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Generalized Convolutions for the Fourier Integral Transforms and Applications

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This paper provides some generalized convolutions for the Fourier integral transforms and treats the applications. Namely, there are six generalized convolutions with weight-function for the Fourier integral transforms. As for applications, the normed ring structures on $L^1(\mathbb{R}^d)$ are constructed, and the explicit solution in $L^1(\mathbb{R}^d)$ of the integral equations with the mixed Toeplitz-Hankel kernel are obtained.

Keywords: generalized convolution, normed ring, integral equation of convolution type.

Introduction

The Fourier transform and its inverse transform are defined as:

$$(Ff)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} f(y) dy, \quad (F^{-1}f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle x, y \rangle} f(y) dy,$$

where $\langle x, y \rangle$ denotes the scalar product of $x, y \in \mathbb{R}^d$, and $f(x)$ is a function (real or complex) defined on \mathbb{R}^d . The integral transform

$$(f \underset{F}{*} g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x-y)g(y) dy$$

is called the Fourier convolution of the two functions f and g , and it is applied in many fields of mathematics.

In 1940, Churchill gave an idea of the generalized convolutions of integral transforms, and found an application for solving boundary value problems (see Churchill [1]). In 1958, Vilenkin formulated the convolution of the integral transforms in the specific space of integrable functions (see Vilenkin [2]). In 1967, the designated methods for convolutions and generalized convolutions of the integral transforms were proposed by Kakichev, and in 1990 a concept of generalized convolutions of the linear operators was first introduced (see [3, 4]). In 1998, the generalized convolution of the Fourier-cosine and Fourier-sine transforms was presented (see [5]).

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In recent years, many papers were devoted to the well-known integral transforms for given convolutions, generalized convolutions, polyconvolutions and their applications (see [6]–[12]). However, there are not so many generalized convolutions for the integral transforms. Generally speaking, each of convolutions is a new transform which can become an object of study. In our view, the generalized convolutions and their applications deserve the interest that they have attracted.

The main purpose of this paper is to construct some generalized convolutions with weight for the Fourier integral transforms.

The paper is divided into three sections and organized as follows.

In Section 1, there are six generalized convolutions with the weight being one of the functions $e^{-i\langle x,h \rangle}$, $e^{i\langle x,h \rangle}$, $\cos xh$, $\sin xh$ for the Fourier integral transforms. We call h the shift in the convolution transform. One interesting fact possessed by the factorization identities of those convolutions is that the shift in the convolution expression is only moved into the weight-function in the right-hand-side. We think this is the main reason for the solvability of the integral equations with different shifts as equation (2.2).

In Subsection 2.1, the linear space $L^1(\mathbb{R}^d)$, equipped with each of the convolution multiplications, becomes the normed ring. In Subsection 2.2, using the convolutions in Section 1 we investigate the integral equations with the mixed Toeplitz-Hankel kernel having shifts and obtain explicit solutions in $L^1(\mathbb{R}^d)$ of those equations.

1. Generalized Convolutions

The concept of generalized convolutions with weight is a nice idea focusing on the so-called factorization identity. We now deal with this concept.

Let U_1, U_2, U_3 be linear spaces over the field of scalars \mathcal{K} , and let V be a commutative algebra over \mathcal{K} . Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are linear operators from U_1, U_2, U_3 to V respectively. Let θ denote an element in algebra V . The following definition is a formulation of the idea of convolutions and generalized convolutions (see [5]).

Definition 1.1. A bilinear map $* : U_1 \times U_2 \rightarrow U_3$ is called convolution with weight-element θ for K_3, K_1, K_2 (that in order) if $K_3(* (f, g)) = \theta K_1(f) K_2(g)$ for any $f \in U_1, g \in U_2$. We call $K_3(* (f, g)) = \theta K_1(f) K_2(g)$ the factorization identity of the convolution.

The image $* (f, g)$ is denoted by $f \underset{K_3, K_1, K_2}{*}^\theta g$. If θ is the unit of V , we say briefly the convolution for K_3, K_1, K_2 . If $U_1 = U_2 = U_3$ and $K_1 = K_2 = K_3$, the convolution is denoted simply $f \underset{K_1}{*}^\theta g$, and $f \underset{K_1}{*} g$ if θ is the unit of V . We think the factorization identity plays the key role in the convolution.

