On Divisibility of Some Sums of Binomial Coefficients Arising From Collection Formulas

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In this paper we establish a series of identities for sums of binomial coefficients to prove their divisibility by prime \( n \). These sums arise from exponents of commutators in collection formula for \((xy)^n\) with some restrictions on variables of the commutators.

Keywords: divisibility, sums of binomial coefficients, collection formulas.

Introduction

In 1932, P. Hall proved the formula (known as Hall’s collection formula) in [1] that makes it possible to investigate interdependently the power and commutator structures of \( p \)-groups. The following statement was proved by P. Hall in the article on the theory of \( p \)-groups [1]. For any two elements \( x \) and \( y \) of any group \( G \) let the formally distinct complex commutators \( R_1, R_2, \ldots \) of \( x \) and \( y \) be arranged in order of increasing weights (the order among the commutators of the same weight is arbitrary). Then for any natural \( n \) the following formula holds:

\[
(xy)^n = x^n y^n R_3^{f_3(n)} \ldots R_i^{f_i(n)} \ldots
\]

where \( f_i(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + \cdots + a_n \binom{n}{n} \), \( w \) is the weight of \( R_i \), and non-negative coefficients \( a_k \) depend only on \( R_i \) but not on \( n \). To compute the exponents is a difficult problem. In that regard, research is conducted in two lines. On the one hand, an explicit form of the exponents is sought (see, for example, [2]), on the other hand, a series of explicit collection formulas [3, 4] was found with some restrictions on the group.

In connection with the research on regularity of Sylow \( p \)-subgroups of group \( GL_n(\mathbb{Z}_{p^n}) \) (problem 8.3 [5]), the following theorem was proved in [6]. Here we use the abridged notation for commutators \([y, x] = y^{-1} x^{-1} y x, [y, i x] = [y, i-1 x], x, i = 1, 2, \ldots\).

**Theorem.** Let \( G \) be a group, \( x, y \in G \), any commutator of \( x \) and \( y \) equals \( 1 \) if it includes more than two \( y \)’s, then we have

\[
(xy)^n = x^n y^n \prod_{u=1}^{n-1} [y, u x]^{a_u} \prod_{u=1}^{n-1} [y, u x, y]^{n-a_{u+1}-(u+1)} \prod_{n-1 \geq u > v \geq 1} [y, u x, y]^{F_n(u, v) + G_n(u, v)},
\]

where

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\[ F_n(u, v) = \sum_{m=0}^{n-1} \sum_{k=0}^{v-m-k} \binom{n-m-i-1}{k-1} \binom{i}{u-k+1} \binom{i}{v-k}, \quad (3) \]

\[ G_n(u, v) = \sum_{m=1}^{n-1} \sum_{k=m}^{n-1} \binom{n}{v} \binom{k}{u} \binom{k}{n}, \quad (4) \]

For any collection formula, there is a divisibility problem for powers of commutators with weights less than \( n \), specifically for prime \( n \). According to Hall’s collection formula (1), these commutators have powers divisible by \( n \). From the properties of binomial coefficients it obviously follows that powers \( \binom{n}{u+1} \) and \( \binom{n}{u+2} \) in (2) are divisible by prime \( n \) if \( u+1 < n \) and \( u+2 < n \), respectively. But the divisibility of \( F_n(u, v) + G_n(u, v) \) is not obvious.

The aim of this paper is to transform \( F_n(u, v) \) and \( G_n(u, v) \) into such expressions that it will be easy to see the divisibility of \( F_n(u, v) \) and \( G_n(u, v) \) individually.

Let us recall that classical binomial coefficient

\[ \binom{n}{k} = \left\{ \begin{array}{ll} \frac{n!}{k!(n-k)!}, & \text{if } n \geq k \geq 0; \\ 0, & \text{if } n < k \text{ or } k < 0 \end{array} \right. \]

is defined for integers \( n \geq 0, k \) and satisfies the following recurrence relation:

\[ \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}. \quad (5) \]

Working with the sums \( F_n(u, v) \) and \( G_n(u, v) \), it is useful to extend the domain of the binomial coefficient to all integers \( n \) and \( k \) so that (5) remains valid. To set such extension we define \( \binom{n}{k} \) for each negative \( n \) with any single integer \( k \) [7]. In this paper the following extension is used: \( \binom{n}{k} = 1 \) for all integers \( i \). In what follows, we will use these extended coefficients without additional qualifications. We also agree that sums of the form \( \sum_{i=a}^{b} c_i \) are equal to zero if \( b < a \).

