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Hypercentral and Monic Automorphisms of Classical Algebras, Rings and Groups

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Up to standard multipliers all non-standard automorphisms of free associative algebras and polynomial algebras are reduced to monic automorphisms of the maximal ideal, which are studied in the present paper. For non-standard automorphisms of some locally nilpotent matrix groups and rings it has turned out to be more efficient to use hypercentral automorphisms.

Keywords: free associative algebra, polynomial algebra, finitary Chevalley group, unipotent subgroup, associated Lie ring, Jordan ring, automorphism.

Introduction

Usually the first step on the road to describing the automorphisms of classical algebras or groups is specifying its standard automorphisms. In this paper we consider two typical cases when up to a multiplication by a standard automorphism every non-standard automorphism can be written as a monic or hypercentral automorphism which were introduced in [1] and [2].

An automorphism (similarly, an endomorphism) of a ring or algebra R is said to be *monic*, if it induces the identity map on each factor R^m/R^{m+1}. The analogous definition of monic endomorphisms of any group uses the factors of its lower central series. If an automorphism acts like the identity modulo the m-th hypercenter $\mathcal{Z}_m(R) \neq R$, then it is called *hypercentral* or *central* for $m = 1$.

Dubish and Perlis [1] described automorphisms of the algebra $NT(n, F)$ of all $n \times n$ matrices with zeros on and above the main diagonal over a field F. Every monic automorphism of this algebra is the product of an inner and central automorphisms. The monic automorphisms of the ring $NT(n, K)$ ($n \geq 3$) over an arbitrary associative ring K with identity have similar description, [3]. However, it is impossible to use monic automorphisms for the description of non-standard automorphisms of the finitary ring $NT(\Gamma, K)$ with any chain Γ of matrix indices (§ 1). It has

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turned out out to be more efficient to use hypercentral automorphisms describing non-standard automorphisms of associated Lie and Jordan rings and of the unitriangular group $UT(\Gamma, K)$. This approach is also used in § 1 for the unipotent subgroups of some finitary Chevalley groups.

The hypercentral (and, similarly, hyperannihilator) series of a polynomial algebra $B_n = F[x_1, \ldots, x_n]$ in commutative variables over an arbitrary field F and of a free associative algebra $A_n = F\langle x_1, x_2, \ldots, x_n\rangle$ with the free generators $x_1, x_2, \ldots, x_n$ is trivial. The same is true for their ideal $R = \langle x_1, \ldots, x_n\rangle$ in spite of the fact that $\bigcap_{k=1}^{\infty} R^k = 0$. The problem of describing the automorphisms of A_n and B_n [4, Question 3.3] is still open. It is agreed to call the standard and non-standard automorphisms of these algebras, respectively, tame and wild, A.Chernyakiewicz [5], P.Cohn [6], etc. Obviously, every automorphism of the ideal R can be uniquely continued to one of the whole algebra. In § 2 we prove

Corollary 1. *Up to a multiplication by a tame automorphism, any automorphism of the algebras A_n and B_n is the continuation of a monic automorphism of R.*

We study some properties of tame and wild automorphisms of the algebra R and the nilpotent factors R/R^k. The problem of automorphism lifting for some free nilpotent groups and algebras is studied in [7], [8]. In § 3 we study certain subgroups of Aut R/R^k and bases of associated linear spaces, see Theorem 3.

1. Automorphisms of Locally Nilpotent Rings and Groups

Firstly, we consider standard automorphisms of certain locally nilpotent rings and groups. Recall that a Lie ring $\Lambda(R) := (R, +, *)$ with the Lie product $\alpha * \beta = \alpha\beta - \beta\alpha$ and, also, a Jordan ring $J(R) := (R, +, \circ)$ with the Jordan product $\alpha \circ \beta = \alpha\beta + \beta\alpha$ are associated to every associative ring R. The map $x \to 1 + x$ of any radical ring R is an isomorphism of the adjoint group $G(R)$. For the automorphism groups it isn't difficult to verify the following equalities:

$$Aut\ R = Aut\ G(R) \cap Aut\ J(R) = Aut\ G(R) \cap Aut\ \Lambda(R).$$

Usually all automorphisms of R are considered as standard automorphisms for the adjoint group and associated rings. Every inner automorphism of the adjoint group gives an *inner* automorphism of the ring R.

By [2], an automorphism of a ring or group is called *hypercentral of height* m (or *central* for $m \leqslant 1$), if it acts like the identity, modulo the m-th hypercenter and even up to multiplication by inner automorphisms such m is the least.