In what follows, we write $\check{F} = F^{-1}$. For any $h \in \mathbb{R}^d$, put $\alpha(x) = e^{-i\langle x,h \rangle}$, $\beta(x) = e^{i\langle x,h \rangle}$, $\gamma(x) = \cos xh$, $\delta(x) = \sin xh$. Note that $(Ff)(x) = (\check{F}\check{f})(x) = (F\check{f})(-x)$.

Theorem 1.1 presents some generalized convolutions with weight-function for the Fourier integral transforms F, \check{F} .

Theorem 1.1. *If $f, g \in L^1(\mathbb{R}^d)$, then each of the integral transforms (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) below defines the generalized convolution followed by its factorization identity:*

$$(f \underset{F}{*}^\alpha g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x - y - h)g(y)dy, \tag{1.1}$$

$$\begin{aligned}
 F(f \underset{F}{*}^{\alpha} g)(x) &= \alpha(x)(Ff)(x)(Fg)(x). \\
 (f \underset{F}{*}^{\beta} g)(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x-y-h)g(y)dy,
 \end{aligned} \tag{1.2}$$

$$\begin{aligned}
 \check{F}(f \underset{F}{*}^{\beta} g)(x) &= \beta(x)(\check{F}f)(x)(\check{F}g)(x). \\
 (f \underset{F, F, \check{F}}{*}^{\alpha} g)(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x+y-h)g(y)dy,
 \end{aligned} \tag{1.3}$$

$$\begin{aligned}
 F(f \underset{F, F, \check{F}}{*}^{\alpha} g)(x) &= \alpha(x)(Ff)(x)(\check{F}g)(x). \\
 (f \underset{\check{F}, \check{F}, F}{*}^{\beta} g)(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x+y-h)g(y)dy,
 \end{aligned} \tag{1.4}$$

$$\begin{aligned}
 \check{F}(f \underset{\check{F}, \check{F}, F}{*}^{\beta} g)(x) &= \beta(x)(\check{F}f)(x)(Fg)(x). \\
 (f \underset{F}{*}^{\gamma} g)(x) &= \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [f(x-y-h) + f(x-y+h)]g(y)dy,
 \end{aligned} \tag{1.5}$$

$$\begin{aligned}
 F(f \underset{F}{*}^{\gamma} g)(x) &= \gamma(x)(Ff)(x)(Fg)(x). \\
 (f \underset{F}{*}^{\delta} g)(x) &= \frac{i}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [f(x-y-h) - f(x-y+h)]g(y)dy,
 \end{aligned} \tag{1.6}$$

$$F(f \underset{F}{*}^{\delta} g)(x) = \delta(x)(Ff)(x)(Fg)(x).$$

Proof of convolution (1.1). Obviously, the integral transform (1.1) is Fourier convolution taken at point $x - h$. Since, $f \underset{F}{*}^{\alpha} g \in L^1(\mathbb{R}^d)$. We now prove the factorization identity. We have

$$\begin{aligned}
 \alpha(x)(Ff)(x)(Fg)(x) &= \frac{e^{-i\langle x, h \rangle}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x, u \rangle} e^{-i\langle x, v \rangle} f(u)g(v)dudv = \\
 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x, u+v+h \rangle} f(u)g(v)dudv = \\
 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x, t \rangle} f(t-y-h)g(y)dtdy = F(f \underset{F}{*}^{\alpha} g)(x).
 \end{aligned}$$

It is easy to see that the convolutions (1.2), (1.3), (1.4) are the immediate consequences of the convolution (1.1).

Proof of convolution (1.5). We prove $f \underset{F}{*}^{\gamma} g \in L^1(\mathbb{R}^d)$. We have

$$\begin{aligned}
 \int_{\mathbb{R}^d} (f \underset{F}{*}^{\gamma} g)(x)dx &\leq \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(u)| [|f(x-u-h)| + |f(x-u+h)|] dudx = \\
 &= \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)| du \left[\int_{\mathbb{R}^d} |f(x-u-h)| dx + \int_{\mathbb{R}^d} |f(x-u+h)| dx \right] = \\
 &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)| du \int_{\mathbb{R}^d} |f(x)| dx < +\infty.
 \end{aligned}$$

We prove the factorization identity. We have

$$\begin{aligned} \gamma(x)(Ff)(x)(Fg)(x) &= \frac{\cos xh}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x,u \rangle} e^{-i\langle x,v \rangle} f(u)g(v) dudv = \\ &= \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [e^{-i\langle x,u+v-h \rangle} + e^{-i\langle x,u+v+h \rangle}] f(u)g(v) dudv = \\ &= \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x,t \rangle} [f(t-y-h) + f(t-y+h)] g(y) dt dy = F(f \underset{F}{\ast} g)(x). \end{aligned}$$

