbekistan,
Vuzgorodok, Tashkent, 700174,
Uzbekistan

Received 28.06.2012, received in revised form 30.08.2012, accepted 31.10.2012

We study the conormal symbol of the singular Bochner-Martinelli integral on a compact closed surface with conical wedges $\mathcal{S}$ in $\mathbb{C}^n$ and evaluate its asymptotic expansion.

Keywords: singular Bochner-Martinelli operator, conormal symbol, conical wedges.

Introduction

Smooth manifolds with conical points are the simplest singular spaces in the hierarchy of stratified varieties. Differential analysis on such manifolds was perhaps initiated by Kondrat'ev [1] who invented the so-called conormal symbol of a differential operator at a singular point.

In the 1980s the analysis encompassed also pseudodifferential operators which has led to diverse algebras of pseudodifferential operators on manifolds with conical points.

When applied to the Cauchy integral on a plane curve with corners, conormal symbols can be efficiently computed.

We study the singular Bochner-Martinelli integral on a compact closed surface with conical wedges S in $\mathbb{C}^n$ and evaluate its conormal symbol at a conical point and it asymptotic expansion. Our computation demonstrates rather strikingly that the conormal symbols are no longer efficient for pseudodifferential operators in dimensions larger than 1.

The singular Bochner-Martinelli integral is of central importance in complex analysis in several variables [2].

As usual, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ under the complex structure $z_j = x_j + \imath x_{n+j}$, for $j = 1, \dots, n$. I.e. $(z_1, ..., z_n) = (x_1, ..., x_n, x_{n+1}, ..., x_{2n}) \in \mathbb{R}^{2n}$. And $x = (x_1, ..., x_{2n})$, $x^{'} = (x_1, ..., x_{p+1})$, $x^{''} = (x_{p+3}, ..., x_{2n})$, $x = (x^{'}, x_{p+2}, x^{''})$. Scalar product on $\mathbb{R}^{p+1}$ Denote by

$$\langle x', y' \rangle = x_1 y_1 + \ldots + x_{p+1} y_{p+1}.$$

We will consider a smooth hypersurface Σ in $\mathbb{R}^{p+2} \setminus \{0\}$ with a singular point at the origin given by

$$\Sigma = \{(rx', r) \in \mathbb{R}^{p+2} : \; x' \in X', \, r \in [0, R)\}. \tag{1}$$

The point $x' = (x_1, \dots, x_{p+1})$ varies over a smooth compact hypersurface $X^{'}$ in $\mathbb{R}^{p+1}$ which does not meet 0.

For instance, X' may be a p-dimensional sphere with centre at the origin.

In any case we assume that $X' = \{x' \in \mathbb{R}^{p+1} : \; \rho(x') = 1\}$, where ρ is a C^1-function on $\mathbb{R}^{p+1} \setminus \{0\}$ with real values, satisfying $\nabla \rho \neq 0$ on X' and $\rho(\lambda x') = \lambda^h \rho(x')$ for all $\lambda > 0$ with some $h > 0$.

*davlat2112@rambler.ru

The origin is a singular point of Σ.

Using (1) it is easy to determine a defining function of the smooth part of Σ. Indeed, write $(x', x_{p+2}) \in \Sigma \setminus \{0\}$. Then readily implies $\rho\left(\dfrac{x'}{x_{p+2}}\right) = 1$, and so the homogeneity of ρ yields

$$\Sigma = \{(x', x_{p+2}) \in \mathbb{R}^{p+2} : \ \psi(x', x_{p+2}) = 0, \ x_{p+2} \in [0, R)\},$$

with $\psi(x', x_{p+2}) = \rho(x') - (x_{p+2})^h$. Let

$$S = \Sigma \times X^{''}, \tag{2}$$

where $X^{''}$ is an open bounded открытое set in $\mathbb{R}^q$, $p+1+q = 2n-1$. Then S is the hypersurface in $\mathbb{C}^n$ with conical wedge $F = O^{'} \times X^{''}$ ($O^{'} = (0, \dots, 0) \in \mathbb{R}^{p+2}$).

Scalar product in $\mathbb{R}^q$ denote by

$$\langle x'', y'' \rangle = x_{p+3}y_{p+3} + \ldots + x_{2n}y_{2n}.$$

Let D be a bounded domain in $\mathbb{C}^n$, with $n > 1$. The boundary of D is assumed to be of the form $Y \cup (S_1 \cup \ldots \cup S_N)$, where Y is a smooth hypersurface and each S_ν is diffeomorphic to a conical hypersurface S (with different p and q), as above. Thus, ∂D is a smooth hypersurface with a finite number of conical wedges.