Let K be an associative ring with identity. Choose a chain (or linearly ordered set) Γ by an order relation $\leqslant$. A niltriangular Γ-matrix $\| a_{ij} \|_{i,j\in\Gamma}$, $a_{uv} = 0$, $u \leqslant v$, over K is said to be *finitary*, if it has a finite number of nonzero elements. The ring $NT(\Gamma, K)$ (or $NT(n, K)$ at $\Gamma = \{1, 2, \cdots, n\}$) of all such Γ-matrices with the usual matrix addition and multiplication is locally nilpotent and hence radical. The adjoint group of the ring $NT(\Gamma, K)$ for a finite chain Γ is isomorphic to the unitriangular group $UT(|\Gamma|, K)$.

Set $R = NT(\Gamma, K)$. Let e_{ij} be the Γ-matrix unit. Evidently, the elementary Γ-matrices xe_{ij} $(x \in K, i > j)$ generate the additive and adjoint groups of the ring R. When a chain Γ is dense, we may always choose $k \in \Gamma$ such that $j < k < i$ and hence $e_{ij} = e_{ik}e_{kj}$. Thus $R = R^2$ and therefore every monic automorphism of the ring R or the associated ring is trivial. Also, see [9], [10].

We now consider the center and certain hypercentral series. Denote by p and q, respectively, the first and the last elements of Γ (if they exist), by $[i, j]$, the segment $\{k \in \Gamma \mid i \leqslant k \leqslant j\}$ of Γ.

Lemma 1. *The center (and the annihilator) of the ring $R = NT(\Gamma, K)$ is nonzero if and only if $p, q \in \Gamma$. The $m-$th hypercenter $\mathcal{Z}_m(R)$ of R coincides with $\langle Ke_{ij} \mid j < i,\ |[p,j]|+|[i,q]| \leqslant m+1\rangle$. Also, it coincides with the m-th hypercenter of the associated Jordan and Lie rings and of the adjoint group.*

Proof. We obtain the statements of the lemma directly, by using the main relations between elementary Γ-matrices. For the adjoint group $G(R)$ and for the rings R, $\Lambda(R)$ see also [3], [9]. □

Evidently, every isomorphism θ of the coefficient ring K induces a ring isomorphism $\| a_{ij} \| \to \| \theta(a_{ij}) \|$ of the ring R. Analogously, every isomorphism (or isometry) of the chain Γ induces a chain isomorphism of R.

Since the ring R is locally nilpotent, $e + \beta + \beta^2 + \cdots = (e - \beta)^{-1} \in R$ for any $\beta \in R$ and the identity Γ-matrix e. It gives an inner automorphism $\alpha \to (e - \beta)\alpha(e - \beta)^{-1}$ $(\alpha \in R)$ of the ring R. Consider a generalization. Choose an arbitrary (lower) triangular Γ-matrix $\gamma = \| \gamma_{ij} \|$ (with $\gamma_{ij} = 0$ for $i < j$) over K in which every row with a number $\neq q$ and every column with a number $\neq p$ have only finitely many nonzero elements. If there exists a similar Γ-matrix γ' satisfying the equalities $\gamma\gamma' = \gamma'\gamma = e$ modulo the center of R, then the map $\alpha \to \gamma\alpha\gamma'$ $(\alpha \in R)$ is an automorphism, which is called *a triangular* (or locally inner, if the main diagonal of γ is zero, or *diagonal* for a diagonal Γ-matrix γ), automorphism of the ring R, [9]. By [3] and [9], we have:

If Γ is a finite chain of order ≥ 3 or K has no zero-divisors, then every automorphism of the ring $R = NT(\Gamma, K)$ is a product of induced ring and chain automorphisms, triangular and central automorphisms.

When either Γ and K satisfy the same restrictions and $|\Gamma| > 4$ or $|\Gamma| = 3, 4$ and K is a commutative ring, the automorphism groups of the Lie ring $\Lambda(R)$ and of the adjoint group $G(R)$ also has been described in [3], [9]. Consider the Jordan automorphisms of the ring R, i.e., automorphisms of the Jordan ring $J(R)$.

By Herstein's classical theorem [11], *every Jordan isomorphism between prime rings of characteristic not* 2 *is an isomorphism or an anti-isomorphism.* The usual goal is to describe all Jordan automorphisms and isomorphisms. The special case has been investigated by X. Wang [12] etc: *every Jordan automorphism of the algebra $NT(n, K)$ over a 2-torsion free commutative ring K with no idempotents except* 0 *and* 1 *is an automorphism or an anti-automorphism.*