Proof of convolution (1.6). The proof of $f \underset{F}{\ast} g \in L^1(\mathbb{R}^d)$ is similar to that of convolution (1.5). So, we prove the factorization identity. We have

$$\begin{aligned} \delta(x)(Ff)(x)(Fg)(x) &= \frac{\sin xh}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x,u \rangle} e^{-i\langle x,v \rangle} f(u)g(v) dudv = \\ &= \frac{i}{2(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [e^{-i\langle x,u+v+h \rangle} - e^{-i\langle x,u+v-h \rangle}] f(u)g(v) dudv = \\ &= \frac{i}{2(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\langle x,t \rangle} [f(t-y-h) - f(t-y+h)] g(y) dt dy = \\ &= F(f \underset{F}{\ast} g)(x). \end{aligned}$$

□

Example. Let $d = 1$. Put $u(x) := 1/\pi x$. It is well-known that the Hilbert transform of a function (or signal) $v(x)$ is given by

$$(Hv)(x) = \text{p.v.} \int_{-\infty}^{+\infty} u(x-y)v(y) dy,$$

provided that this integral exists as Cauchy's principal value. This is precisely the convolution of v with the tempered distribution p.v. $u(x)$.

Now we put $r(x) := \frac{x}{\sqrt{2\pi}(x^2 - h^2)}$, $s(x) := \frac{h}{\sqrt{2\pi}(x^2 - h^2)}$. By the convolution transforms (1.5) and (1.6) we have the informal identities:

$$\begin{aligned} \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{1}{x-y-h} + \frac{1}{x-y+h} \right] g(y) dy &= \text{p.v.} \int_{-\infty}^{+\infty} r(x-y)g(y) dy, \\ \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\frac{1}{x-y-h} - \frac{1}{x-y+h} \right] g(y) dy &= \text{p.v.} \int_{-\infty}^{+\infty} s(x-y)g(y) dy. \end{aligned}$$

2. Application

2.1. Normed Ring Structures on $L^1(\mathbb{R}^d)$

Definition 2.1 (see Naimark [13]). A vector space V with a ring structure and a vector norm is called the normed ring if $\|vw\| \leq \|v\|\|w\|$, for all $v, w \in V$.

If V has a multiplicative unit element e , it is also required that $\|e\| = 1$.

Let X denote the linear space $L^1(\mathbb{R}^d)$. Now we define norms for $f \in X$. For each of all convolutions in Section 1, the norm is chosen as

$$\|f\| = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)| dx.$$

Theorem 2.1. *The space X , equipped with each of the convolution multiplications, becomes a normed ring with no unit.*

Proof. The proof is divided into two steps.

Step 1. X has a normed ring structure. It is clear that X , equipped with each of the convolution multiplications listed above, has the ring structure. We have to prove the multiplicative inequality. We now prove that for convolution (1.5), the proof that for the others is similar. We have

$$\begin{aligned} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f \underset{F}{*}^{\gamma} g(x) dx &\leq \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-u+h)| |g(u)| dx du + \\ &+ \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-u-h)| |g(u)| dx du = \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int_{\mathbb{R}^d} |f(x+u-h)| dx \right) \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)| du \right) = \|f\| \|g\|. \end{aligned}$$

Hence $\|f \underset{F}{*}^{\gamma} g\| \leq \|f\| \cdot \|g\|$.

Step 2. The space X has no unit. For briefness of our proof, let us use the common symbols: $*$ for the convolutions and γ_0 for the weight function of $\alpha, \beta, \gamma, \delta$. Suppose that there exists an $e \in X$ such that $f = f * e = e * f$ for every $f \in X$. Choose $\delta(x) := e^{-\frac{1}{2}|x|^2} \in L^1(\mathbb{R}^d)$. We then have $(F\delta)(x) = (\check{F}\delta)(x) = \delta(x)$ (see [14, Theorem 7.6]). By $\delta = \delta * e = e * \delta$ and the factorization identities of the convolutions, we have

$$\mathcal{F}_j(\delta) = \gamma_0 \mathcal{F}_k(\delta) \mathcal{F}_\ell(e), \tag{2.1}$$

where $\mathcal{F}_j, \mathcal{F}_k, \mathcal{F}_\ell \in \{F, \check{F}\}$ (note that it may be $\mathcal{F}_j = \mathcal{F}_k = \mathcal{F}_\ell = F$, etc.).