Our main result is the following

**Theorem 1.** For any integers \( n \geq 1, u \geq 0, v \geq 1 \) we have

\[ F_n(u, v) + G_n(u, v) = \sum_{k=1}^{v} \sum_{s=0}^{u-k} \binom{u-k}{s-1} \binom{v-k}{s} \binom{n}{u+s+2} + \sum_{i=0}^{v-1} (-1)^{i} \binom{n+i}{u+i+1} \binom{n}{v-i+1}. \]

In particular, if \( n \) is prime and \( u+v+2 < n \), then \( F_n(u, v) + G_n(u, v) \) is divisible by \( n \).

1. **Auxiliary statements**

Before we start working with \( F_n(u, v) \) and \( G_n(u, v) \) we will prove some statements about the extended binomial coefficient. Those statements are valid and well known for the classical binomial coefficient. The following statement will be used most often.

**Lemma 1.** If \( n, k \) are integers and \( n < k \), then \( \binom{n}{k} = 0 \).

**Proof.** If \( n < k \) then \( k = n+s \), where \( s > 0 \). We use the induction on \( s \) to prove that \( \binom{n}{n+s} = 0 \) for any integer \( n \). For \( s = 1 \) and any \( n \) according to (5), we have \( \binom{n}{n+1} = \binom{n+1}{n+1} - \binom{n}{n} = 0 \). Let \( \binom{n}{n+s} = 0 \) for some \( s \) and any \( n \), then \( \binom{n}{n+s+1} = \binom{n+1}{n+s+1} - \binom{n}{n+s} = \binom{n+1}{n+s+1} = 0 \).
Lemma 2. For any integers $n$ and $a$ we have the summation formula

$$
\sum_{i=a}^{n} \binom{i}{a} = \frac{n + 1}{a + 1}.
$$

Proof. If $n < a$ then both sides of (6) are equal to zero. Further on, we fix an arbitrary integer $a$ and use the induction on $n \geq a$. For $n = a$, the formula (6) is valid because $\binom{n}{a} = \binom{a+1}{a+1} = 1$. Assume that (6) is valid for some $n \geq a$, then we get

$$
\sum_{i=a}^{n+1} \binom{i}{a} = \sum_{i=a}^{n} \binom{i}{a} + \binom{n+1}{a} + \binom{n+1}{a+1} + \binom{n+1}{a+1} = \binom{n+2}{a+1}.
$$

Lemma 3. For any non-negative integers $n, b, a$ we have the summation formula

$$
\sum_{i=b}^{n-a} \binom{n-i}{a} \binom{i}{b} = \frac{n+1}{a+b+1}.
$$

Proof. If $n-a < b$ then both sides of (7) are equal to zero. Now assume that $n-a \geq b$. Let us transform $\binom{n-i}{a} \binom{i}{b}$ using (5).

$$
\binom{n-i}{a} \binom{i}{b} = \binom{n-i}{a} \binom{i+1}{b+1} - \binom{n-i}{a} \binom{i}{b+1} = \binom{n-i}{a} \binom{i+1}{b+1} - \binom{n-i+1}{b+1} \binom{i}{b+1}.
$$

By induction on integer $s \geq 0$ we obtain

$$
\binom{n-i}{a} \binom{i}{b} = \binom{n-i}{a-s} \binom{i}{b+s} + \sum_{j=1}^{s} \left( \binom{n-i}{a-j+1} \binom{i+1}{b+j} - \binom{n-i+1}{a-j+1} \binom{i}{b+j} \right).
$$