In the general case we may construct non-trivial Jordan isomorphisms of the ring $R = NT(\Gamma, K)$ by analogy with idempotent isomorphisms of $\Lambda(R)$ and $G(R)$. Let S be a ring and f be a central idempotent of the ring K. An isomorphism $\theta:\ K^+ \to S^+$ of the additive groups is called *an idempotent isomorphism* of the ring K, if it induces an isomorphism of the ideal fK and an anti-isomorphism of the ideal $(1 - f)K$ and also $\theta(1_K) = 1_S$, [3]. If $'$ is an anti-automorphism of the chain Γ, we obtain induced an idempotent Jordan isomorphism R:

$$\alpha \ \to\ \theta(f\alpha) + \theta[(1 - f)\alpha'] \quad (\alpha = \| \alpha_{ij} \| \in R, \quad \alpha'_{ij} = \theta(\alpha_{j'i'})).$$

Jordan automorphisms of the ring $R = NT(\Gamma, K)$ can be described by analogy with $\operatorname{Aut}\Lambda(R)$ and $\operatorname{Aut} G(R)$ in [3], [9] (see also [13]). A Jordan automorphism of R is called *standard*, if it is a product of an automorphism of R and an idempotent automorphism of $J(R)$. The following theorem holds.

Theorem 1. *Let $R = NT(\Gamma, K)$, $|\Gamma| > 4$. If Γ is a finite chain or K is a ring without zero-divisors, then every automorphism of the Jordan ring $J(R)$ (analogously, of $\Lambda(R)$ or $G(R)$) is a product of some standard and hypercentral of height $\leqslant 3$ automorphisms of $J(R)$.*

Note that automorphisms are also described in the exceptional cases $|\Gamma| = 3, 4$, in particularl, for any commutative ring of coefficients.

Example. We now show that there exist non-standard Jordan hypercentral automorphisms of R. Let $p, q \in \Gamma$ and let there exists the direct successor k of p (i.e., the first element of the subset $\{j \in \Gamma \mid p < j\}$) and the direct successor m of k in Γ. Choose an element $c \in K$ with $c(K \circ K) = 0$; this is equivalent to the restrictions $2c = 0$ and $c(K * K) = 0$. Then the map $xe_{kp} \to (e_{kp} + ce_{qk})x$, $x \in K$ (other elementary matrices xe_{uv} are fixed) and, analogously, the map

$$xe_{kp} \to (e_{kp} + ce_{qm})x, \quad xe_{mp} \to (e_{mp} + ce_{qk})x \quad (x \in K)$$

determines automorphisms of the Jordan ring R. Such automorphisms together with symmetrical ones generate all Jordan hypercentral automorphisms up to multiplication by inner and central automorphisms.

Recall that the adjoint group of the ring $NT(n, K)$ is isomorphic to the unitriangular group $UT(n, K)$ which is also isomorphic to the unipotent subgroup $UG(K)$ of the Chevalley group of alone Lie type $G = A_{n-1}$. The well-known question about automorphisms of all unipotent subgroups over finite fields [14, Problem (1.5)] has been solved in 90-s. In the general case the following theorem has been proved (see [2]).

Theorem 2. *Every automorphism of the unipotent subgroup $UG(K)$ of Lie rank ≥ 3 over an arbitrary field K is a product of some standard and hypercentral of height $\leqslant 5$ automorphisms.*

For the unipotent group of the classical types $G = B_n,\ C_n,\ D_n,\ {}^2A_n,\ {}^2D_n$ finitary generalizations of the types

$$B_\Gamma,\ C_\Gamma,\ D_\Gamma,\ {}^2A_\Gamma,\ {}^2D_\Gamma, \tag{1}$$

respectively, with an arbitrary chain Γ has been investigated, see [2], [15] etc.

Hypothesis. *Does theorem 2 hold for the finitary unipotent group $UG(K)$ of types* (1) *with any infinite chain Γ ?*

In [9] theorem 1 is proved inr the case of an infinite chain Γ by using the description of maximal abelian ideals of associated rings. There exist close structural connections of normal subgroups of $UG(K)$ and ideals of the associated Lie ring. The normal structure and maximal abelian normal subgroups of $UG(K)$ are described in a uniform form by V.Levchuk and G.Suleimanova [15].

Also, the last description allows one to find the large (or the highest order) abelian subgroups of $UG(K)$ over finite fields in explicit form, see A.S.Kondratyev's question [14, Problem (1.6)]. We obtain all large abelian normal subgroups of $UG(K)$ directly from [15]. As it was recently shown by G.Suleimanova, there exist large abelian subgroups of some groups $UG(K)$ which are not conjugated in the Chevalley group $G(K)$ with a normal subgroup of $UG(K)$. Therefore it is natural to investigate *the question of description all such exceptional cases.*

2. Automorphisms of the Algebras A_n and B_n

We consider a free associative algebra $A_n = F\langle x_1, x_2, \ldots, x_n\rangle$ over an arbitrary field F with the free generators $x_1 x_2, \ldots, x_n$ and a polynomial algebra $B_n = F[x_1, \ldots, x_n]$ in n commutative variables. If φ is an endomorphism of such an algebra and $x_i^\varphi = \varphi_i$ for all i, then we write $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$.