Since the analysis at singular points is local, one can assume without loss of generality that $N = 1$, i.e., $\partial D = Y \cup S$, where

$$S = \{z \in \mathbb{C}^n : \ z = (rx', r, x''), \ x' \in X', \ x'' \in X'', \ r \in [0, R)\}. \tag{3}$$

Given an integrable function f on ∂D ($f \in \mathcal{L}^1(\partial D)$), the Bochner-Martinelli integral of f is defined by

$$F(z) = \int_{\partial D} f(\zeta) U(\zeta, z),$$

where $z \notin \partial D$.

For points $z \in \partial D$ the singular Bochner-Martinelli integral of f is defined by

$$M_S[f](z) = \text{v.p.} \int_{\partial D} f(\zeta) U(\zeta, z) = \lim_{\varepsilon \to +0} \int_{\partial D \setminus B(z,\varepsilon)} f(\zeta) U(\zeta, z), \tag{4}$$

where

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} \, d\bar{\zeta}[k] \wedge d\zeta,$$

$B(z, \varepsilon) = \{\zeta \in \mathbb{C}^n : \ |\zeta - z| < \varepsilon\}$, and $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, while $d\bar{\zeta}[k]$ is the wedge product of all differentials $d\bar{\zeta}_1, \ldots, d\bar{\zeta}_n$ but $d\bar{\zeta}_k$. In the sequel, we drop the designation 'p.v.' for short.

The properties of the Bochner-Martinelli singular integral operator on smooth hypersurfaces are well understood [2].

We are aimed at investigating the asymptotic expansion of the conormal symbol of the operator M_S on hypersurfaces with singular wedges. For domains with conical singular points this problem has been solved in [3].

1. Known Results

We need some result from paper [4].

Theorem 1. *The restriction of the Bochner-Martinelli kernel to the hypersurface S has the form*

$$U(\zeta,z)|_S = \frac{1}{\sigma_{2n}} \frac{\langle(\nu(y'),\nu_{p+2}(y')),(sy'-rx',s-r)\rangle}{\left(|sy'-rx'|^2+(s-r)^2+|y''-x''|^2\right)^n}\, s^p\cdot ds\, d\sigma(y')\, dy''+$$
$$+\imath\frac{1}{\sigma_{2n}} \frac{\langle(\nu(y'),\nu_{p+2}(y',s)),\mu(x,y,r,s)\rangle}{\left(|sy'-rx'|^2+(s-r)^2+|y''-x''|^2\right)^n}\, s^p\cdot ds\, d\sigma(y')dy'',$$

where vectors $\nu(y')=\dfrac{\nabla\rho(y')}{|\nabla\rho(y')|}$ *for* $y'=(y_1,\dots,y_{p+1})\in X'$ *u* $\nu_{p+2}(y')=-h\dfrac{1}{|\nabla\rho(y')|}$, *and*
$\mu(x',x'',y',y'',r,s)=-(y_{n+1}-x_{n+1},\dots,y_{p+2}-x_{p+2})$, *if* $p+1<n$;
$\mu(x',x'',y',y'',r,s)=-(s-r,y_{n+1}-x_{n+1},\dots,y_{n+p+1}-x_{n+p+1},rx_1-sy_1)$, *if* $p+1=n$;
$\mu(x',x'',y',y'',r,s) = -(sy_{n+1}-rx_{n+1},\dots,sy_{p+1}-rx_{p+1},s-r,y_{p+3}-x_{p+3},\dots,y_{2n}-x_{2n},rx_1-sy_1,\dots,rx_{p+2-n}-sy_{p+2-n})$, *if* $p+1>n$.

Consider the results from [5]. We rewrite the hypersurface S as followes

$$S=\{z\in\mathbb{C}^n:\ z=(rx',r,rx''),\ x'\in X',\ x''\in X''_r,\ r\in(0,R)\}, \tag{5}$$

where $X''_r=\dfrac{1}{r}X''$.