Suppose $s = n-b+1$ ($n-b+1 \geq a+1 > 0$), then $\binom{i}{n+1} = 0$ because $i < n+1$. Therefore, if we extend the summation in (7) to $n$ ($\binom{n}{n+1} = 0$ for $i > n-a$) and substitute (8) into (7), we get

$$
\sum_{i=b}^{n-a} \binom{n-i}{a} \binom{i}{b} = \sum_{i=b}^{n-b+1} \sum_{j=1}^{s} \left( \binom{n-i}{a-j+1} \binom{i+1}{b+j} - \binom{n-i+1}{a-j+1} \binom{i}{b+j} \right) =
$$

$$
= \sum_{j=1}^{n-b+1} \left( \binom{n-b}{a-j+1} \binom{b+1}{b+j} - \binom{n-b+1}{a-j+1} \binom{b}{b+j} \right) +
$$

$$
+ \sum_{j=1}^{n-b+1} \left( \binom{n-b-1}{a-j+1} \binom{b+2}{b+j} - \binom{n-b}{a-j+1} \binom{b+1}{b+j} \right) +
$$

$$
\cdots
$$

$$
+ \sum_{j=1}^{n-b+1} \left( \binom{1}{a-j+1} \binom{n}{b+j} - \binom{2}{a-j+1} \binom{n-1}{b+j} \right) +
$$

$$
+ \sum_{j=1}^{n-b+1} \left( \binom{0}{a-j+1} \binom{n+1}{b+j} - \binom{1}{a-j+1} \binom{n}{b+j} \right).
$$


After collecting similar terms we have

\[
\sum_{i=b}^{n-a} \binom{n-i}{a} \frac{i}{b} = - \sum_{j=1}^{n-b+1} \binom{n-b+1}{a-j+1} \frac{b}{b+j} + \sum_{j=1}^{n-b+1} \binom{0}{a-j+1} \frac{1}{b+j}.
\]

On the right-hand side of the obtained equality, the first sum is equal to zero because \(\binom{b}{b+j} = 0\) for \(j \geq 1\). In the second sum \(\binom{0}{a-j+1} \neq 0\) only for \(j = a + 1\), so all terms except one (for \(j = a + 1\)) are equal to zero. The sum always includes this term because \(1 \leq a + 1 \leq n - b + 1\).

Finally, we get

\[
\sum_{i=0}^{n-a} \binom{n-i}{a} \frac{i}{b} = \binom{0}{0} \binom{n+1}{a+b+1} = \binom{n+1}{a+b+1}.
\]

\(\square\)

2. Transformation of the sum \(F_n(u, v)\)

Let us simplify the expression \(F_n(u, v)\) as follows. At first, we extend the summation over \(i\) to \(n - k - 1\) (added terms are equal to zero because \(\binom{n-m-i-1}{k-1} = 0\) for \(i > n - m - k\)), then we change the order of summation.

\[
F_n(u, v) = \sum_{k=1}^{v} \sum_{i=v-k}^{n-k-1} \sum_{m=1}^{n-i-1} \binom{n-m-i-1}{k-1} \binom{i}{u-k+1} \binom{i}{v-k}.
\]

Now we replace \(n-m-i-1\) by \(m\).

\[
F_n(u, v) = \sum_{k=1}^{v} \sum_{i=v-k}^{n-k-1} \sum_{m=2}^{n-i-2} \binom{m}{k-1} \binom{i}{u-k+1} \binom{i}{v-k}.
\]

The starting value of \(m\) can be changed to \(k - 1\) because \(-i \leq 0\), \(k - 1 \geq 0\) and \(\binom{m}{k-1} = 0\) for \(m < k - 1\). So, we apply the formula (6) to the summation over \(m\) and, finally, get

\[
F_n(u, v) = \sum_{k=1}^{v} \sum_{i=v-k}^{n-k-1} \binom{n-i-1}{k} \binom{i}{u-k+1} \binom{i}{v-k}.
\] (9)

Further on, we will consider the summation over \(i\), so it will be useful to introduce the notation:

\[
f_n(u, v, k) = \sum_{i=v-k}^{n-k} \binom{n-i}{k} \binom{i}{u-k} \binom{i}{v-k}.
\]

Lemma 4. For any nonnegative integers \(n, u, v, k\) the following relation holds:

\[
f_n(u, v, k) = f_n(u+1, v, k+1) + f_n(u, v, k+1) + f_n(u, v+1, k+1).
\] (10)

Proof. If \(n < v\) then both sides of (10) are equal to zero.