We call an automorphism elementary if it has the form

$$(x_1, \ldots, x_{k-1}, \alpha x_k + f, x_{k+1}, \ldots, x_n), \tag{2}$$

where f is a polynomial not containing x_k. An automorphism is said to be *tame*, if it is a product of some elementary automorphisms; all other automorphisms are called *wild*. Tame and wild automorphisms, in particular, the Anik automorphism of A_n and the Nagata automorphism of B_n were considered in [5], [16], [6], etc. It was recently proved that the Nagata automorphism and the Anik automorphism are wild, [17], [18]. Thus, the tame automorphism groups $\mathrm{TAut}\, A_n$ and $\mathrm{TAut}\, B_n$ are proper subgroups of groups $\mathrm{Aut}\, A_n$ and $\mathrm{Aut}\, B_n$, respectively.

In both algebras we choose the ideal R generated by the elements $x_1, x_2, \ldots, x_n$. Obviously, for every automorphism of the ideal R there exists a unique continuation on the whole algebra.

We now show that the study of the wild automorphisms of the algebras A_n and B_n comes to the study of wild monic automorphisms of the ideal R. We consider the following automorphisms of the algebra A_n and B_n:

$$(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n), \quad c_i \in F. \tag{3}$$

Also for every $\alpha \in GL_n(F)$ we define the automorphism of R:

$$\tilde{\alpha} : X \to X\alpha, \quad X = (x_1, x_2, \ldots, x_n).$$

Proposition 1. *Let φ be an automorphism of the algebra A_n or B_n. Then φ is a product of the continuation of some monic automorphism of R, $\tilde{\alpha}$ for some $\alpha \in GL_n(F)$ and an automorphism* (3).

Proof. Since $x_i^\varphi = c_i$, $1 \leqslant i \leqslant n$, modulo R, up to a multiplication by an automorphism (3) we may suppose that φ preserves R. Therefore modulo R^2 we have

$$\varphi(X) = X\alpha, \quad \varphi^{-1}(X) = X\beta$$

for some $n \times n$ matrices α, β over F. It follows that $\alpha = \beta^{-1} \in GL_n(F)$ and hence $\tilde{\alpha} \in \mathrm{Aut}\, R$. So φ is a continuation of an automorphism of R which is a product of $\tilde{\alpha}$ and some monic automorphism of the ideal R. □

It is well-known that every invertible matrix over any field is a product of elementary matrices. Therefore we obtain

Corollary 1. *Up to a multiplication by a tame automorphism any automorphism of the algebras A_n and B_n is the continuation of a monic automorphism of R.*

Note that monic endomorphisms of the ideal R and its factors R/R^k are always monomorphisms. Every n-tuple $(\varphi_1, \ldots, \varphi_n)$ over R^2 determines an endomorphism

$$(x_1 + \varphi_1, \ldots, x_n + \varphi_n), \quad \varphi_i \in R^2. \tag{4}$$

of R. This endomorphism is not always an automorphism.

Example. We show that a monic endomorphism $\varphi = (x_1 + x_1x_1, x_2, \ldots, x_n)$ of R is not an automorphism. Suppose it is an automorphism. Using the Fox derivation [19] we calculate the Jacobi matrix J_φ.Then $J_\varphi = \mathrm{diag}(2+x_1, 1, 1, \ldots, 1)$, and hence J_φ is a matrix over a commutative ring $F[x_1]$. Since φ is an automorphism, there exists an inverse matrix J_φ^{-1}. In particular, the determinant $|J_\varphi| = 2+x_1$ is an invertible element. However, the element $2+x_1$ is not invertible in the free algebra $F[x_1]$. This gives us a contradiction. Thus φ is not an automorphism. A criterion for an endomorphism of the algebra A_n to be an automorphism, based on the concept of a Fox derivation, is proved in [20].

Let us consider some properties of endomorphisms of the ideal R and a factor algebra R/R^k for a fixed integer $k > 1$. The following lemma is evident.

Lemma 2. *Any tame automorphism of R induces a tame automorphism of a free nilpotent algebra R/R^k. Any tame automorphism of R/R^k is induced by a tame automorphism of an ideal R.*

Clearly, a monic endomorphism φ of the algebra R or R/R^k is an automorphism if and only if this algebra is equal to its φ-image. By definition, φ induces the identity automorphisms on the factors R^m/R^{m+1}, $m = k-1, k-2, \ldots, 1$. Consequently, $\varphi(R/R^k) = R/R^k$. Thus the following lemma holds.

