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On the Fredholm property for the steady Navier-Stokes equations in weighted Hölder spaces

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We prove that the steady Navier-Stokes equations induce a Fredholm non-linear map on the scale of Hölder spaces weighted at the infinity.

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The theory of *nonlinear Fredholm operators* by S. Smale [3] provides an approach to obtain generic results on the uniqueness and/or existence for nonlinear equations in Banach spaces. We recall that a bounded linear operator $\mathcal{L}$ in Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is called Fredholm if its *kernel* and *cokernel* are finite-dimensional and its *range* is closed. Then a nonlinear operator $\mathcal{N}$ is Fredholm if at every point $x \in \mathcal{X}$ its derivative (i.e. the principal linear part) $\mathcal{N}'_x$ possesses the Fredholm property. The most advanced results were obtained for the so called *proper* operators (a map is proper if the inverse image of a compact set is compact). For instance, using results on proper Fredholm maps from [3], J.C. Saut and R. Temam proved the generic uniqueness theorem for the steady version of the Navier-Stokes equations on the scale of the Sobolev spaces, see [4]. Recently, A. Shlapunov and N. Tarkhanov [2] proved that the evolution Navier-Stokes equations induce a Fredholm open injective map on the scale of the Hölder spaces over the strip $\mathbb{R}^n \times [0,T]$, $T > 0$, $n \geqslant 2$, weighted at the infinity with respect to the space variables. In the present short note we prove that the steady Navier-Stokes type equations induce a Fredholm map on the scale of the Hölder spaces over $\mathbb{R}^n$, $n \geqslant 3$, weighted at the infinity.

Namely, let $\mathbb{Z}_{\geqslant 0}$ be the set of all natural numbers including zero, and let $\mathbb{R}^n$ be the Euclidean space of dimension $n \geqslant 3$ with coordinates $x = (x^1, \dots, x^n)$. Following [2], we denote by $C^{s,0}_\delta$ the space of all s times continuously differentiable functions on $\mathbb{R}^n$ with finite norm

$$\|u\|_{C^{s,0}_\delta} = \sum_{|\alpha| \leqslant s} \sup_{x \in \mathbb{R}^n} (1+|x|)^{(\delta+|\alpha|)/2} |\partial^\alpha u(x)|.$$

For $0 < \lambda < 1$, we introduce

$$\langle u \rangle_{\lambda,\delta} = \sup_{\substack{x \neq y \\ |x-y| \leqslant |x|/2}} \Big(\max\,(1+|x|^2, 1+|y|^2) \Big)^{(\delta+\lambda)/2} \frac{|u(x)-u(y)|}{|x-y|^\lambda}.$$

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We define $C^{s,\lambda}_\delta$ to consist of all s times continuously differentiable functions on $\mathbb{R}^n$, such that

$$\|u\|_{C^{s,\lambda}_\delta} = \|u\|_{C^{s,0}_\delta} + \sum_{|\alpha|\leqslant s} \langle \partial^\alpha u\rangle_{\lambda,\delta+|\alpha|} + \|u\|_{C^{s,\lambda}(\overline{B}_1)} < \infty$$

where $C^{s,\lambda}(\overline{B}_1)$ is the space of Hölder functions in the unit closed ball $\overline{B}_1$ centered at the origin. These are Banach spaces for all $s \in \mathbb{Z}_{\geqslant 0}$ and all $0 \leqslant \lambda < 1$.

Denote by Λ^q the bundle of the differential forms of degree q and let $C^{s,\lambda}_{\delta,\Lambda^q}$ stand for the spaces of the differential forms of degree q with the coefficients in $C^{s,\lambda}_\delta$.

Problem 1 *Let $s \geqslant 2$ and $\delta > 0$. Given form $f \in C^{s-2,\lambda}_{\delta+2,\Lambda^1}$ find a form $u \in C^{s,\lambda}_{\delta,\Lambda^1}$ and a function $p \in C^{s-1,\lambda}_{\delta+1}$ satisfying*

$$\begin{cases} -\mu\Delta u + \mathbb{D}u + d_0 p &= f, \\ d_0^* v &= 0 \end{cases}$$

where μ is a positive real number, Δ is the Laplace operator, d_q is the de Rham differential on q-forms, d_q^* is its formal adjoint and $\mathbb{D}u = \sum_{j=1}^n u_j \partial_j u$.

For $n = 3$ the de Rham differentials give: $d_0 = \nabla$, $d_1 = \mathrm{rot}$, $d_2 = \mathrm{div}$ where ∇ is the gradient operator, rot is the rotation operator and div is the divergence operator in $\mathbb{R}^n$. Hence Problem 1 is precisely the steady Navier-Stokes equations on the scale $C^{s,\lambda}_\delta$ if $n = 3$.