Introduce the function $k(x',y',x'',y'',t)$, defined for $(x',x'')\in X'\times X''_r$, $(y',y'')\in X'\times X''_s$, $t>0$, by formula

$$k(x',y',x'',y'',t)=\frac{1}{\sigma_{2n}}\frac{\langle(\nu(y'),\nu_{p+2}(y')),(y'-tx',1-t)\rangle}{\left(|y'-tx'|^2+(1-t)^2+|y''-tx''|^2\right)^n}+$$
$$+\frac{\imath}{\sigma_{2n}}\frac{\langle\nu(y'),\nu_{p+2}(y'),\mu'(x',x'',y',y'',t)\rangle}{\left(|y'-tx'|^2+(1-t)^2+|y''-tx''|^2\right)^n},$$

where $\tilde\mu(x',x'',y',y'',t)=-(y_{n+1}-tx_{n+1},\dots,y_{p+2}-tx_{p+2})$, if $p+1<n$;
$\tilde\mu(x',x'',y',y'',t)=-(1-t,y_{n+1}-tx_{n+1},\dots,y_{n+p+1}-tx_{n+p+1},tx_1-y_1)$, if $p+1=n$;
$\tilde\mu(x',x'',y',y'',t)=-(y_{n+1}-tx_{n+1},\dots,y_{p+1}-tx_{p+1},1-t,y_{p+3}-tx_{p+3},\dots,y_{2n}-tx_{2n},tx_1-y_1,\dots,tx_{p+2-n}-y_{p+2-n})$, if $p+1>n$.

Using this kernel we can write singular Bochner-Martinelli integral by the form

$$M_S f\,(x',x'',r)=\int_0^\infty s\frac{ds}{s}\int_{X'\times X''_s} k\Big(x',y',x'',y'',\frac{r}{s}\Big)f(y',y'',s)\,d\sigma(y')dy'', \tag{6}$$

where (x',x'',r) and (y',y'',s) are identified with $z=(rx',r,rx'')$ and with $\zeta=(sy',s,sy'')$, respectively.

Note that integral by $X'\times X''_s$ is singular since $k\left(x',y',x'',y'',\dfrac{r}{s}\right)$ has singularity under $y'=x'$, $x''=y''$ and $s=r$.

Denote by $\mathcal{M}_{r\mapsto\lambda}$ the Mellin transform defined on functions $f(r)$ on the semi-axis. It is given by

$$\mathcal{M}_{r\mapsto\lambda}f=\int_0^\infty r^{-\imath\lambda}f(r)\frac{dr}{r}$$

for $\lambda\in\mathbb{C}$.

Composing the singular Bochner-Martinelli operator (6) with the Mellin transform yields

$$\mathcal{M}_{r\mapsto\lambda} M_S f(x',x'',r) = \int_0^\infty r^{-\imath\lambda}\,\frac{dr}{r}\int_0^\infty \frac{ds}{s}\int_{X'\times X''_s} k\Big(x',x'',y',y'';\frac{r}{s}\Big) f(y',y'',s)\,d\sigma(y')dy'' =$$
$$= \int_0^\infty \frac{ds}{s}\int_{X'\times X''_s}\Big(\int_0^\infty r^{-\imath\lambda}\,k\Big(x',x'',y',y'';\frac{r}{s}\Big)\frac{dr}{r}\Big) f(y',y'',s)\,d\sigma(y')dy''.$$

In the integral over $r\in(0,\infty)$ we change the variables by $r = st$, where t runs over $(0,\infty)$. This gives

$$\mathcal{M}_{r\mapsto\lambda} M_S f(x',x'',r) =$$
$$= \int_0^\infty s^{-\imath\lambda}\,\frac{ds}{s}\int_{X'\times X''_s}\Big(\int_0^\infty t^{-\imath\lambda}\,k(x',x'',y',y'';t)\frac{dt}{t}\Big) f(y',y'',s)\,d\sigma(y')dy'' =$$
$$= \int\limits_{X'\times X''_s} \mathcal{M}_{t\mapsto\lambda} k(x',x'',y',y'';t)\,\mathcal{M}_{s\mapsto\lambda} f(y',y'',s)\,d\sigma(y')dy''$$

for $x'\in X'$, $x''\in X''_s$ и $\lambda\in\mathbb{C}$. It follows that

$$M_S f(r) = \mathcal{M}^{-1}_{\lambda\mapsto r} a(\lambda)\mathcal{M}_{r'\mapsto\lambda} f(r'), \tag{7}$$

where $f(r) := f(x',x'',r)$ is thought of as a function of $r\in(0,\infty)$ with values in functions of $(x',x'')\in X'\times X''_s$, and $a(\lambda)$ is a family of singular integral operators on $X'\times X''_s$, parametrised by λ varying on a horizontal line in the complex plane. The action of $a(\lambda)$ is specified by

$$a(\lambda) f(x',x'',t) = \int\limits_{X'\times X''_t} \mathcal{M}_{t\mapsto\lambda} k(x',x'',y',y'';t)\, f(y',y'',t)\,d\sigma(y')dy''.$$

The family $a(\lambda)$ is usually referred to as the *conormal symbol* of the pseudodifferential operator (6) based on the Mellin transform.