Lemma 3. *Every monic endomorphism of a free nilpotent algebra is always an automorphism.*

As above, every matrix $\alpha \in GL_n(F)$ defines an automorphism $\tilde{\alpha}$ of the ideal R. This notation is preserved for the induced automorphisms of R/R^k. The following lemma gives a canonical form of tame automorphisms.

Lemma 4. *Let φ be a tame monic automorphism of R. Then there exist monomials $m_i \in R^2$ and $\tilde{\lambda}_i \in \widetilde{GL}_n(F)$, $i = 1, 2, \ldots, v$, such that*

$$\varphi = \prod_{i=1}^{v} \tilde{\lambda}_i^{-1} e_i \tilde{\lambda}_i, \tag{5}$$

$$e_i = (x_1 + m_i, x_2, \ldots, x_n). \tag{6}$$

Proof. By definition any tame automorphism can be written as a product of elementary automorphisms. Up to a substitution, any elementary automorphism may be written as a product of elementary automorphisms (6). So we have:

$$\varphi = \tilde{\mu}_1 e_1 \ldots \tilde{\mu}_v e_v \tilde{\mu}_{v+1} = \tilde{\mu}_1 e_1 \tilde{\mu}_1^{-1} \tilde{\mu}_1 \tilde{\mu}_2 e_2 \ldots \tilde{\mu}_1 \ldots \tilde{\mu}_v e_v \tilde{\mu}_v^{-1} \ldots \tilde{\mu}_1^{-1} \tilde{\mu}_1 \ldots \tilde{\mu}_{v+1},$$

where $\tilde{\mu}_i \in \widetilde{GL}_n(F)$, e_i is like (6) for some monomials m_i. Let $\tilde{\lambda}_i = \tilde{\mu}_1 \ldots \tilde{\mu}_i$. Since φ acts identically modulo R^2 we have $x_i^\varphi = x_i \mod R^2$. On the other hand $x_i^\varphi = x_i^{\tilde{\lambda}_{v+1}}$. Hence $\tilde{\lambda}_{v+1}$ is an identity. □

We should also mention the following corollary of Lemma 4.

Lemma 5. *If φ is a tame automorphism of a free nilpotent algebra R/R^k over a field F which is a subfield of the field D then φ is a tame automorphism of a free nilpotent algebra over D.*

Thus, if some automorphism of the algebra A_n over a field F induces a wild automorphism of a free nilpotent algebra over a field D and F is a subfield of D then the former automorphism of A_n is wild.

3. Bases of the Abelian Factors of $\mathrm{Aut}\,\mathbf{R}/\mathbf{R}^{\mathbf{k}}$

In this paragraph we study the subgroup Γ_s $(1 < s < k)$ of all automorphisms of R/R^k which act identically modulo R^s/R^k. We find an algorithm constructing the bases of a subgroup of tame automorphisms of the factor Γ_s/Γ_{s+1} as a linear space.

Lemma 6. $\mathrm{Aut}\, R/R^k \simeq \Gamma_2 \leftthreetimes \widetilde{GL}_n(F)$.

Proof. It is an obvious consequence of Proposition 1. □

To shorten our expressions we use the tensor notation for summation and ordered n-tuples:

$$(a_i) := (a_1, a_2, \dots, a_n), \quad a^i b_i := \sum_{i=1}^{n} a^i b_i. \tag{7}$$

Let F be an algebraically closed field. Every element of Γ_s/Γ_{s+1} has the form (4) with an n-tuple $(\varphi_1, \dots, \varphi_n)$ over R^s/R^{s+1}. The multiplication of two elements of Γ_s/Γ_{s+1} is equivalent to the addition of the corresponding tuples. If $\varphi \in \Gamma_s/\Gamma_{s+1}$ then using the notation (7) we write $\varphi = (x_i + a_i^{j_1\dots j_s} x_{j_1} \dots x_{j_s})$. Let $g \in F,\ \gamma = gE \in GL_n(F)$. Then,

$$\tilde{\gamma}^{-1}\varphi\tilde{\gamma} = (x_i + g^{s-1} a_i^{j_1\dots j_s} x_{j_1} \dots x_{j_s}). \tag{8}$$

The conjugation (8) is equivalent to the multiplication of the tuple $(\varphi_1, \dots, \varphi_n)$ by a constant. We may define a bijection $\varphi \to \hat{\varphi}$ between the elements of Γ_s/Γ_{s+1} and the linear space V_s of the ordered n-tuples over R^s/R^{s+1} with the operation of addition and multiplication we have just described. Obviously, those operations preserve the property of the operands to correspond to tame automorphisms.

