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Sequential Empirical Process of Independence

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Uniform strong laws of large numbers and the central limit theorem for special sequential empirical process of independence for a certain class of measurable functions are considered in the paper.

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1. Introduction and preliminaries

Let us consider a sequence of experiments in which observed data consist of independent pairs $\{(X_k, A_k), k \geqslant 1\}$, where X_k are random variables (r.v.-s) on a probability space $(\Omega, A, \mathbb{P})$ with values in a measurable space $(\mathfrak{X}; \mathcal{B})$ and A_k are events with common probability $p = P(A_k) \in (0,1)$. Let $\delta_k = I(A_k)$ be an indicator of the event A_k. At the n-th stage of experiment the observed data are $S^{(n)} = \{(X_k, \delta_k), 1 \leqslant k \leqslant n\}$. Each pair (X_k, δ_k) induces a statistical model with sample space $\mathfrak{X} \otimes \{0,1\}$ with σ-algebra $\mathcal{G}$ of sets $B \otimes D$ and distribution $\mathbb{Q}^*(\cdot)$ on $(\mathfrak{X} \otimes \{0,1\}, \mathcal{G})$:

$$\mathbb{Q}^*(B \otimes D) = P(X_k \in B, \delta_k \in D), B \in \mathcal{B}, D \subset \{0,1\}.$$

We consider submeasures $\mathbb{Q}_m(B) = \mathbb{Q}^*(B \otimes \{m\})$, $m = 0,1$ and $\mathbb{Q}(B) = \mathbb{Q}_0(B) + \mathbb{Q}_1(B) = \mathbb{Q}^*(B \otimes \{0,1\}), B \in \mathcal{B}$. From a practical point of view, it is important to test the validity of hypothesis $\mathcal{H}$ for independence of r.v. X_k and event A_k for each $k \geqslant 1$. In order to verify this we use the signed measure $\Lambda(B) = \mathbb{Q}_1(B) - p\mathbb{Q}(B), B \in \mathcal{B}$, where $p = \mathbb{Q}_1(\mathfrak{X})$ and the validity of $\mathcal{H}$ is equivalent to the equality $\Lambda(B) = 0$ for any $B \in \mathcal{B}$. We introduce the empirical estimates of the above introduced measures for $B \in \mathcal{B}$ from sample $S^{(n)}$:

$$\mathbb{Q}_{0n}(B) = \frac{1}{n}\sum_{k=1}^{n}(1-\delta_k) I(X_k \in B), \quad \mathbb{Q}_{1n}(B) = \frac{1}{n}\sum_{k=1}^{n}\delta_k I(X_k \in B),$$

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$$\mathbb{Q}_n(B) = \mathbb{Q}_{0n}(B) + \mathbb{Q}_{1n}(B) = \frac{1}{n}\sum_{k=1}^{n} I(X_k \in B), \tag{1}$$

$$\Lambda_n(B) = \mathbb{Q}_{1n}(B) - p_n\mathbb{Q}_n(B), p_n = \mathbb{Q}_{1n}(\mathfrak{X}).$$

By the strong law of large numbers (SLLN) we have for a fixed set B that $\mathbb{Q}_{mn}(B) \xrightarrow[n\to\infty]{a.s} \mathbb{Q}_m(B)$, $m = 0,1$; $\mathbb{Q}_n(B) \xrightarrow[n\to\infty]{a.s} \mathbb{Q}(B)$ and $\Lambda_n(B) \xrightarrow[n\to\infty]{a.s} \Lambda(B)$. If hypothesis $\mathcal{H}$ is valid then $\Lambda_n(B) \xrightarrow[n\to\infty]{a.s} 0$. Then we arrive at the study of limit behaviour of normalized process $\{\chi_n = a_n(\Lambda_n(B) - \Lambda(B)), B \in \mathcal{G}\}$, where $\{a_n, n \geqslant 1\}$ is a (possible random) sequence of positive numbers, and $\mathcal{G}$ is a certain class of sets from $\mathcal{B}$. The specially normalized empirical process of independence indexed by the class $\mathcal{F}$ of measurable functions $f \in \mathcal{F}$ was studied [1]. Class $\mathcal{F}$ coincides with χ_n when $f = I(\cdot)$ is the indicator. In this paper we extend these results for the sequential analogue of that process.