Now assume that \(n \geq v\). Replacing \(\binom{n-i}{k+1}\) on the difference \(\binom{n-i+1}{k+1} - \binom{n-i}{k+1}\) in \(f_n(u, v, k)\) and taking into account that

\[
- \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} = - \sum_{i=v-k}^{n-k-1} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} = \]

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If we can write:

\[ n \]

Now we transform two sums in the obtained equality. If \( n \)

Therefore

This remains true if \( n \)

Because

\[ f_u(u, v, k) = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} + f_u(u, v+1, k+1), \]

we get

\[ f_u(u, v, k) = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} - \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i+1}{u-k} \binom{i}{v-k} + + f_u(u, v+1, k+1). \]

Because

\[ - \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i+1}{u-k} \binom{i}{v-k} = \]

\[ = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i+1}{u-k} \binom{i}{v-k} + \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i+1}{u-k} \binom{i}{v-k-1} = \]

we can write:

\[ f_u(u, v, k) = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} - \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k} + + \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i+1}{u-k} \binom{i}{v-k-1} + f_u(u, v+1, k+1). \]

If \( n \geq v + 1 \) we see that the difference of the first two sums is equal to \( \binom{n-(v-k)+1}{k+1} \binom{v-k}{v-k} \). This remains true if \( n = v \) (then the second sum equals zero, while the first consists of only one term). Therefore

\[ f_u(u, v, k) = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i+1}{u-k} \binom{i}{v-k-1} = \]

\[ = \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k-1} + \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k-1} \binom{i}{v-k-1} + \]

\[ + \binom{n-(v-k)+1}{k+1} \binom{v-k}{u-k} \binom{v-k}{v-k} + f_u(u, v+1, k+1). \]

Now we transform two sums in the obtained equality. If \( n - 1 \geq v \) then, firstly:

\[ \sum_{i=v-k}^{n-k} \binom{n-i}{k+1} \binom{i}{u-k} \binom{i}{v-k-1} = \]
Lemma 5. For any non-negative integers $p, n, u, v, k$ the following relation holds:

$$f_n(u, v, k) = \sum_{s=0}^{p} \sum_{l=0}^{p-s} \binom{p}{s, l} f_n(u + s, v + l, k + p).$$  

Proof. The proof is by induction on $p$. For $p = 0$ there is nothing to prove. Suppose (12) is valid for some $p$; then we apply (10) to $f_n(u + s, v + l, k + p)$ and get three sums on the right-hand side of (12). In each of these sums we will replace the indices.

In the first sum $s$ is replaced by $s - 1$.

$$\sum_{s=0}^{p} \sum_{l=0}^{p-s} \binom{p}{s, l} f_n(u + s + 1, v + l, k + p + 1) = \sum_{s=0}^{p+1} \sum_{l=0}^{p-s+1} \binom{p}{s-1, l} f_n(u + s, v + l, k + p + 1).$$
As before, the start value of $s$ remains zero to make the further transformations easier (for $s = 0$ we have $(\binom{p}{s-1, l}) = (\binom{p}{s-l}) = (\binom{p}{s-1}) = (\binom{p}{p}) = 0$).

In the second sum we increase the finish values of $s$ and $l$ to $p + 1$ and $p - s + 1$, respectively (if $s = p + 1$ or $l = p - s + 1$, then $(\binom{p}{s, l}) = 0$).

$$\sum_{s=0}^{p-s} \sum_{l=0}^{p-s} \binom{p}{s, l} f_n(u + s, v + l, k + p + 1) = \sum_{s=0}^{p+1} \sum_{l=0}^{p-s+1} \binom{p}{s, l} f_n(u + s, v + l, k + p + 1).$$

In the third sum we replace $l$ by $l - 1$ (the starting value of $l$ remains zero) and increase the finish value of $s$ to $p + 1$.

$$\sum_{s=0}^{p-s} \sum_{l=0}^{p-s} \binom{p}{s, l} f_n(u + s, v + l + 1, k + p + 1) = \sum_{s=0}^{p+1} \sum_{l=0}^{p-s+1} \binom{p}{s, l - 1} f_n(u + s, v + l, k + p + 1).$$

If $s = p + 1$ or $l = 0$, then $(\binom{p}{s, l}) = 0$, so the last equality is valid.