By (2.1) we have $\delta = \gamma_0 \delta \mathcal{F}_\ell(e)$. Due to $\delta(x) \neq 0$ for every $x \in \mathbb{R}^d$, $\gamma_0(x) (\mathcal{F}_\ell e)(x) = 1$ for every $x \in \mathbb{R}^d$. Since $|\gamma_0(x)| \leq 1$, the last identity contradicts the Riemann-Lebesgue lemma: $\lim_{x \rightarrow \infty} (\mathcal{F}_\ell e)(x) = 0$ (see Rudin [14, Theorem 7.5]). Hence, X has no unit. \square

2.2. Integral Equations of the Convolution Type

Consider the integral equation with the mixed Toeplitz-Hankel kernel having shifts

$$\lambda \varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [k_1(x+y-h_1) + k_2(x-y-h_2)] \varphi(y) dy = p(x), \tag{2.2}$$

where $\lambda \in \mathbb{C}$ is predetermined, k_1, k_2, p are given, the shifts $h_1, h_2 \in \mathbb{R}^d$, and $\varphi(x)$ is to be determined.

Since the convolutions in Section 1 are considered in $L^1(\mathbb{R}^d)$, given functions are assumed to be elements of $L^1(\mathbb{R}^d)$, and unknown function will be determined there. In what follows, the function identity $f(x) = g(x)$ means that it is valid for almost every $x \in \mathbb{R}^d$. However, if both functions f, g are continuous, the identity $f(x) = g(x)$ must be valid for every $x \in \mathbb{R}^d$.

For any $f \in L^1(\mathbb{R}^d)$, we write $\check{f}(x) := f(-x)$. Put

$$\begin{aligned} \gamma_1(x) &= e^{-i\langle x, h_1 \rangle}, \quad \gamma_2(x) = e^{-i\langle x, h_2 \rangle}, \\ A(x) &:= \lambda + \gamma_2(x)(Fk_2)(x), \quad B(x) := \gamma_1(x)(Fk_1)(x), \end{aligned} \tag{2.3}$$

$$\begin{aligned} D_{F, \check{F}}(x) &:= \lambda^2 + \lambda[\gamma_2(x)(Fk_2)(x) + \check{\gamma}_2(x)(\check{F}k_2)(x)] + \\ &\quad + \check{\gamma}_2(x)F[k_2 \underset{F, F, \check{F}}{\overset{\check{\gamma}_2}{*}} k_2](x) - \check{\gamma}_1(x)F[k_1 \underset{F, F, \check{F}}{\overset{\check{\gamma}_1}{*}} k_1](x), \end{aligned} \tag{2.4}$$

$$D_F(x) := \lambda(Fp)(x) + F[(p \underset{F, F, \check{F}}{\overset{\check{\gamma}_2}{*}} k_2) - (k_1 \underset{F, F, \check{F}}{\overset{\check{\gamma}_1}{*}} p)](x), \tag{2.5}$$

$$D_{\check{F}}(x) := \lambda(\check{F}p)(x) + F[(k_2 \underset{F, F, \check{F}}{\overset{\check{\gamma}_2}{*}} p) - (p \underset{F, F, \check{F}}{\overset{\check{\gamma}_1}{*}} k_1)](x). \tag{2.6}$$

Actually,

$$\begin{aligned} D_{F, \check{F}}(x) &:= \lambda^2 + \lambda[\gamma_2(x)(Fk_2)(x) + \gamma_2(-x)(Fk_2)(-x)] + \\ &\quad + (Fk_2)(x)(Fk_2)(-x) - (Fk_1)(x)(Fk_1)(-x), \end{aligned} \tag{2.7}$$

$$\begin{aligned} D_F(x) &:= \lambda(Fp)(x) + \gamma_2(-x)(Fk_2)(-x)(Fp)(x) - \\ &\quad - \gamma_1(x)(Fk_1)(x)(Fp)(-x), \end{aligned} \tag{2.8}$$

$$\begin{aligned} D_{\check{F}}(x) &:= \lambda(Fp)(-x) + \gamma_2(x)(Fk_2)(x)(Fp)(-x) - \\ &\quad - \gamma_1(-x)(Fk_1)(-x)(Fp)(x). \end{aligned} \tag{2.9}$$