2. Sequential uniform law of large numbers

For a measure G and class $\mathcal{F}$ of Borel measurable functions $f : \mathfrak{X} \to \mathbb{R}$ we introduce the following integral

$$Gf = \int_{\mathfrak{X}} f dG, f \in \mathcal{F}.$$

Let us introduce the following $\mathcal{F}$-indexed extensions of (1) for $f \in \mathcal{F}$:

$$\mathbb{Q}_{0n}f = \frac{1}{n}\sum_{k=1}^{n}(1-\delta_k) f(X_k), \ \ \mathbb{Q}_{1n}f = \frac{1}{n}\sum_{k=1}^{n}\delta_k f(X_k),$$

$$\mathbb{Q}_n f = \mathbb{Q}_{0n}f + \mathbb{Q}_{1n}f = \frac{1}{n}\sum_{k=1}^{n} f(X_k), \tag{2}$$

and $\Lambda_n f = \mathbb{Q}_{1n}f - p_n\mathbb{Q}_n f$, where $p_n = \mathbb{Q}_{1n}1 = \mathbb{Q}_{1n}(\mathfrak{X}) = \frac{1}{n}\sum_{k=1}^{n}\delta_k$. Relations (1) are special cases of (2) when $\mathcal{F} = \{I(B), B \in \mathcal{G}\}$. We define $\mathcal{F}$-indexed empirical process $G_n : \mathcal{F} \to \mathbb{R}$ as

$$f \mapsto G_n f = \sqrt{n}(\mathbb{Q}_n - \mathbb{Q}) f = n^{-1/2}\sum_{k=1}^{n}(f(X_k) - \mathbb{Q}f), f \in \mathcal{F}. \tag{3}$$

Here $G_n f = G_{0n}f + G_{1n}f$ with subempirical processes

$$G_{jn}f = \sqrt{n}(\mathbb{Q}_{jn} - \mathbb{Q}_j) f, \ \ j = 0,1, f \in \mathcal{F}. \tag{4}$$

For a given f by SLLN and central limit theorem (CLT) we have

$$(a) \ \ \mathbb{Q}_n f \xrightarrow[n\to\infty]{a.s.} \mathbb{Q}f \ \text{ as } \ \mathbb{Q}|f| < \infty; \tag{5}$$

$$(b) \ \ G_n f \Rightarrow Gf \overset{d}{=} N\left(0, \sigma^2_{\mathbb{Q}}(f)\right), \ \ n \to \infty \ \text{ as } \ \mathbb{Q}f^2 < \infty, \tag{6}$$

where $\sigma^2_{\mathbb{Q}}(f) = \mathbb{Q}(f - \mathbb{Q}f)^2$.

There is theory for uniform variants of special classes $\mathcal{F}$ of measurable functions in (5) and (6) (see, for example, [2–4]). There are various extensions of the Glivenko-Cantelli theorem and the Donsker theorem for $\mathcal{F}$-indexed empirical processes (3) under certain conditions on the set $\mathcal{F}$ of measurable functions. These conditions ensure that $n^{-1/2}\|G_n f\|_{\mathcal{F}} = \sup\left\{n^{-1/2}\left|G_n f\right|,\ f \in \mathcal{F}\right\}$ converges either in probability or almost surely to zero. These classes $\mathcal{F}$ are called the weak or strong Glivenko-Cantelli classes, respectively. Donsker-type theorems provide general conditions on $\mathcal{F}$ in order to get weak convergence

$$G_n f \Rightarrow Gf \quad \text{in} \quad l^{\infty}\left(\mathcal{F}\right), \tag{7}$$

where $l^{\infty}\left(\mathcal{F}\right)$ is the space of all bounded functions $f : X \to \mathbb{R}$ with the supremum-norm $\|.\|_{\mathcal{F}}$ (see [3], p. 81). Class $\mathcal{F}$ with condition (7) is called the Donsker class. The limiting field $\{Gf,\ f \in \mathcal{F}\}$ in (7) is called $\mathbb{Q}$-Brownian bridge. Let us introduce tight Borel measurable element of $l^{\infty}\left(\mathcal{F}\right)$ and Gaussian field with zero mean and covariance function

$$\operatorname{cov}\left(Gf, Gg\right) = \mathbb{Q}fg - \mathbb{Q}f\mathbb{Q}g, \quad f, g \in \mathcal{F}. \tag{8}$$

Remind that $\mathbb{Q}$-Brownian bridge $\{Gf, f \in \mathcal{F}\}$ can be represented in terms of $\mathbb{Q}$-Brownian sheet $\{W\left(f\right), f \in \mathcal{F}\}$ with zero mean and covariance

$$\operatorname{cov}\left(W\left(f\right), W\left(g\right)\right) = \mathbb{Q}fg, \quad f, g \in \mathcal{F}, \tag{9}$$

by distribution equality

$$Gf \overset{d}{=} W\left(f\right) - W\left(1\right)\mathbb{Q}f, \ \ f \in \mathcal{F}. \tag{10}$$