Finally, we obtain

$$f_n(u, v, k) = \sum_{s=0}^{p+1} \sum_{l=0}^{p-s+1} \binom{p}{s-1, l} f_n(u + s, v + l, k + p + 1) +$$

$$\sum_{s=0}^{p+1} \sum_{l=0}^{p-s+1} \binom{p}{s, l} f_n(u + s, v + l, k + p + 1) + \sum_{s=0}^{p+1} \sum_{l=0}^{p-s+1} \binom{p}{s, l - 1} f_n(u + s, v + l, k + p + 1).$$

To conclude the proof, it remains to note that $(\binom{p}{s, l}) + (\binom{p}{s, l}) = (\binom{p+1}{s, l})$ according to (11).

**Lemma 6.** For any non-negative integers $n, p, q, u, v, k$ we have the relation

$$f_n(u, v, k) = \sum_{l=0}^{p} \sum_{s=0}^{p-l} \binom{p + q}{s, l + q} f_n(u + s, v + l + q, k + p + q) +$$

$$\sum_{l=0}^{q} \sum_{s=0}^{p} \binom{p + l - 1}{s, p - 1} \binom{p}{s} f_n(u + s, v + l, k + p + l). \quad (13)$$

**Proof.** The proof is by induction on $q$. For $q = 0$ the equality (13) takes the form of (12) (with changed summation order). Suppose (13) holds true for some $q \geq 0$, then we transform the first sum on the right-hand side of (13). The application of (10) to $f_n(u + s, v + l + q, k + p + q)$ divides the sum into three sums. We transform two of them to the form of the third one:

$$\sum_{l=0}^{p} \sum_{s=0}^{p-l} \binom{p + q}{s, l + q} f_n(u + s, v + l + q + 1, r + 1),$$

for convenience, the temporary notation $r = k + p + q$ will be used.

In the first sum we separate terms for $l = 1$, then, in the remaining expression, replace $s$ by $s - 1$, $l$ by $l + 1$. The starting value of $s$ and the final value of $l$ remain zero and $p$, respectively (for $s = 0$ or $l = p$ we have $(\binom{p+q}{s-1, l+q+1}) = 0$).

$$\sum_{l=1}^{p} \sum_{s=0}^{p-l} \binom{p + q}{s, l + q} f_n(u + s + 1, v + l + q, r + 1) = \sum_{s=0}^{p-1} \binom{p + q}{s, q + 1} f_n(u + s + 1, v + q + 1, r + 1) +$$

$$\sum_{l=0}^{q} \sum_{s=0}^{p} \binom{p + l - 1}{s, p - 1} \binom{p}{s} f_n(u + s, v + l, k + p + l).$$
In the second sum we also separate terms for \( l = 1 \) and, in the remaining expression, replace \( l \) by \( l + 1 \). The final values of \( l \) and \( s \) remain \( p \) and \( p - l \), respectively (for \( l = p \) or \( s = p - l \) we have \( \binom{p}{s,l+q+1} = 0 \)).

\[
\begin{align*}
\sum_{l=1}^{p} \sum_{s=0}^{p-l} \binom{p+q}{s,l+q} f_n(u+s+1,v+l+q,r+1) &= \sum_{s=0}^{p-1} \binom{p+q}{s,q+1} f_n(u+s+1,v+q+1,r+1) + \\
&+ \sum_{l=2}^{p} \sum_{s=0}^{p-l} \binom{p+q}{s,l+q} f_n(u+s,v+l+q+1,r+1).
\end{align*}
\]

