We prove that the steady Navier-Stokes equations induce a Fredholm non-linear map on the scale of Hölder spaces weighted at the infinity.

Keywords: steady Navier-Stokes Equations, non-linear Fredholm operators, weighted Hölder spaces.


The theory of nonlinear Fredholm operators by S. Smale [3] provides an approach to obtain generic results on the uniqueness and/or existence for nonlinear equations in Banach spaces. We recall that a bounded linear operator \( L \) in Banach spaces \( X \) and \( Y \) is called Fredholm if its kernel and cokernel are finite-dimensional and its range is closed. Then a nonlinear operator \( N \) is Fredholm if at every point \( x \in X \) its derivative (i.e. the principal linear part) \( N'_x \) possesses the Fredholm property. The most advanced results were obtained for the so called proper operators (a map is proper if the inverse image of a compact set is compact). For instance, using results on proper Fredholm maps from [3], J.C. Saut and R. Temam proved the generic uniqueness theorem for the steady version of the Navier-Stokes equations on the scale of the Sobolev spaces, see [4]. Recently, A. Shlapunov and N. Tarkhanov [2] proved that the evolution Navier-Stokes equations induce a Fredholm open injective map on the scale of the Hölder spaces over the strip \( \mathbb{R}^n \times [0, T] \), \( T > 0, n \geq 2 \), weighted at the infinity with respect to the space variables. In the present short note we prove that the steady Navier-Stokes-type equations induce a Fredholm map on the scale of the Hölder spaces over \( \mathbb{R}^n \), \( n \geq 3 \), weighted at the infinity.

Namely, let \( \mathbb{Z}_{\geq 0} \) be the set of all natural numbers including zero, and let \( \mathbb{R}^n \) be the Euclidean space of dimension \( n \geq 3 \) with coordinates \( x = (x_1, \ldots, x_n) \). Following [2], we denote by \( C^{s,0}_\delta \) the space of all \( s \) times continuously differentiable functions on \( \mathbb{R}^n \) with finite norm

\[
\|u\|_{C^{s,0}_\delta} = \sum_{|\alpha| \leq s} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{(\delta + |\alpha|)/2} |\partial^\alpha u(x)|.
\]

For \( 0 < \lambda < 1 \), we introduce

\[
\langle u \rangle_{\lambda, \delta} = \sup_{x \neq y \atop |x-y| \leq |\alpha|/2} \left( \max_{1 \leq j \leq n} (1 + |x_j|^2, 1 + |y_j|^2) \right)^{(\delta + \lambda)/2} |u(x) - u(y)| |x-y|^\lambda.
\]
We define $C^{s,\lambda}_\delta$ to consist of all $s$ times continuously differentiable functions on $\mathbb{R}^n$, such that

$$\|u\|_{C^{s,\lambda}_\delta} = \|u\|_{C^{s,0}_\delta} + \sum_{|\alpha| \leq s} (\partial^\alpha u)_\lambda, \delta + |\alpha| + \|u\|_{C^{s,\lambda}_\delta(\overline{B}_1)} < \infty$$

where $C^{s,\lambda}(\overline{B}_1)$ is the space of Hölder functions in the unit closed ball $\overline{B}_1$ centered at the origin. These are Banach spaces for all $s \in \mathbb{Z}_{\geq 0}$ and all $0 \leq \lambda < 1$.

Denote by $\Lambda^q$ the bundle of the differential forms of degree $q$ and let $C^{s,\lambda}_\delta$ stand for the spaces of the differential forms of degree $q$ with the coefficients in $C^{s,\lambda}_\delta$.

**Problem 1** Let $s \geq 2$ and $\delta > 0$. Given form $f \in C^{s-2,\lambda}_\delta$, find a form $u \in C^{s,\lambda}_\delta$ and a function $p \in C^{s-1,\lambda}_\delta$ satisfying

$$\begin{aligned}
-\mu \Delta u + d_0 p &= f, \\
\partial_0^* v &= 0
\end{aligned}$$

where $\mu$ is a positive real number, $\Delta$ is the Laplace operator, $d_q$ is the de Rham differential on $q$-forms, $d_0^*$ is its formal adjoint and $\Delta u = \sum_{j=1}^n u_j \partial_j u$.

For $n = 3$ the de Rham differentials give: $d_0 = \nabla, d_1 = \text{rot}, d_2 = \text{div}$ where $\nabla$ is the gradient operator, $\text{rot}$ is the rotation operator and $\text{div}$ is the divergence operator in $\mathbb{R}^n$. Hence Problem 1 is precisely the steady Navier-Stokes equations on the scale $C^{s,\lambda}_\delta$ if $n = 3$.