Theorem 2.2. *Assume that the following conditions are fulfilled: $D_{F, \check{F}}(x) \neq 0$ for every $x \in \mathbb{R}^d$, and $\frac{D_F}{D_{F, \check{F}}} \in L^1(\mathbb{R}^d)$. Then the equation (2.2) has a solution in $L^1(\mathbb{R}^d)$ if and only if*

$$F^{-1} \left(\frac{D_F}{D_{F, \check{F}}} \right) \in L^1(\mathbb{R}^d). \tag{2.10}$$

If (2.10) is satisfied, then its solution can be obtained in the explicit form

$$\varphi(x) = F^{-1} \left(\frac{D_F}{D_{F, \check{F}}} \right) (x).$$

Proof. Note that the shift h in the convolutions (1.1), (1.2), (1.3), (1.4) is separate. Thus from those convolutions it follows that

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x + y - h_1)g(y)dy = (f \underset{F, F, \check{F}}{\overset{\check{\gamma}_1}{*}} g)(x) = (f \underset{\check{F}, \check{F}, F}{\overset{\check{\gamma}_1}{*}} g)(x),$$

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x - y - h_2)g(y)dy = (f \underset{F}{\overset{\check{\gamma}_2}{*}} g)(x) = (f \underset{\check{F}}{\overset{\check{\gamma}_2}{*}} g)(x).$$

By the factorization identities of those convolutions we get

$$F \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x+y-h_1)g(y)dy \right) = \gamma_1(x)Ff(x)\check{F}g(x), \tag{2.11}$$

$$F \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x-y-h_2)g(y)dy \right) = \gamma_2(x)Ff(x)Fg(x), \tag{2.12}$$

$$\check{F} \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x+y-h_1)g(y)dy \right) = \check{\gamma}_1(x)\check{F}f(x)Fg(x), \tag{2.13}$$

and

$$\check{F} \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x-y-h_2)g(y)dy \right) = \check{\gamma}_2(x)\check{F}f(x)\check{F}g(x), \tag{2.14}$$

for any $f, g \in L^1(\mathbb{R}^d)$.

Necessity. Suppose that the equation (2.2) has a solution $\varphi \in L^1(\mathbb{R}^d)$, i.e.,

$$\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [k_1(x+y-h_1) + k_2(x-y-h_2)]\varphi(y)dy = p(x).$$

Applying each of the transforms F and \check{F} in turn to both sides of this identity and using (2.11), (2.12), (2.13), (2.14), we obtain the system of two linear equations

$$\begin{cases} A(x)(F\varphi)(x) + B(x)(\check{F}\varphi)(x) = (Fp)(x), \\ B(-x)(F\varphi)(x) + A(-x)(\check{F}\varphi)(x) = (\check{F}p)(x), \end{cases} \tag{2.15}$$

where $A(x), B(x)$ are defined by (2.3), and $(F\varphi)(x), (\check{F}\varphi)(x)$ are unknown functions. The determinants of (2.15) denoted by $D_{F,\check{F}}(x), D_F(x), D_{\check{F}}(x)$ are defined by (2.4), (2.5), (2.6) respectively. Due to $D_{F,\check{F}}(x) \neq 0$ for every $x \in \mathbb{R}^d$ we get

$$F\varphi(x) = \frac{D_F(x)}{D_{F,\check{F}}(x)}, \quad \text{and} \quad \check{F}\varphi(x) = \frac{D_{\check{F}}(x)}{D_{F,\check{F}}(x)}.$$

As $\frac{D_F(x)}{D_{F,\check{F}}(x)} \in L^1(\mathbb{R}^d)$, we can apply the inversion theorem of the Fourier transform (see [14,

Theorem 7.7]) to obtain $\varphi(x) = F^{-1} \left(\frac{D_F}{D_{F,\check{F}}} \right) (x)$. The necessity is proved.