Thus, the tame automorphisms from Γ_s/Γ_{s+1} induce some subspace TV_s of V_s. In the group $\mathrm{TAut}\ R/R^k$ we introduce the following subset

$$\Phi = \{\varphi_i^j \mid i = 2, 3, \dots, k-1, j = 1, 2, \dots, d_i\} \tag{9}$$

such that the automorphisms $\{\varphi_s^j\}_{j=1}^{d_s}$ induce a basis of TV_s for each s. Evidently, we may choose a subset Φ containing the automorphisms of the form (6) for all monomials m_i in R^2/R^k. Then it is easy to construct an algorithm to check if a given automorphism of R/R^k is tame or wild by consecutively expressing it in each factor Γ_s/Γ_{s+1} using the base of the linear space TV_s.

We construct an algorithm expressing a given automorphism $\psi(r)$ $(r \in F)$ in a given subset $\Phi' = \{\varphi_i^j\}, i = 2, 3, \dots, k-1, \quad j = 1, 2, \dots, d_i'$ of Φ.

Algorithm 1.

1. Let $\psi_2 = \psi$.
2. If $\psi_s \in \Gamma_s \setminus \Gamma_{s+1}$ $(2 \leqslant s < k)$ is the identity, then the test is a success. Otherwise let $\{\hat{\varphi}_s^j\}_{j=1}^{d_s'}$ be elements of TV_s induced by $\varphi_s^j, i = 1, 2, \dots, d_s'$.
3. Regard $\hat{\psi}_s(r)$ as a polynomial in r with vector-coefficients c_s^u. Since r is an arbitrary element of the field, we express each vector-coefficient c_s^u in $\hat{\varphi}_s^i$, and find

$$\hat{\psi}_s(r) = \sum_{i=1}^{d_s'} \sum_u \alpha_i^u r^u \hat{\varphi}_s^i, \quad c_s^u = \sum_{i=1}^{d_s'} \alpha_i^u \hat{\varphi}_s^i. \tag{10}$$

4. If the procedure of step 3 did not succeed, then ψ_s is not expressed in Φ'.

5. If step 3 gives $\tilde{\psi}_s$, then set $\psi_{s+1} = \psi_s \prod_{p=1}^{t}(\gamma_p^{-1}x_i)\varphi_{j_p}(\gamma_p x_i)$, where $\gamma_p^s = \sum_u \alpha_i^u r^u$. Since ψ_{s+1} acts identically modulo Γ_{s+1}, we proceed to step 2 with $s+1$.

For each $\tilde{\lambda} \in \widetilde{GL}_n(F)$ and $\varphi \in \Phi$ an automorphism $\varphi' = \tilde{\lambda}^{-1}\varphi_i^j\tilde{\lambda}$ can be expressed in Φ using algorithm 1. It should hold for all elementary and diagonal matrices λ, in particular, for the following $n \times n$ matrices

$$\gamma_p(r) = \text{diag}(1, 1, \ldots, r, 1, \ldots, 1) \tag{11}$$

where r is the p-th element of the diagonal.

Algorithm 2 below is based on the latter requirement and constructs the sought set Φ.

Algorithm 2.

1. Let $\{m_i^s\}_{i=1}^{M_s}$ be the set of all monomials of degree $s+1$ over $x_1, x_2, \ldots, x_s$. Initially, $\Phi' = (x_1 + m_1^t, x_2, \ldots, x_n), \ldots, (x_1 + m_{M_s}^t, x_2, \ldots, x_n)_{t=1}^{k-2}$.

2. For each fixed constant r there is a finite number of elementary matrices $t(r)$ and a finite number of diagonal matrices $\gamma_p(r)$. If for each elementary matrix $t(r)$, diagonal matrix $\gamma_p(r)$ and for each $\varphi \in \Phi'$ automorphisms $t(r)^{-1}\varphi t(r)$ and $\gamma_p(r)^{-1}\varphi\gamma_p(r)$ are expressible in Φ', then $\Phi = \Phi'$ is the sought set.

3. Let $\psi(r) \in \Gamma_s \setminus \Gamma_{s+1}$ be not expressible in Φ' with algorithm 1. Regarding $\hat{\psi}(r)$ as a polynomial in r of degree d, choose a subset $\{0, f_1, \ldots, f_d\}$ of F of the power $d+1$. Add those $\psi(f_i)$ to Φ' for which $\hat{\psi}(f_i)$ can not be linearly expressed in $\{\hat{\varphi}_s^j\}_{j=1}^{d'_s}$, and go to step 2.

The following well-known Lemma 7 together with Lemma 8 prove the correctness of algorithm 2.