To evaluate it more explicitly, we denote by Z the unique root of

$$\langle y'-tx', y'-tx'\rangle + \langle y''-tx'', y''-tx''\rangle + (1-t)^2 = 0$$

in the upper half-plane, i.e.,

$$Z = \frac{1+\langle x',y'\rangle + \langle x'',y''\rangle\rangle}{1+|x'|^2+|x''|^2} + \tag{8}$$
$$+\frac{\imath\sqrt{|y'-x'|^2+|x''-y''|^2+(|x'|^2+|x''|^2)(|y'|^2+|y''|^2)-(\langle x',y'\rangle+\langle x'',y''\rangle)^2}}{1+|x'|^2+|x''|^2}$$

Lemma 1. *In the strip* $0<\operatorname{Im}\lambda<2n-1$*, the Mellin transform of* $k(x',x'',y',y'';t)$ *has the form*

$$\mathcal{M}_{t\mapsto\lambda} k(x',x'',y',y'';t) =$$
$$= \pi\imath\,\frac{(-1)^{n-1}}{(n-1)!}\,\frac{\exp\pi\lambda}{\operatorname{sh}\pi\lambda}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}(\imath\lambda+1)(\imath\lambda+2)\dots(\imath\lambda+j-1)\times$$
$$\times\frac{((\imath\lambda+j)A-\imath\lambda ZB)Z^{-\imath\lambda-j-1}+(-1)^{j-1}((\imath\lambda+j)A-\imath\lambda\bar{Z}B)\bar{Z}^{-\imath\lambda-j-1}}{(1+|x|^2)^n\,(Z-\bar{Z})^{2n-1-j}},$$

where

$$A = \frac{1}{\sigma_{2n}} \langle (\nu(y'), \nu_{p+2}(y')), (y', 1)\rangle + \frac{\imath}{\sigma_{2n}} \langle (\nu(y'), \nu_{p+2}(y')), (\tilde{\mu}(x', x'', y', y'', 0)\rangle,$$
$$B = -\frac{1}{\sigma_{2n}} \langle (\nu(y'), \nu_{p+2}(y')), (x', 1)\rangle + \frac{\imath}{\sigma_{2n}} \langle (\nu(y'), \nu_{p+2}(y')), \tilde{\mu}'_t(x', x'', y', y'', 0)\rangle.$$

Lemma is based on formulas

$$\operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z}) =$$
$$= \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!\,(n-1-j)!}\, p(p-1)\dots(p-j+1)\, \frac{(-1)^{j+1} Z^{p-j} + \bar{Z}^{p-j}}{(Z-\bar{Z})^{2n-1-j}}, \tag{9}$$

where

$$f(t) = \frac{t^p}{(t-Z)^n (t-\bar{Z})^n} \tag{10}$$

and

$$\int_0^\infty t^{-\imath\lambda} k(x', x'', y', y''; t) \frac{dt}{t} =$$
$$= \pi i \, \frac{\exp \pi\lambda}{\sinh \pi\lambda} \Big(\operatorname{res}(t^{-\imath\lambda-1} k(x', x'', y', y''; t); Z) + \operatorname{res}(t^{-\imath\lambda-1} k(x', x'', y', y''; t); \bar{Z}) \Big). \tag{11}$$

Denote

$$G(t) = t^{-\imath\lambda-1} k(x', x'', y', y''; t) = \frac{1}{a^n} \frac{A t^{-\imath\lambda-1} - B t^{-\imath\lambda}}{(t-Z)^n (t-\bar{Z})^n}.$$

Theorem 2. *For $|\gamma| < n-1/2$ the singular Bochner-Martinelli integral admits the representation*

$$M_S f\,(r) = \frac{1}{2\pi} \int\limits_{\{\operatorname{Im}\lambda = (n-1/2)-\gamma\}} r^{\imath\lambda} a(\lambda) \mathcal{M}_{r' \mapsto \lambda} f(r')\, d\lambda.$$

2. Main Results

We first find asymptotics of the sum of residues of the function $f(t)$ given by formula (10).