Now, integration by parts yields that, for $\delta > n - 2$, we have

$$(f, 1)_{L^2(\mathbb{R}^n)} = 0,$$  

(1)

for any form $f \in C^{s-2,\lambda}_\delta$, admitting a solution $(u, p)$ to Problem 1. Clearly, the weighted space $C^{s,\lambda}_\delta$ with $\delta > 0$ corresponds to the one point compactification of $\mathbb{R}^n$ and then Problem 1 is similar to the steady Navier-Stokes equations in the periodic case (or, the same, on a torus) and (1) is similar to [4, condition (2.3)]. We will always assume that (1) is fulfilled if $\delta > n - 2$.

On the next step, using Hodge theory for the de Rham complex over weighted spaces (see [2]), we reduce the Navier-Stokes equation for the velocity $u$ to the equation with respect to vorticity $d_1 u$ (cf. [1] on the scale of Sobolev spaces). With this purpose, we note that $\ker d$ will stand for solutions of the equation $du = 0$ in $\mathbb{R}^n$ in the sense of distributions. We denote by $R^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}}(d)$ the range of the operator $d : C^{s,\lambda}_{\delta+1,\Lambda^q} \to C^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}}$. According to [2, Corollary 3.11], the space $R^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}}(d)$ coincides with $C^{s-1,\lambda}_{\delta+1} \cap \ker d_{q+1}$ if $\delta \in (0, n-1)$; it consists of elements $f \in C^{s-1,\lambda}_{\delta+1} \cap \ker d_{q+1}$ satisfying

$$(f, d_q h_j)_{L^2(\mathbb{R}^n)} = 0,$$  

(2)

for all harmonic homogeneous polynomials $h_j$ of degree $j$ in $\mathbb{R}^n$ with $0 \leq j \leq m + 1$ if $\delta \in (n - 1 + m, n + m)$, $m \in \mathbb{Z}_{\geq 0}$.

Now, let $\Lambda$ be the exterior product on the differential forms and let $\ast$ be the Hodge star operator on the differential forms, induced by identity $dx_1 \wedge \cdots \wedge dx_q = dx$ for each differential $dx_1 = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$ with $|I| = q$. Now for $q$-form $F(x) = \sum_{|I| = q} F_I(x) dx_I$ we set

$$(\varphi F)(x) = \int_{\mathbb{R}^n} \sum_{|I| = q} F_I(y) \varphi_n(x - y) dy, \quad \Phi_F(x) = \int_{\mathbb{R}^n} F(y) \wedge (d_{m-q-1})^* \left( \sum_{|I| = q} \varphi_n(x - y) \ast dy_I \right),$$

where $\varphi_n(x - y)$ is the standard fundamental solution of the Laplace operator in $\mathbb{R}^n$. The behaviour of these potentials on the weighted Hölder spaces were investigated in [2, §3]. Now, for 1-form $g$ we set

$$G_{\ast g}(x) = \ast(g \wedge \Phi_F).$$
Problem 2. Let $s \geq 3$ and $\delta > 0$. Given form $f \in C^{s-2,\lambda}_{\delta + 2, A^1}$ (satisfying (1) if $\delta > n - 2$) find a form $g \in R^{s-1,\lambda}_{\delta + 1, A^2}$ satisfying
\[ g - (1/\mu)\varphi G g = \varphi d_1 f. \]

Now we may achieve the main theorems of our note.

Theorem 1. Let $n \geq 3$, $s \in \mathbb{N}$, $s \geq 3$ and $1 < \delta < n/2$ with $2\delta + n + 3 \notin \mathbb{Z}_{\geq 0}$. Then Problems 1 and 2 are equivalent.

Proof. Indeed, we may write the nonlinear term $D$ in the Lamb form (see, for instance, [1] or [2, Lemma 1.2]):
\[ Du = d_0 |u|^2 + s(d_1 u \wedge u). \]
According to [2, Corollary 3.11], the operator $\Phi$ maps $R^{s,\lambda}_{\delta + 1, A^2}$ continuously to $C^{s,\lambda}_{\delta, A^2}$ for the chosen $\delta$. Then, as $u = \Phi g$ is the unique form from $C^{s,\lambda}_{\delta, A^2}$ satisfying $d_1 \Phi g = g$, $d_2^* \Phi g = 0$ in $\mathbb{R}^n$, see [2], we conclude that $dD = Gd$. As $d^2 = 0$, this proves that Problem 1 is equivalent to the following equation
\[ -\mu \Delta g + G g = d_1 f \]
(3)
on the discussed scales of spaces. Besides, the potential $\varphi$ induces bounded linear map from the range $R^{s-2,\lambda}_{\delta + 2}(\Delta)$ of the bounded operator $\Delta : C^{s,\lambda}_{\delta, A^2} \to C^{s-2,\lambda}_{\delta, A^2}$ where $R^{s-2,\lambda}_{\delta + 2}(\Delta)$ coincides with $C^{s-2,\lambda}_{\delta + 2}$ if $0 < \delta < n - 2$; it consists of all the elements $F \in C^{s-2,\lambda}_{\delta + 2}$ satisfying
\[ (F, h_j)_{L_2(\mathbb{R}^n)} = 0, \]
(4)for all harmonic homogeneous polynomials $h_j$ of degree $j$ in $\mathbb{R}^n$ with $0 \leq j \leq m$ if $\delta \in (n - 2 + m, n + m - 1)$, $m \in \mathbb{Z}_{\geq 0}$ (see, for instance, [2, Theorem 3.1]).