According to (11), \( \binom{p+q}{s-1,l+q+1} + \binom{p+q}{s,l+q+1} + \binom{p+q}{s,q+1} = \binom{p+q+1}{s,l+q+1} \), so summing up two transformed sums and the third one, we get

\[
\begin{align*}
\sum_{l=1}^{p} \sum_{s=0}^{p-l} \binom{p+q}{s,l+q} f_n(u+s+1,v+l+q,r) &= \sum_{l=1}^{p-1} \sum_{s=0}^{p-l} \binom{p+q+1}{s,l+q+1} f_n(u+s,v+l+q+1,r+1) + \\
&+ \sum_{s=0}^{p-1} \binom{p+q}{s,q+1} f_n(u+s+1,v+q+1,r+1) + \sum_{s=0}^{p-1} \binom{p+q}{s,q+1} f_n(u+s,v+q+1,r+1).
\end{align*}
\]

In the second sum we replace \( s \) by \( s - 1 \) but, as before, we sum from \( s = 0 \) (if \( \binom{p+q}{s-1,q+1} = 0 \) for \( s = 0 \)).

\[
\sum_{s=0}^{p-1} \binom{p+q}{s,q+1} f_n(u+s+1,v+q+1,k+p+q+1) = \sum_{s=0}^{p-1} \binom{p+q}{s-1,q+1} f_n(u+s,v+q+1,k+p+q+1).
\]

In the third sum we increase the final value of \( s \) to \( p \) (if \( \binom{p+q}{s,q+1} = 0 \) for \( s = p \)).

\[
\sum_{s=0}^{p-1} \binom{p+q}{s,q+1} f_n(u+s,v+q+1,r+1) = \sum_{s=0}^{p} \binom{p+q}{s,q+1} f_n(u+s,v+q+1,r+1).
\]

From the definition of the trinomial coefficient and (5) it follows that

\[
\binom{p+q}{s-1,q+1} + \binom{p+q}{s,q+1} = \binom{p+q+1}{p-1} \binom{p-1}{s-1} + \binom{p+q}{p-1} \binom{p-1}{s} = \binom{p+q}{p-1} \binom{p}{s},
\]

so, finally we get

\[
\sum_{l=1}^{p} \sum_{s=0}^{p-l} \binom{p+q}{s,l+q} f_n(u+s,v+l+q,r) = \sum_{l=1}^{p} \sum_{s=0}^{p-l} \binom{p+q+1}{s,l+q+1} f_n(u+s,v+l+q+1,r+1) +
\]

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are equal to zero. Indeed, all terms in $f$ are equal to zero because $\binom{s}{v} = 0$ for $i \geq 0$.

**Theorem 2.** For any non-negative integers $n, u, v, k$ such that $k \leq v$ or $k \leq u$ we have

$$f_n(u, v, k) = \sum_{s=0}^{v-k} \binom{n}{v-k} \binom{n-1}{s} \binom{v-k}{u+s} f_{n-s}(u+s, v, k).$$

**Proof.** Let us consider the case $k \leq v$ and $k \leq u$. Suppose $p = v - k \geq 0$, $q = u - k \geq 0$ in (13). Then, since by definition

$$f_n(u, v, k) = \sum_{s=0}^{v-k} \binom{v-k+s}{v-k} \binom{n-1}{s} \binom{v-k}{u+s} f_{n-s}(u+s, v, k),$$

because $\binom{s}{v} = 0$ for $i \geq l + 1$ and $s - (v-k) < 0$, we obtain

$$f_n(u, v, k) = \sum_{s=0}^{u-k} \binom{v-k+l-1}{v-k-1} \binom{v-k}{s} f_{n-s}(u+s, v+l).$$

Moreover, by definition, we have

$$f_n(u+s, v, k) = \sum_{i=0}^{n-(v+l)} \binom{n+1}{i} (u+s-(v+l), v),$$

because $\binom{n+1}{i} = 1$ for $i \geq 0$.