Lemma 7. *Let $f(r) = \alpha_0 + \alpha_1 r + \ldots + \alpha_p r^p$, where α_i are vector-coefficients. If $0, a_1, \ldots, a_p$ are different elements of the field F then $f(r)$ can be linearly expressed in $f(0), f(a_1), \ldots, f(a_p)$ for any r.*

Lemma 8. *If F is an algebraically closed field then algorithm 2 terminates in a finite number of steps and constructs such set Φ that any tame automorphism may be expressed in $\Phi \cup \widetilde{GL}_n(F)$ in a finite number of steps using algorithm 1.*

Proof. V_s is an n^{s+1}-dimensional linear space, elements of $\Phi \cap \Gamma_s$ induce linearly independent elements of TV_s.

Evidently, the set $\{0, f_1, \ldots, f_d\}$, required on step 3 of algorithm 2 always exists. By Lemma 7, if $\hat{\psi}(r)$ can not be linearly expressed in $\{\hat{\varphi}_s^j\}_{j=1}^{d'_s}$ then one of the vectors $\hat{\psi}(0), \hat{\psi}(f_1), \ldots, \hat{\psi}(f_d)$ also can not be linearly expressed in $\{\hat{\varphi}_s^j\}_{j=1}^{d'_s}$. Thus, algorithm 2 adds exactly one element to Φ' each time step 3 is executed. Hence the algorithm finishes in a finite number of steps.

The obtained set Φ satisfies the following condition: for each $\varphi \in \Phi$ and any elementary matrix $t(r)$ or a diagonal matrix $\gamma_p(r)$ (11) automorphisms $\tilde{t}(r)^{-1}\varphi\tilde{t}(r)$ and $\tilde{\gamma}_p(r)^{-1}\varphi\tilde{\gamma}_p(r)$ can be expressed in Φ using algorithm 1 since it was the condition of the termination.

Let us show that any automorphism $\tilde{\lambda}^{-1}e\tilde{\lambda}$, where $\tilde{\lambda} \in \widetilde{GL}_n(F)$, $e = (x_i + \delta_i^1 m)$, m being some monomial over $x_2, \ldots, x_n$, may be expressed in Φ using algorithm 1. Elements $\varphi_i^j \in \Phi \cap \Gamma_s \setminus \Gamma_{s+1}$ induce some linearly independent subset $\{\hat{\varphi}_s^j\}$ of TV_s. Let us prove that $\tilde{\lambda}^{-1}e\tilde{\lambda}$ can be expressed in this subset modulo R^{s+1}. Any matrix λ may be decomposed in a product of q elementary matrices $t_1, \dots, t_q$ for some q and a diagonal matrix γ. Obviously $e \in \Phi$. We have:

$$\tilde{\lambda}^{-1} e \tilde{\lambda} = \tilde{\gamma}^{-1} \tilde{t}_q^{-1} \dots \tilde{t}_1^{-1} e \tilde{t}_1 \dots \tilde{t}_q \tilde{\gamma}. \quad (12)$$

Using algorithm 1 and the expression of $\tilde{t}_1^{-1} e \tilde{t}_1$ in Φ, as above, we obtain:

$$\tilde{\lambda}^{-1} e \tilde{\lambda} = \prod_{j=1}^{w} \left(\tilde{\gamma}^{-1} \tilde{t}_q^{-1} \dots \tilde{t}_2^{-1} (\alpha_j^{-1} x_i) \varphi_s^j (\alpha_j^{-1} x_i) \tilde{t}_2 \dots \tilde{t}_q \tilde{\gamma} \right) \mod \Gamma_{s+1}.$$

Since the element $(\alpha_j x_i)$ commutes with any element of $\widetilde{GL_n}(F)$,

$$\tilde{\lambda}^{-1} e \tilde{\lambda} = \prod_{j=1}^{w} \left(\tilde{\gamma}'^{-1} \tilde{t}_q^{-1} \dots \tilde{t}_2^{-1} \varphi_s^j \tilde{t}_2 \dots \tilde{t}_q \tilde{\gamma}' \right) \mod \Gamma_{s+1}, \quad (13)$$

where $\tilde{\gamma}' = \tilde{\gamma}(\alpha_j x_i)$. Each multiplier in (13) is similar to $\tilde{\lambda}^{-1} e \tilde{\lambda}$ in (12) only with $q-1$ elementary matrices remaining. By induction we decompose $\tilde{\gamma} \tilde{\lambda}^{-1} e \tilde{\lambda} \tilde{\gamma}^{-1}$ in $\{\hat{\varphi}_s^j\}$ modulo R^{s+1}. Since γ is a product of diagonal matrices of the form $\gamma_p(r)$, we repeat the same reasoning for diagonal matrices. □

The following theorem gives a sufficient condition of an automorphism to be wild. Let A_n be a free associative algebra over a field F, as above, and A'_n be a free associative algebra over the algebraic closure of the field F. Denote by R' an ideal $\langle x_1, x_2, \dots, x_n \rangle$ of A'_n.