Now, integration by parts yields that, for $\delta > n-2$, we have

$$(f, 1)_{L^2(\mathbb{R}^n)} = 0, \tag{1}$$

for any form $f \in C^{s-2,\lambda}_{\delta+2,\Lambda^1}$ admitting a solution (u,p) to Problem 1. Clearly, the weighted space $C^{s,\lambda}_\delta$ with $\delta > 0$ corresponds to the one point compactification of $\mathbb{R}^n$ and then Problem 1 is similar to the steady Navier-Stokes equations in the periodic case (or, the same, on a torus) and (1) is similar to [4, condition (2.3)]. We will always assume that (1) is fulfilled if $\delta > n-2$.

On the next step, using Hodge theory for the de Rham complex over weighted spaces (see [2]), we reduce the Navier-Stokes equation for the velocity u to the equation with respect to vorticity $d_1 u$ (cf. [1] on the scale of Sobolev spaces). With this purpose, we note that *ker d* will stand for solutions of the equation $du = 0$ in $\mathbb{R}^n$ in the sense of distributions. We denote by $R^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}}(d)$ the range of the operator $d : C^{s,\lambda}_{\delta,\Lambda^q} \to C^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}}$. According to [2, Corollary 3.11], the space $R^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}}(d)$ coincides with $C^{s-1,\lambda}_{\delta+1} \cap \ker d_{q+1}$ if $\delta \in (0, n-1)$; it consists of elements $f \in C^{s-1,\lambda}_{\delta+1} \cap \ker d_{q+1}$ satisfying

$$(f, d_q h_j)_{L^2(\mathbb{R}^n)} = 0, \tag{2}$$

for all harmonic homogeneous polynomials h_j of degree j in $\mathbb{R}^n$ with $0 \leqslant j \leqslant m+1$ if $\delta \in (n-1+m, n+m)$, $m \in \mathbb{Z}_{\geqslant 0}$.

Now, let $\wedge$ be the exterior product on the differential forms and let $\star$ be the Hodge star operator on the differential forms, induced by identity $dx_I \wedge \star dx_I = dx$ for each differential $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$ with $|I| = q$. Now for q-form $F(x) = \sum_{|I|=q} F_I(x) dx_I$ we set

$$(\varphi F)(x) = \int_{\mathbb{R}^n} \sum_{|I|=q} F_I(y)\varphi_n(x-y)dy, \;\; \Phi v(x) = \int_{\mathbb{R}^n} F(y) \wedge (d_{n-q-1})^*_y \Big(\sum_{|I|=q} \varphi_n(x-y) \star dy_I \Big),$$

where $\varphi_n(x-y)$ is the standard fundamental solution of the Laplace operator in $\mathbb{R}^n$. The behaviour of these potentials on the weighted Hölder spaces were investigated in [2, §3]. Now, for 1-for g we set

$$\mathbb{G}g(x) = \star(\star g \wedge \Phi g).$$

Problem 2 *Let $s \geqslant 3$ and $\delta > 0$. Given form $f \in C^{s-2,\lambda}_{\delta+2,\Lambda^1}$ (satisfying (1) if $\delta > n-2$) find a form $g \in R^{s-1,\lambda}_{\delta+1,\Lambda^2}$ satisfying*

$$g - (1/\mu)\varphi\mathbb{G}g = \varphi d_1 f.$$

Now we may achieve the main theorems of our note.

Theorem 1. *Let $n \geqslant 3$, $s \in \mathbb{N}$, $s \geqslant 3$ and $1 < \delta < n/2$ with $2\delta - n + 3 \notin \mathbb{Z}_{\geqslant 0}$. Then Problems 1 and 2 are equivalent.*

Proof. Indeed, we may write the nonlinear term $\mathbb{D}$ in the Lamb form (see, for instance, [1] or [2, Lemma 1.2]:

$$\mathbb{D}u = d_0|u|^2 + \star(\star d_1 u \wedge u).$$

According to [2, Corollary 3.11], the operator Φ maps $R^{s,\lambda}_{\delta+1,\Lambda^2}$ continuously to $C^{s,\lambda}_{\delta,\Lambda^1}$ for the chosen δ. Then, as $u = \Phi g$ is the unique form from $C^{s,\lambda}_{\delta,\Lambda^1}$ satisfying $d_1\Phi g = g$, $d_0^*\Phi g = 0$ in $\mathbb{R}^n$, see [2], we conclude that $d\mathbb{D} = \mathbb{G}d$. As $d^2 = 0$, this proves that Problem 1 is equivalent to the following equation

$$-\mu\Delta g + \mathbb{G}g = d_1 f \tag{3}$$

on the discussed scales of spaces. Besides, the potential φ induces bounded linear map from the range $R^{s-2,\lambda}_{\delta+2}(\Delta)$ of the bounded operator $\Delta : C^{s,\lambda}_{\delta} \to C^{s-2,\lambda}_{\delta+2}$ to $C^{s,\lambda}_{\delta}$ where $R^{s-2,\lambda}_{\delta+2}(\Delta)$ coincides with $C^{s-2,\lambda}_{\delta+2}$ if $0 < \delta < n-2$; it consists of all the elements $F \in C^{s-2,\lambda}_{\delta+2}$ satisfying

$$(F, h_j)_{L^2(\mathbb{R}^n)} = 0, \tag{4}$$

for all harmonic homogeneous polynomials h_j of degree j in $\mathbb{R}^n$ with $0 \leqslant j \leqslant m$ if $\delta \in (n-2+m, n+m-1)$, $m \in \mathbb{Z}_{\geqslant 0}$ (see, for instance, [2, Theorem 3.1].