Note that if $u+s-(v+l) < 0$, i.e. $l > u+s-v$, then all terms in $f_n(u+s, v+l, v+l)$ equal zero because $\binom{n+1}{i} = 0$ for $i \geq 0$. So, after changing the order of the sums in (15) and removing zero terms in the summation over $l$, we get

$$f_n(u, v, k) = \sum_{s=0}^{u-k} \sum_{l=0}^{u+s-v} \binom{v-k+l-1}{v-k-1} \binom{v-k}{s} f_{n-s}(u+s, v+l).$$

Finally, we remove zero terms in (16) and apply the formula (7) to it.

$$f_n(u+s, v+l, v+l) = \sum_{i=0}^{n-(v+l)} \binom{n-1}{i} (u+s-(v+l), v)_l.$$
and \( \binom{i}{u-k} = 0 \) for \( i < 0 \). If \( k \leq v \) and \( k > u \), then \( f_n(u, v, k) \) is equal to zero again because \( \binom{i}{u-k} = 0 \) for \( i > v - k \geq 0 \). The right-hand side is equal to zero because \( \binom{u+s-k}{v-k} = 0 \) for \( s \leq v - k \).

**Corollary 1.** For any natural \( n \) and non-negative integers \( u, v \) we have

\[
F_n(u, v) = \sum_{k=1}^{v} \sum_{s=0}^{v-k} \binom{u-k+1+s}{s} \binom{v-k}{u+s+2}. \tag{17}
\]

In particular, if \( u + v + 1 < n \) then \( F_n(u, v) \) is divisible by prime \( n \).

**Proof.** Applying (14) to \( f_{n-1}(u+1, v, k) \) in (9), we obtain (17) (condition \( k \leq v \) is satisfied). If \( n \) is prime, \( u + v + 1 < n \), then \( u + s + 2 \leq u + v + 1 < n \), so the binomial coefficient \( \binom{n}{u+s+2} \) is divisible by \( n \) for any \( s \). \( \Box \)

3. Transformation of the sum \( G_n(u, v) \)

**Lemma 7.** For any natural \( n \) and integers \( u \geq 0, v \geq 1 \) we have

\[
G_n(u, v) = \sum_{i=1}^{n-1} \binom{i}{u} \binom{i}{v}. \tag{18}
\]

**Proof.** We change the order of summation in \( G_n(u, v) \).

\[
G_n(u, v) = \sum_{m=1}^{n-2} \sum_{k=m+1}^{n-1} \binom{k}{m} \binom{m}{u} = \sum_{m=2}^{n-1} \sum_{k=m}^{n-1} \binom{k}{m} \binom{m}{u}.
\]

Because \( v \geq 1 \) we remove zero terms in the summation over \( k \) and apply the formula (6).

\[
\sum_{m=2}^{n-1} \sum_{k=m}^{n-1} \binom{k}{m} \binom{m}{u} = \sum_{m=2}^{n-1} \sum_{k=m}^{n-1} \binom{k}{m} \binom{m}{u} = \sum_{m=2}^{n-1} \binom{m}{u+1} \binom{m}{v+1}.
\]

Adding to the last sum the term \( \binom{1}{v+1} \binom{1}{u} = 0 \) we get (18). \( \Box \)

**Theorem 3.** For any natural \( n \) and non-negative integers \( u \geq 1, v \geq 1 \) we have

\[
\sum_{i=1}^{n-1} \binom{i}{u} \binom{i}{v} = \sum_{i=0}^{v} (-1)^i \binom{n+i}{u+i+1} \binom{n}{v-i}. \tag{19}
\]

**Proof.** Let us transform \( \binom{i}{u} \binom{i}{v} \) using (5).

\[
\binom{i}{u} \binom{i}{v} = \binom{i+1}{u+1} \binom{i}{v} - \binom{i}{u+1} \binom{i}{v} = \binom{i+1}{u+1} \binom{i}{v} - \binom{i+1}{u+1} \binom{i}{v-1} - \binom{i}{u+1} \binom{i}{v}.
\]