Lemma 2. *The sum of residues of the function $f(t)$ at Z and $\bar{Z}$ has no singularity as $\operatorname{Im} Z \to 0$, and*

$$\lim_{\operatorname{Im} Z \to 0} \Big(\operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z}) \Big) = \frac{p(p-1)\dots(p-2n+2)}{(2n-1)!}\, Z^{p-2n+1}.$$

And also

$$\operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z}) =$$
$$= \frac{1}{2(n-1)!} \sum_{s=0}^{\infty} \frac{p \cdots (p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \Big(\bar{Z}^{p-2n-s+1} + (-1)^s Z^{p-2n-s+1} \Big).$$

Proof. Set $\varSigma = \operatorname{res}(f; Z) + \operatorname{res}(f; \bar{Z})$. By formula (9),

$$\varSigma = \frac{(-1)^n}{(n-1)!} \sum_{j=0}^{n-1} \frac{(2n-2-j)!}{j!\,(n-1-j)!}\, p(p-1)\dots(p-j+1)\, Z^{p-2n+1} \frac{(-1)^{j+1} + \Big(\dfrac{\bar{Z}}{Z}\Big)^{p-j}}{\Big(1 - \dfrac{\bar{Z}}{Z}\Big)^{2n-1-j}}.$$

Setting $Q := 1 - \bar{Z}/Z$ we rewrite Σ in the form

$$\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,Z^{p-2n+1}\frac{(-1)^{j+1}+(1-Q)^{p-j}}{Q^{2n-1-j}},$$

which splits into two sums

$$Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{(-1)^{j+1}}{Q^{2n-1-j}},$$

$$Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{\sum\limits_{k=0}^{\infty}\binom{p-j}{k}(-Q)^k}{Q^{2n-1-j}}.$$

The binomial series in the latter sum converges only for $|Q|<1$. If $|Q|=1$ it should be replaced by a Taylor polynomial of sufficiently large degree N along with a remainder $((\operatorname{Im} Z)^{N+1})$.

Set $l = j+k$ in the second sum and transform it. We obtain

$$Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=0}^{n-1}\,\sum_{l=j}^{\infty}\,\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,\frac{p(p-1)\dots(p-l+1)}{(l-j)!}\,\frac{(-1)^{l-j}Q^l}{Q^{2n-1}}.$$

Interchanging the order of summation and substituting j for l and k for j immediately yields

$$\begin{aligned}
&Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=0}^{n-1}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\,\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}+\\
+\quad&Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=n}^{\infty}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\,\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}.
\end{aligned}$$

Summarizing we get

$$\begin{aligned}
\Sigma \quad = \quad & Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{(-1)^{j+1}}{Q^{2n-1-j}}+\\
+\quad & Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=0}^{n-1}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\,\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}+\\
+\quad & Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=n}^{\infty}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\,\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}.
\end{aligned}$$

Lemma 2 will be proved once we prove the lemma below. This latter is of independent interest.

Lemma 3. *We have*

$$\begin{array}{rcl}
\sum_{k=0}^{j}(-1)^k\binom{j}{k}\dfrac{(2n-2-k)!}{(n-1-k)!} &=& \dfrac{(2n-2-j)!}{(n-1-j)!}, \qquad \textit{if} \quad j=0,1,\dots,n-1;\\
\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\dfrac{(2n-2-k)!}{(n-1-k)!} &=& 0, \qquad \textit{if} \quad j=n,n+1,\dots,2n-2;\\
\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\dfrac{(2n-2-k)!}{(n-1-k)!} &=& (-1)^{n-1}(n-1)!, \qquad \textit{if} \quad j=2n-1;\\
\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\dfrac{(2n-2-k)!}{(n-1-k)!} &=& \sum_{l=2n-1}^{j}\dfrac{(-1)^{l+n}(l-2n+2)\cdots(l-n)}{l!(j-l)!}, \qquad \textit{if} \quad j>2n-1.
\end{array}$$

Proof. Consider the function $F(z)=\sum\limits_{k=0}^{j}(-1)^k\binom{j}{k}\dfrac{(2n-2-k)!}{(n-1-k)!}z^{n-1-k}$.

A trivial verification shows that

$$\begin{array}{rcl}
F(z) &=& \left(\dfrac{\partial}{\partial z}\right)^{n-1}\sum_{k=0}^{j}(-1)^k\binom{j}{k}z^{2n-2-k} =\\
&=& \left(\dfrac{\partial}{\partial z}\right)^{n-1} z^{2n-2}\left(1-\dfrac{1}{z}\right)^j = \left(\dfrac{\partial}{\partial z}\right)^{n-1}\left(z^{2n-2-j}(z-1)^j\right).
\end{array}$$

For $z=1$ we then get

$$\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}+ \;=\; \binom{n-1}{j}j!\,(2n-2-j)(2n-3-j)\dots n=\frac{(2n-2-j)!}{(n-1-j)!}$$

whenever $j=0,1,\dots,n-1$.

In just the same way we evaluate the second sum. Suppose $j=n,n+1,\dots,2n-2$. Consider the function

$$F(z)=\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}z^{n-k-1}$$

which is actually equal $\left(\dfrac{\partial}{\partial z}\right)^{n-1}\left(z^{2n-2-j}(z-1)^j\right)$, as is easy to check.