Since $\delta > 1$ the potential $\varphi d_1 f$ is a convergent integral. Moreover, as $f$ satisfies (1) if $\delta > n - 2$, then using integration by parts we see that $d_1 f$ satisfies (4) with $j = 0$ and $j = 1$. Hence if $\delta \neq n - 2, \delta \neq n - 1, \delta \neq n, 1 < \delta < n + 1$, the form $\varphi d_1 f$ belongs to $C^{s-1,\lambda}_{\delta + 1}$. However, as $1 < \delta < n/2$ and $n \geq 3$ all these conditions are fulfilled.

Similarly, integration by parts yields $D u$ satisfies (1) if $2\delta + 1 > n - 2$. On the other hand, according to [2, Lemma 2.9] on the multiplication of the weighted functions, we see that $D$ maps $C^{s,\lambda}_{\delta, A^2}$ continuously to $C^{s-1,\lambda}_{\delta + 1, A^2}$. Hence, as $2\delta - n + 3 \notin \mathbb{Z}_{\geq 0}$, the operator $d_1 D$ maps $C^{s,\lambda}_{\delta, A^2}$ continuously to $C^{s-1,\lambda}_{\delta + 1, A^2}$ if $2\delta < n$. Similarly, $\varphi G$ maps $R^{s-1,\lambda}_{\delta + 1, A^2}$ (d) continuously to $C^{s,\lambda}_{\delta, A^2}$ if $2\delta < n$ (and, by the very definition, to $R^{s,\lambda}_{\delta + 2, A^2}(d)$).

Finally, as $\varphi D u = u$ for each $u \in C^{s,\lambda}_{\delta, A^2}$ (see, for instance, [1] or [2, Theorem 3.1]) and the space $C^{s,\lambda}_{\delta, A^2}$ is continuously embedded to the space $C^{s-1,\lambda}_{\delta', A^2}$ for any $\delta \geq \delta'$ (see [2, Theorem 2.3]), applying the potential $\varphi$ to (3) we conclude that Problems 1 and 2 are equivalent, which was to be proved.

Theorem 2. Let $n \geq 3$, $s \in \mathbb{N}$, $s \geq 3$ and $1 < \delta < n/2$ with $2\delta - n + 3 \notin \mathbb{Z}_{\geq 0}$. Then the continuous operator
\[ I - (1/\mu) \varphi G : R^{s-1,\lambda}_{\delta + 1}(d) \to R^{s-1,\lambda}_{\delta + 1}(d) \]
is Fredholm one and the operator
\[ (1/\mu) \varphi G : R^{s-1,\lambda}_{\delta + 1}(d) \to R^{s-1,\lambda}_{\delta + 1}(d) \]
is continuous and compact.

Proof. We already proved in Theorem 1 that $\varphi G$ maps $R^{s-1,\lambda}_{\delta + 1, A^2}(d)$ continuously to $R^{s,\lambda}_{2\delta, A^2}(d)$ if $2\delta < n$. On the other hand, as the embedding $C^{s,\lambda}_{\delta, A^2}$ to the space $C^{s-1,\lambda}_{\delta', A^2}$ is compact for any
\( \delta > \delta' \) we see that \( \varphi G \) maps \( R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d) \) compactly to \( R^{s,\Lambda}_{\delta+1,\Lambda^2}(d) \) if \( 2 < 2\delta < n \), i.e. the second statement of the theorem is true.

Finally, as
\[
G'_{|g=g_0} = d * (g_0 \wedge \Phi F) + d * (F \wedge \Phi g_0)
\]
for each \( F, g_0 \in R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d) \), we may argue as before to conclude that
\[
(\varphi G'_{|g=g_0} : R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d) \to R^{s,\Lambda}_{\delta+1,\Lambda^2}(d))
\]
is a compact linear operator. Thus the operator \( I - (1/\mu)\varphi G'_{|g=g_0} \) is Fredholm for each \( g_0 \in R^{s-1,\lambda}_{\delta+1,\Lambda^2}(d) \) because of the famous Fredholm theorems. \( \square \)

The work was supported by the grant of the Ministry of Education and Science of the Russian Federation N 1.2604.2017/PCh.

References