By induction on integer \( s \geq 0 \) we obtain

\[
\binom{i}{u} \binom{i}{v} = (-1)^s \binom{i+s}{u+s} \binom{i}{v-s} + \sum_{j=0}^{s-1} (-1)^j \left( \binom{i+j+1}{u+j+1} \binom{i+1}{v-j} - \binom{i+j}{u+j+1} \binom{i}{v-j} \right).
\]
Suppose $s = v + 1$, then $\binom{i}{v-1} = \binom{i}{i} = 0$ because $i \geq 1$, so we get

$$\binom{i}{u} \binom{i}{v} = \sum_{j=0}^{v} (-1)^j \left[ \binom{i+j+1}{u+j+1} \binom{i+1}{v-j} - \binom{i+j}{u+j+1} \binom{i}{v-j} \right].$$

Therefore,

$$\sum_{i=1}^{n-1} \binom{i}{u} \binom{i}{v} = \sum_{i=1}^{n-1} \sum_{j=0}^{v} (-1)^j \left[ \binom{i+j+1}{u+j+1} \binom{i+1}{v-j} - \binom{i+j}{u+j+1} \binom{i}{v-j} \right] =$$

$$= \sum_{j=0}^{v} (-1)^j \left[ \binom{j+2}{u+j+1} \binom{2}{v-j} - \binom{j+1}{u+j+1} \binom{1}{v-j} \right] +$$

$$+ \sum_{j=0}^{v} (-1)^j \left[ \binom{j+3}{u+j+1} \binom{3}{v-j} - \binom{j+2}{u+j+1} \binom{2}{v-j} \right] +$$

$$+ \sum_{j=0}^{v} (-1)^j \left[ \binom{n-1+j}{u+j+1} \binom{n-1}{v-j} - \binom{n-2+j}{u+j+1} \binom{n-2}{v-j} \right] +$$

$$+ \sum_{j=0}^{v} (-1)^j \left[ \binom{n+j}{u+j+1} \binom{n}{v-j} - \binom{n-1+j}{u+j+1} \binom{n-1}{v-j} \right].$$

After collecting similar terms we have

$$\sum_{i=1}^{n-1} \binom{i}{u} \binom{i}{v} = -\sum_{j=0}^{v} (-1)^j \binom{j+1}{u+j+1} \binom{1}{v-j} + \sum_{j=0}^{v} (-1)^j \binom{n+j}{u+j+1} \binom{n}{v-j},$$

where the first sum is equal to zero if $u \geq 1$ or $v \geq 1$. In the second case we have

$$\sum_{j=0}^{v} (-1)^j \binom{j+1}{u+j+1} \binom{1}{v-j} = (-1)^{v-1} \binom{v+1}{u+v+1} \binom{1}{1} + (-1)^v \binom{v+1}{u+v+1} \binom{1}{0} = 0.$$ 

Thus, we obtain (19).

**Corollary 2.** For any integers $n \geq 1, u \geq 0, v \geq 1$ we have

$$G_n(u, v) = \sum_{i=0}^{v+1} (-1)^i \binom{n+i}{u+i+1} \binom{n}{v-i+1}.$$  \hspace{1cm} (20)

In particular, if $u + v + 2 < n$ then $G_n(u, v)$ is divisible by prime $n$.

**Proof.** Because $v+1 \geq 2$ we apply (19) to (18) in order get (20).

Suppose $n$ is prime, $u + v + 2 < n$. The binomial coefficient $(v+1)_{u+i+1}$ is divisible by $n$ for any $0 \leq i \leq v$, and $(v+1)_{u+i+1}$ is divisible by $n$ for $i = v + 1$. \hfill \Box

Corollaries 1 and 2 are a proof of Theorem 1.

**Remark 1.** If we impose the conditions $n - 1 \geq u > v \geq 1$ on the expression $F_n(u, v) + G_n(u, v)$ in Theorem 1, then (1) contains only classical binomial coefficients, i.e. it does not depend on the extension of binomial coefficient.

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References


О делимости некоторых сумм биномиальных коэффициентов, возникающих в собирательных формулах

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В данной работе установлен ряд тождеств для сумм биномиальных коэффициентов, чтобы доказать их делимость на простое \( n \). Эти суммы возникают в степенях коммутаторов из собирательной формулы для \((xy)^n\) при некоторых ограничениях на входные переменные в коммутаторы.

Ключевые слова: делимость, суммы биномиальных коэффициентов, собирательные формулы.