Hence we readily deduce that $F(1)=0$, as desired.

Let us prove the thired equality corresponding to $j=2n-1$. For this purpose, consider the function $F(z)=\sum\limits_{k=0}^{n-1}(-1)^k\binom{2n-1}{k}\dfrac{(2n-2-k)!}{(n-1-k)!}z^{n-k-1}$.

An easy computation shows that

$$\begin{array}{c}
F(z)=\left(\dfrac{\partial}{\partial z}\right)^{n-1}\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{k}z^{2n-2-k}=\\
=\left(\dfrac{\partial}{\partial z}\right)^{n-1}\left(\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}z^{2n-2-k}-(-1)^{2n-1}\dfrac{1}{z}\right)=\\
=\left(\dfrac{\partial}{\partial z}\right)^{n-1}\left(\dfrac{(z-1)^{2n-1}}{z}\right)+\left(\dfrac{\partial}{\partial z}\right)^{n-1}\dfrac{1}{z}=\left(\dfrac{\partial}{\partial z}\right)^{n-1}\left(\dfrac{(z-1)^{2n-1}}{z}\right)+(-1)^{n-1}(n-1)!\,z^{-n}.
\end{array}$$

For $z=1$ the first term vanishes, and so $F(1)=(-1)^{n-1}(n-1)!$.

Consider the last equality for $j > 2n-1$. We have

$$F(z) = \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{l=0}^{n-1}\frac{(-1)^l z^{2n-l-2}}{l!(j-l)!}\Big) = \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{l=0}^{j}\frac{(-1)^l z^{2n-l-2}}{l!(j-l)!} - \sum_{l=2n-1}^{j}\frac{(-1)^l z^{2n-l-2}}{l!(j-l)!}\Big) =$$
$$= \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\frac{z^{2n-j-2}(z-1)j}{j!}\Big) + \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{l=2n-1}^{j}\frac{(-1)^{l+1} z^{2n-l-2}}{l!(j-l)!}\Big).$$

Thus
$$F(1) = \sum_{2n-1}^{j}\frac{(-1)^{l+n}(l-2n+2)\cdots(l-n)}{l!(j-l)!},$$
which proves the lemma. □

We are now in a position to complete the proof of Lemma 2. To this end, we observe that the first and the second sums in the expression for Σ cancel. In the third sum only the terms corresponding to $j \geqslant 2n-1$ do not vanish. Hence it follows that $\lim\limits_{\operatorname{Im} Z\to 0}\Sigma =$

$$= Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\binom{p}{2n-1}\frac{(-1)^{2n-1}Q^{2n-1}}{Q^{2n-1}} = (-1)^{n-1}(n-1)! = \binom{p}{2n-1}Z^{p-2n+1}.$$

Then by Lemma 3 we have

$$\Sigma = Z^{p-2n+1}\sum_{j=2n-1}^{\infty}(-1)^j p(p-1)\cdots(p-j+1)Q^{j-2n-1}\sum_{l=2n-1}^{j}\frac{(-1)^{l+n}(l-2n+2)\cdots(l-n)}{l!(j-l)!}.$$

Substituting $j = k+2n-1$ and $l = s+2n-1$, we get

$$\Sigma = Z^{p-2n+1}\sum_{k=0}^{\infty}(-1)^{k+1}p\cdots(p-k-2n+2)Q^k\sum_{s=0}^{k}\frac{(-1)^{s+n-1}(s+1)\cdots(s+n-1)}{(s+2n-1)!(k-s)!} =$$
$$= Z^{p-2n-1}\sum_{s=0}^{\infty}\sum_{k=s}^{\infty}\frac{(-1)^{k+s+n}p\cdots(p-k-2n+2)(s+1)\cdots(s+n-1)Q^k}{(s+2n-1)!(k-s)!} =$$
$$= Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^{s+n}(j+1)\cdots(j+n-1)}{(s+2n-1)!}\sum_{k=s}^{\infty}\frac{(-1)^k p\cdots(p-k-2n+2)Q^k}{(k-s)!} =$$
$$= Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^{s+n}(j+1)\cdots(j+n-1)}{(s+2n-1)!}\sum_{m=0}^{\infty}\frac{(-1)^{s+m}p\cdots(p-s-m-2n+2)Q^{s+m}}{m!} =$$
$$= Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^n(j+1)\cdots(j+n-1)Q^s}{(s+2n-1)!}\sum_{m=0}^{\infty}\frac{(-1)^m p\cdots(p-s-m-2n+2)Q^m}{m!}.$$