Theorem 3. *Let $\varphi \in \operatorname{Aut} A_n$, $\tilde{\varphi}$ be the induced automorphism of R'/R'^k, and let Φ be the set obtained by means of the algorithm 2. If $\tilde{\varphi}$ can not be expressed in Φ using algorithm 1, then φ is a wild automorphism of A_n.*

In the case of algebraically closed fields, the constructed set (9) allows us to describe all wild automorphisms of the free nilpotent algebra R/R^k. The problem of automorphism lifting for some free nilpotent groups and algebras is studied in different papers, see [7], [8], etc.

The following lemma describes all automorphisms of a free nilpotent algebra R/R^3 modulo the tame subgroup $\operatorname{TAut} R/R^3$. We choose endomorphisms of R:

$$\sigma_1 = (x_1 + x_1 x_2, x_2, \dots, x_n), \quad \sigma_2 = \tau(\sigma_1)$$

for the opposite anti-automorphism $\tau : x_i x_j \dots x_l \to x_l \dots x_j x_i$ of the ideal R. Also set $\tau(\varphi_1, \dots, \varphi_n) = (\tau(\varphi_1), \dots, \tau(\varphi_n))$.

Lemma 9. *If $\varphi \in \operatorname{Aut} R$ and $\varphi = \sigma_1$ or σ_2 modulo R^3, then φ is wild. Moreover,*

$$\operatorname{Aut} R = \langle \varphi, \ \tau(\varphi), \ \operatorname{TAut} R \rangle \mod R^3.$$

Proof. Let us consider the five endomorphisms

$$\phi_1 = (x_1 + x_2 x_2, x_2, \dots, x_n), \quad \phi_3 = (x_1 + x_1 x_3, x_2 - x_2 x_3, x_3, \dots, x_n),$$
$$\phi_2 = (x_1 + x_2 x_3, x_2, \dots, x_n), \quad \phi_4 = \tau(\phi_3),$$
$$\phi_5 = (x_1 + x_1 x_3, x_2 + x_3 x_2, x_3 - x_3 x_3, x_4, \dots, x_n).$$

Modulo R^3 we have

$$\phi_3 = (x_1, x_2 + x_1, x_3, \dots, x_n)^{-1}\phi_2(x_1, x_2 + x_1, x_3, \dots, x_n)\cdot$$
$$\cdot (x_1 - x_2x_3, x_2, \dots, x_n)(x_1, x_2 + x_1x_3, x_3, \dots, x_n),$$
$$\phi_5 = (x_3, x_2 + x_3, x_1 + x_3, x_4, \dots, x^n)^{-1}\phi_2(x_3, x_2 + x_3, x_1 + x_3, x_4, \dots, x_n)\cdot$$
$$\cdot (x_1 + x_2x_3 + x_3x_3, x_2, \dots, x_n)(x_1, x_2 + x_3x_1 + x_3x_3, x_3, \dots, x_n)\cdot$$
$$\cdot (x_1, x_2, x_3 - x_2x_1, x_4, \dots, x_n)(x_1, x_3, x_2, x_4, \dots, x_n)\phi_4(x_1, x_3, x_2, x_4, \dots, x_n)\cdot$$
$$\cdot (x_3, x_1, x_2, x_4, \dots, x_n)\phi_3(x_2, x_3, x_1, x_4, \dots, x_n).$$

It is obvious that ϕ_1 and ϕ_2 are the initial elements of Φ in algorithm 2, and the remaining elements are constructed by conjugating those elements with elementary matrices, i.e., using the transformations from algorithms 1 and 2. The substitutions over $x_1, \dots, x_n$ may also be considered as the products of elementary matrices. The whole set Φ linearly generating TV_2 would consist of elements up to a substitution equal to ϕ_i, $i = \overline{1,5}$.

One may notice that any element $\xi_0^{-1}\phi_i\xi_0$ in the set $\{\xi^{-1}\phi_i\xi | \xi \in S_n\}$ is uniquely defined by not more than 3 indices k, l, m. Hence to check that Φ and $GL_n(F)$ generate TAut R/R^3 we don't have to consider all cases of the position (i, j) of r in the elementary matrix $t(r)$. We may let the indices be $1, 2, 3$ (since there exists a transformation mapping k, l, m in $1, 2, 3$) and consider 4×4 elementary matrices. Hence we can in a finite number of steps prove that for each n such set Φ really satisfies the condition of algorithm 2 termination.

The dimension of V_2 is, obviously, n^3. Let τ_i be an automorphism, exchanging x_1 and x_i. Then it is easy to verify that $|\Phi| = n^3 - 2n$ and $2n$ corresponds to the remaining wild automorphisms $\tau_i\sigma_1\tau_i$ and $\tau_i\sigma_2\tau_i$. □

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