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Some Problems in the Theory of Analytic Multifunctions

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In this paper we give a series of open problems in the theory of pseudoconcave sets.

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A set-valued function (multifunction) $F : z \to S_z$, where $z \in D \subset \mathbb{C}^n,\ S_z \subset \mathbb{C}_w$, is called an *analytic multifunction*, if the graph $S = \{(z, w) \in \mathbb{C}^{n+1} : z \in D, w \in S_z\}$ is pseudoconcave in $G = D \times \mathbb{C}$, i.e., the set S is closed and the set G/S is pseudoconvex in the neighborhood of S.

This definition is motivated by the fact that the graph of an analytic and algebraic function $S = \{a_0(z)w^k + a_1(z)w^{k-1} + \ldots + a_k(z) = 0\}$ where $k \geqslant 1, a_j(z) \in O(D),\ j = 0, 1, \ldots, k$, are analytic multifunctions in D.

Analytic multifunctions (or, equivalently, pseudoconcave sets) are natural object study in the theory of analytic extension, analytic structures of singular sets of holomorphic functions etc. Functional properties and the analytic structure of such functions were investigated in the works of Oka [1], Nishino [2], Yamaguchi [3], Slodkowski [4]-[7], Berndtsson-Ransford [8], Alexander and Wermer [9]-[12], and others.

We used analytic multifunctions in the rational approximation theory, in the problems of extension of holomorphic and pluriharmonic functions [13]-[14]. Analyticity of an arbitrary multifunction $F : z \to S_z$ is connected with plurisubharmonicity (Psh) of, $V(z, w) = -\ln \rho(w, S_z)$ where $\rho(w, S_z)$ is the distance from the point w to the set S_z, at a fixed $z \in D$. The function $-\ln \rho(w, S_z)$ is easily determined and convenient for the investigation of F.

If $z \to S_z$ is analytic, i.e., if S is pseudoconcave, then the function $V(z, w)$ is Psh in $G \setminus S$. The converse is not clear, because the function $-\ln \rho(w, S_z)$ in contrast with $-\ln \rho((z, w), S_z)$ does not, in general, tend to $+\infty$ at $(z, w) \to S$.

Example 1. We consider a $u(z) \in Sh(B)$ in the unit disk $B \subset \mathbb{C} : u < 0$ and let $\{u(z) = -\infty\}$ be dense on B, $u(0) = -1$. Then $S = \{|w| \leqslant e^{u(z)}\}$ is bounded and analytic in $B \times \mathbb{C} \subset \mathbb{C}^2$.

We have $V(z, w) = -\ln \rho(w, S_z) = -\ln(|w| - e^{u(z)})$ and if $w_0 = \dfrac{1}{2e},\ z_v \to 0$ at $\nu \to \infty$, where $u(z_v) = -\infty$, then $V(z_v, w_0) = -\ln \dfrac{1}{2e} \neq -\infty$.

For $n = 1$, S is analytic if and only if $-\ln \rho(w, S_z) \in Psh(D)$ (see Slodkowski [4]). On the other side, for $S = (\mathbb{C}^2 \times \{|w| = 1\}) \cup (\{0\} \times \{|w| \leqslant 1\}) \subset \mathbb{C}^3$ the function $-\ln \rho(w, S_z) \in Psh(\mathbb{C}^3/S)$, but S is not pseudoconcave, i.e., $z \to S_z$ is not analytic (at a point $\{0\} \times \{|w| < 1\}$).

The class of multifunctions S, when S is pluripolar in C^{n+1} is more interesting in complex analysis. For this kind of "good" multifunctions the following theorem is valid:

Theorem 1 (A. Sadullaev [15])**.** *A "good" multifunction $S \subset D \times \mathbb{C}$ is analytic if and only if the function $V(z, w) = -\ln \rho(w, S_z) \in Psh(G/S)$.*

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We have the following series of open problems for "good" analytic multifunctions

1. Is the analog of the Hartogs theorem true for a "good" analytic multifunction: If $F : z \to S_z, z \in D \subset \mathbb{C}^n$, is separately analytic in the variables $z_1, z_2, \ldots, z_n$, is then F analytic in D?

2. Let $z \to S_z$ be a multifunction such that S_z is polynomially convex, or, more generally, G/S is connected. Is $z \to S_z$ analytic if and only if $V(z,w) \in Psh(G/S)$?

3. Let $z \to S_z$ be a bounded "good" analytic multifunction in $D \subset \mathbb{C}^n$, that is, $S \subset D \times \mathbb{C}_w,\ \ S \subset \{|w| < R\}$ be a pseudoconcave, and pluripolar set. Is there for any $\varepsilon > 0$ and any $K \subset G, K \cap S = \varnothing$ a pseudopolynomial $P_N(z,w) = a_0 + a_1(z)w + \ldots + a_N(z)w^N$ such that $S \subset \{|P_N(z)|^{\frac{1}{N}} < \varepsilon\}$ and $P|_K \neq 0$?

   The positive solution of this problem is very important in the theory of approximation by rational functions. For if we try to approximate a holomorphic function f by a thin singular set S, then S is pseudoconcave as a singular set as well. And we need the polynomials $P_N(z)$ for construction of a rational sequence $r(z) = \dfrac{q_m(z)}{P_N^j(z)}$, tending to f (see [14]).

4. Let $S \subset \mathbb{C}^{n+1}$ be a pluripolar, pseudoconcave set. Is $M = \mathbb{C}^{n+1}/S$ a parabolic manifold? It means that there exists an exhaustion function such that

   – $\forall C$ the set $\{z \in M : \rho(z) < C\}$ is compact;

   – $\exists K \subset M$ such that $(dd^c\rho)^{n+1} = 0$ outside of K.

   Note that in the theory of Nevanlinna we use this kind of manifolds. If S is the graph of an analytic function, then the problem has a positive solution.

5. Is the graph of a "good (pluripolar)" analytic multifunction $z \to S_z$ complete pluripolar set, i.e., $\exists u \in Psh(D \times \mathbb{C}) : S = \{u(z) = -\infty\}$?

6. If $F : z \to S_z$ and $F' : z \to S_z^{r}$ are two pluripolar analytic multifunctions such that $F = F'$ in a neighborhood $U \subset D, U \neq \varnothing$ will then $F = F'$ everywere in D?

7. N.Shcherbina has recently proved, that if $S = \{a_0(z) + a_1(z)w + \ldots + a_k(z)w^k = 0\}$ is continuous and pluripolar, then S is analytic. Is its analog true for an arbitrary continuous multifunction?

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