The sum
$$\sum_{m=0}^{\infty}\frac{(-1)^m p\cdots(p-s-m-2n+2)Q^m}{m!} =$$
$$= p\cdots(p-s-2n+2)\sum_{m=0}^{\infty}\frac{(-1)^m(p-2n-s+1)\cdots(p-s-m-2n+2)Q^m}{m!} =$$
$$= p\cdots(p-s-2n+2)(1-Q)^{p-2n-s-1}.$$

Thus

$$\Sigma = Z^{p-2n+1} \sum_{s=0}^{\infty} \frac{(-1)^n (s+1)\cdots(s+n-1)p\cdots(p-2n-s+2)Q^s(1-Q)^{p-2n-s+1}}{(s+2n-1)!} =$$

$$= (-1)^n \cdot Z^{p-2n+1} \cdot (1-Q)^{p-2n+1} \sum_{s=0}^{\infty} \frac{(s+1)\cdots(s+n-1)p\cdots(p-2n-s+2)Q^s(1-Q)^{-s}}{(s+2n-1)!}.$$

Since $Q = 1 - \bar{Z}/Z$, then $1 - Q = \bar{Z}/Z$ and $\dfrac{Q}{1-Q} = \dfrac{Z-\bar{Z}}{\bar{Z}}$.

From here $\Sigma = (-1)^n \bar{Z}^{p-2n-1} \sum\limits_{s=0}^{\infty} \dfrac{p\cdots(p-2n-s+2)}{s!(s+n)\cdots(s+2n-1)} \cdot \dfrac{(Z-\bar{Z})^s}{\bar{Z}^s}$.

Therefore

$$\operatorname{res}(f;Z) + \operatorname{res}(f;\bar{Z}) = \frac{\bar{Z}^{p-2n+1}}{(n-1)!} \sum_{s=0}^{\infty} \frac{p\cdots(p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)\bar{Z}^s} =$$

$$= \frac{1}{2(n-1)!} \sum_{s=0}^{\infty} \frac{p\cdots(p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \cdot \Big(\bar{Z}^{p-2n-s+1} + (-1)^s Z^{p-2n-s+1}\Big),$$

as desired. □

Theorem 3. *The function $\mathcal{M}_{t\mapsto\lambda} k(x', x'', y', y''; t)$ admits an asymptotic expansion*

$$\begin{aligned} &\mathcal{M}_{t\mapsto\lambda} k(x', x'', y', y''; t) = \\ &= \pi \frac{(i\lambda+1)\ldots(i\lambda+2n-2)}{(2n-1)!} \frac{\exp \pi\lambda}{\sinh \pi\lambda} \frac{((i\lambda+2n-1)\operatorname{Im} A - i\lambda \operatorname{Re} Z \operatorname{Im} B) Z^{-i\lambda-2n}}{(1+|x|^2)^n} + \\ &+ O(\operatorname{Im} Z) \end{aligned}$$

as $\operatorname{Im} Z \to 0$.

More then $\mathcal{M}_{t\mapsto\lambda} k(x', x'', y', y''; t) =$

$$= \frac{i\pi}{2(2n-1)!} \frac{\exp \pi\lambda}{(1+|x|^2)^n \sinh \pi\lambda} \sum_{s=0}^{\infty} \frac{(i\lambda+1)\cdots(i\lambda+2n+s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \times$$

$$\times \left((-1)^{s+1} \bar{Z}^{-i\lambda-2n-s}(i\lambda B\bar{Z} + A(i\lambda+2n+s-1)) - \bar{Z}^{-i\lambda-2n-s}(i\lambda BZ + A(i\lambda+2n+s-1))\right).$$

Proof. Using Lemmas 1 and 2 we obtain $\lim\limits_{\operatorname{Im} Z\to 0} \mathcal{M}_{t\mapsto\lambda} k(x', x'', y', y''; t) =$

$$= -\pi i \frac{(i\lambda+1)\ldots(i\lambda+2n-2)}{(2n-1)!} \frac{\exp \pi\lambda}{\sinh \pi\lambda} \frac{((i\lambda+2n-1)A - i\lambda Z B) Z^{-i\lambda-2n}}{(1+|x|^2)^n}.$$

Let us estimate the sum $B + \bar{B}$. Since $\langle \nabla_y \rho, y\rangle = h$ it follows that the real part of B is $\dfrac{B+\bar{B}}{2} = \dfrac{2}{\sigma_{2n}} \dfrac{\langle \nabla_y \rho, x\rangle - h}{|\nabla_y \rho|} = \dfrac{2}{\sigma_{2n}} \dfrac{\langle \nabla_y \rho, x - y\rangle}{|\nabla_y \rho|}$, which is $O(\operatorname{Im} Z)$ as $\operatorname{Im} Z \to 0$.