Since $\delta > 1$ the potential $\varphi d_1 f$ is a convergent integral. Moreover, as f satisfies (1) if $\delta > n-2$, then using integration by parts we see that $d_1 f$ satisfies (4) with $j = 0$ and $j = 1$. Hence if $\delta \neq n-2$, $\delta \neq n-1$, $\delta \neq n$, $1 < \delta < n+1$, the form $\varphi d_1 f$ belongs to $C^{s-1,\lambda}_{\delta+1}$. However, as $1 < \delta < n/2$ and $n \geqslant 3$ all these conditions are fulfilled.

Similarly, integration by parts yields $\mathbb{D}u$ satisfies (1) if $2\delta + 1 > n - 2$. On the other hand, according to [2, Lemma 2.9] on the multiplication of the weighted functions, we see that $\mathbb{D}$ maps $C^{s,\lambda}_{\delta,\Lambda^1}$ continuously to $C^{s-1,\lambda}_{2\delta+1,\Lambda^1}$. Hence, as $2\delta - n + 3 \notin \mathbb{Z}_{\geqslant 0}$, the operator $\varphi d_1 \mathbb{D}$ maps $C^{s,\lambda}_{\delta,\Lambda^1}$ continuously to $C^{s,\lambda}_{2\delta,\Lambda^2}$ if $2\delta < n$. Similarly, $\varphi\mathbb{G}$ maps $R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d)$ continuously to $C^{s,\lambda}_{2\delta,\Lambda^2}$ if $2\delta < n$ (and, by the very definition, to $R^{s,\lambda}_{2\delta,\Lambda^2}(d)$).

Finally, as $\varphi\Delta u = u$ for each $u \in C^{s,\lambda}_{\delta,\Lambda^q}$ (see, for instance, [1] or [2, Theorem 3.1]) and the space $C^{s,\lambda}_{\delta,\Lambda^q}$ is continuously embedded to the space $C^{s-1,\lambda}_{\delta',\Lambda^q}$ for any $\delta \geqslant \delta'$ (see [2, Theorem 2.3]), applying the potential φ to (3) we conclude that Problems 1 and 2 are equivalent, which was to be proved. □

Theorem 2. *Let $n \geqslant 3$, $s \in \mathbb{N}$, $s \geqslant 3$ and $1 < \delta < n/2$ with $2\delta - n + 3 \notin \mathbb{Z}_{\geqslant 0}$. Then the continuous operator*

$$I - (1/\mu)\varphi\mathbb{G} : R^{s-1,\lambda}_{\delta+1}(d) \to R^{s-1,\lambda}_{\delta+1}(d)$$

is Fredholm one and the operator

$$(1/\mu)\varphi\mathbb{G} : R^{s-1,\lambda}_{\delta+1}(d) \to R^{s-1,\lambda}_{\delta+1}(d)$$

is continuous and compact.

Proof. We already proved in Theorem that $\varphi\mathbb{G}$ maps $R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d)$ continuously to $R^{s,\lambda}_{2\delta,\Lambda^2}(d)$ if $2\delta < n$. On the other hand, as the embedding $C^{s,\lambda}_{\delta,\Lambda^q}$ to the space $C^{s-1,\lambda}_{\delta',\Lambda^q}$ is compact for any

$\delta > \delta'$ we see that $\varphi\mathbb{G}$ maps $R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d)$ compactly to $R^{s,\lambda}_{\delta+1,\Lambda^2}(d)$ if $2 < 2\delta < n$, i.e. the second statement of the theorem is true.

Finally, as

$$\mathbb{G}'_{|g=g_0}F = d\star(\star g_0 \wedge \Phi F) + d\star(\star F \wedge \Phi g_0)$$

for each $F, g_0 \in R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d)$, we may argue as before to conclude that

$$(\varphi\mathbb{G})'_{|g=g_0} : R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d) \to R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d)$$

is a compact linear operator. Thus the operator $I - (1/\mu)\varphi\mathbb{G}'_{|g=g_0}$ is Fredholm for each $g_0 \in R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d)$ because of the famous Fredholm theorems. □

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О свойстве Фредгольма для стационарных уравнений Навье-Стокса в весовых пространствах Гельдера

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Мы доказываем, что стационарные уравнения Навье-Стокса индуцирует нелинейный оператор фредгольмовского типа в весовых пространствах Гельдера.

Ключевые слова: стационарные уравнения Навье-Стокса, нелинейные фредгольмовы операторы, весовые пространства Гельдера.