Sufficiency. From (2.7), (2.8), (2.9) it follows that $D_{F,\check{F}}(x) \equiv D_{F,\check{F}}(-x)$, and $D_F(x) \equiv D_{\check{F}}(-x)$. It is easy to see that

$$F^{-1} \left(\frac{D_F}{D_{F,\check{F}}} \right) (x) \equiv F \left(\frac{D_{\check{F}}}{D_{F,\check{F}}} \right) (x).$$

Consider the function $\varphi(x) = F^{-1} \left(\frac{D_F}{D_{F,\check{F}}} \right) (x) = F \left(\frac{D_{\check{F}}}{D_{F,\check{F}}} \right) (x)$. This implies that $\varphi \in L^1(\mathbb{R}^d)$. We apply the inversion theorem of the Fourier integral transform to get $(F\varphi)(x) =$

$\frac{D_F(x)}{D_{F,\check{F}}(x)}$, and $(\check{F}\varphi)(x) = \frac{D_{\check{F}}(x)}{D_{F,\check{F}}(x)}$. We can see that the two functions $(F\varphi)(x)$ and $(\check{F}\varphi)(x)$ satisfy (2.15). Thus

$$A(x)(F\varphi)(x) + B(x)(\check{F}\varphi)(x) = (Fp)(x).$$

Using the factorization identities of convolutions (1.1), (1.3) we get

$$F\left[\lambda\varphi + (k_1 \underset{F,F,\check{F}}{*}^{\gamma_1} \varphi) + (k_2 \underset{F}{*}^{\gamma_2} \varphi)\right](x) = (Fp)(x).$$

Equivalently,

$$F\left[\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [k_1(x+y-h_1) + k_2(x-y-h_2)]\varphi(y)dy\right] = (Fp)(x).$$

By the inversion theorem of the Fourier transform, we conclude that $\varphi(x)$ satisfies the equation (2.2) for almost every $x \in \mathbb{R}^d$. □

Let \mathcal{S} denote the space of rapidly decreasing functions on \mathbb{R}^d (see [14]).

Proposition 2.1. *Let $\lambda \neq 0$. Then*

- (a) $D_{F,\check{F}} \neq 0$ for every x outside a ball with a finite radius.
- (b) Assume that $k_1, k_2 \in \mathcal{S}$ and $p \in L^1(\mathbb{R}^d)$. Then $\frac{D_F}{D_{F,\check{F}}} \in L^1(\mathbb{R}^d)$, provided $D_{F,\check{F}} \neq 0$ for every $x \in \mathbb{R}^d$ and $Fp \in L^1(\mathbb{R}^d)$.

Proof. (a) Combining the facts that two functions γ_1, γ_2 are continuous and bounded on \mathbb{R}^d and the Riemann-Lebesgue lemma for the Fourier integral transform, we conclude that the function $D_{F,\check{F}}(x)$ is continuous on \mathbb{R}^d and $\lim_{|x| \rightarrow \infty} D_{F,\check{F}}(x) = \lambda$. (see [14, Theorem 7.5]). Now (a) follows from the fact that $\lambda \neq 0$ and the continuity of $D_{F,\check{F}}(x)$.

(b) By the continuity of $D_{F,\check{F}}(x)$ and $\lim_{|x| \rightarrow \infty} D_{F,\check{F}}(x) = \lambda \neq 0$, there exist $R > 0, \epsilon_1 > 0$ so that $\inf_{|x| > R} |D_{F,\check{F}}(x)| > \epsilon_1$. Since $D_{F,\check{F}}(x)$ is continuous and does not vanish in the compact set $S(0, R) = \{x \in \mathbb{R}^d : |x| \leq R\}$, there exists $\epsilon_2 > 0$ so that $\inf_{|x| \leq R} |D_{F,\check{F}}(x)| > \epsilon_2$. We then have

$\sup_{x \in \mathbb{R}^d} \frac{1}{|D_{F,\check{F}}(x)|} \leq \max\left\{\frac{1}{\epsilon_1}, \frac{1}{\epsilon_2}\right\} < \infty$. Hence the function $\frac{1}{|D_{F,\check{F}}(x)|}$ is continuous and bounded on \mathbb{R}^d . Therefore, $\frac{D_F}{D_{F,\check{F}}} \in L^1(\mathbb{R}^d)$, provided $D_F \in L^1(\mathbb{R}^d)$. We now prove that if $Fp \in L^1(\mathbb{R}^d)$,

then $D_F \in L^1(\mathbb{R}^d)$. Indeed, we have $\gamma, Fk_1, Fk_2 \in \mathcal{S}$ (see [14, Theorem 7.7]). It follows that the functions $\gamma(x), (Fk_1)(x), (Fk_2)(x)$ are continuous and bounded on \mathbb{R}^d . Now we can conclude that if Fp belongs to $L^1(\mathbb{R}^d)$, then so does each of the three terms in the right side of (2.8). Therefore, $D_F \in L^1(\mathbb{R}^d)$, provided that $Fp \in L^1(\mathbb{R}^d)$. □

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