On the other hand, A is purely imaginary, for $\langle (\nu(y), \nu_{2n}(y)), (y, 1)\rangle = 0$.

This establishes the first formula.

Consider the last formula. We have $\operatorname{res}(G; Z) + \operatorname{res}(G; \bar{Z}) =$

$$= \frac{B}{2(n-1)!a^n} \sum_{s=0}^{\infty} \frac{(-i\lambda)\cdots(-i\lambda-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \left(\bar{Z}^{-i\lambda-2n-s+1} + (-1)^s Z^{-i\lambda-2n-s+1}\right) +$$

$$+\frac{A}{2(n-1)!a^n}\sum_{s=0}^{\infty}\frac{(-i\lambda-1)\cdots(-i\lambda-2n-s+1)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\left(\bar{Z}^{-i\lambda-2n-s}+(-1)^s Z^{-i\lambda-2n-s}\right)=$$

$$=\frac{1}{2(n-1)!a^n}\sum_{s=0}^{\infty}\frac{(-i\lambda+1)\cdots(-i\lambda-2n-s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\times$$

$$\times\left((-1)^{s+1}(i\lambda B\bar{Z}^{-i\lambda-2n-s+1}+A(i\lambda+2n+s-1)\bar{Z}^{-i\lambda-2n-s})-\right.$$

$$\left.-\left(i\lambda BZ^{-i\lambda-2n-s+1}+A(i\lambda+2n+s-1)Z^{-i\lambda-2n-s}\right)\right)=$$

$$=\frac{1}{2(n-1)!a^n}\sum_{s=0}^{\infty}\frac{(-i\lambda+1)\cdots(-i\lambda-2n-s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\times$$

$$\times\left((-1)^{s+1}\bar{Z}^{-i\lambda-2n-s}(\lambda B\bar{Z}+A(i\lambda+2n+3-1))\ -Z^{-i\lambda-2n-s}(\lambda BZ+A(i\lambda+2n+3-1))\right).$$

Then using the equality (11), we get

$$\int_0^{\infty}t^{-\imath\lambda}k(x',x'',y',y'';t)\frac{dt}{t}=\pi i\,\frac{\exp\pi\lambda}{\sinh\pi\lambda}\Big(\operatorname{res}(G(t);Z)+\operatorname{res}(G(t);\bar{Z})\Big)=$$

$$=\frac{i\pi\exp\pi\lambda}{2(n-1)!a^n\sinh\pi\lambda}\sum_{s=0}^{\infty}\frac{(-i\lambda+1)\cdots(-i\lambda-2n-s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\times$$

$$\times\left((-1)^{s+1}\bar{Z}^{-i\lambda-2n-s}(\lambda B\bar{Z}+A(i\lambda+2n+3-1))\ -Z^{-i\lambda-2n-s}(\lambda BZ+A(i\lambda+2n+3-1))\right).$$

□

References

[1] V.A.Kondrat'ev, Boundary value problems for elliptic equations in domains with conical points, *Trudy Mosk. Mat. Obshch.*, **16**(1967), 209–292 (in Russian).

[2] A.Kytmanov, The Bochner-Martinelli Integral, and Its Applications, Birkhäuser Verlag, Basel et al., 1995.

[3] A.M.Kytmanov, Myslivets S.G. On Asymptotic Expansion of the Conormal Symbol of the Singular Bochner-Martinelli Operator on the Surfaces with Singular Points, *J. Sib. Fed. Univ. Mathematics and Physics*, **1**(2008), no. 1, 3–12.

[4] G.Khudaiberganov, D.Kh.Djumabaev, Integral Bochner-Martinelli on the singular hypersurfaces, *Uzbeksky Mat, Zh.*, (2011), no. 2, 162–173 (in Russian).

[5] D.Kh.Djumabaev, On conormal symbol of the singular integral Bochner-Martinelli in the domains with conical wedges, *Uzbeksky Mat, Zh.*, (2012), no. 1, 29–37 (in Russian).

Об асимптотическом разложении конормального символа сингулярного интегрального оператора Бохнера-Мартинелли на поверхностях с коническими ребрами

Давлат Х. Джумабаев

В работе изучен конормальный символ сингулярного интеграла Бохнера-Мартинелли на компактных закрытых поверхностях с коническими ребрами $\mathcal{S}$ в $\mathbb{C}^n$ и вычислено его асимптотическое разложение.

Ключевые слова: сингулярный оператор Бохнера-Мартинелли, конормальный символ, коническое ребро.