For a given f with the conditions $\mathbb{Q}_j\left|f\right| < \infty,\ j = 0, 1$ by SLLN we have

$$\Lambda_n f \underset{n\to\infty}{\overset{a.s.}{\to}} \Lambda f \overset{under\ \mathcal{H}}{=} 0 \tag{11}$$

Moreover, for a given f variable $\sqrt{n}\left(\Lambda_n - \Lambda\right) f$ is a linear functional of subempirical processes (4) with the condition $\mathbb{Q}_j f^2 < \infty$, $j = 0, 1$. It has limiting normal distribution $N\left(0, \sigma^2_{\mathbb{Q}}\left(f\right)\right)$. Uniform SLLN and CLT for the specially normalized empirical $\mathcal{F}$-indexed process

$$\left\{\Delta_n f = \left(\frac{n}{p_n\left(1 - p_n\right)}\right)^{1/2}(\Lambda_n - \Lambda)f,\ f \in \mathcal{F}\right\},$$

Were proved [1].It was shown that the limiting distribution is $\mathbb{Q}$-Brownian bridge $\{Gf,\ f \in \mathcal{F}\}$ with covariance (8). Let us consider the following sequential extension of $\{\Delta_n f,\ f \in \mathcal{F}\}$

$$\left\{\Delta_n\left(s; f\right) = \left(p_n\left(1 - p_n\right)\right)^{-1/2} n^{-1/2}\left[ns\right]\left(\Lambda_{[ns]} - \Lambda\right) f,\ (s; f) \in \mathcal{D}\right\}, \tag{12}$$

where $\mathcal{D} = T \otimes \mathcal{F}$, $T = [0, 1]$, $\Lambda_{[ns]} = \mathbb{Q}_{1[ns]} - p_{[ns]}\mathbb{Q}_{[ns]}$ and $[a]$ denotes the integer part of a. Then $\Delta_n f = \Delta_n\left(1; f\right)$. Let $\|\psi\left(s\right)\|_T = \sup\left\{\left|\psi\left(s\right)\right|, 0 \leqslant s \leqslant 1\right\}$ and $\|\Delta_n\left(s; f\right)\|_{\mathcal{D}} = \sup\left\{\left|\Delta_n\left(s; f\right)\right|,\ (s; f) \in \mathcal{D}\right\}$. We will prove uniform strong and weak LLN's for process

$$\left\{\frac{[ns]}{n}\left(\Lambda_{[ns]} - \Lambda\right) f, \ \ (s; f) \in \mathcal{D}\right\}.$$

Sequential SLLN is considered in the following theorem.

Theorem 2.1. *Let us assume that* $\mathbb{Q}_j f^2 < \infty,\ j = 0, 1,\ f \in \mathcal{F}$. *Then*

$$\left\|\frac{[ns]}{n}\left(\Lambda_{[ns]} - \Lambda\right) f\right\|_T \xrightarrow[n\to\infty]{a.s} 0. \tag{13}$$

Proof. It is easy to see that

$$\frac{[ns]}{n}\left(\Lambda_{[ns]}-\Lambda\right)f=\frac{\left(1-p_{[ns]}\right)}{n}\sum_{k=1}^{[ns]}\left(\delta_k f\left(X_k\right)-\mathbb{Q}_1 f\right)-$$
$$-\frac{p_{[ns]}}{n}\sum_{k=1}^{[ns]}\left(\left(1-\delta_k\right)f\left(X_k\right)-\mathbb{Q}_0 f\right)-\frac{1}{n}\sum_{k=1}^{[ns]}\left(\delta_k-p\right)\mathbb{Q}f. \quad (14)$$

Assuming $\mathbb{Q}_1 1=p$, from (14) we have

$$\left\|\frac{[ns]}{n}\left(\Lambda_{[ns]}-\Lambda\right)f\right\|_T\leqslant\mathbb{Q}\left|f\right|\cdot\left|\frac{1}{n}\sum_{k=1}^{[ns]}\left(\delta_k-\mathbb{Q}_1 1\right)\right|+$$
$$+\left\|\frac{1}{n}\sum_{k=1}^{[ns]}\left(\delta_k f\left(X_k\right)-\mathbb{Q}_1 f\right)\right\|_T+\left\|\frac{1}{n}\sum_{k=1}^{[ns]}\left(\left(1-\delta_k\right)f\left(X_k\right)-\mathbb{Q}_0 f\right)\right\|_T. \quad (15)$$

Using sequential SLLN (Theorem 1.1 in [2]), we obtain (13) for all three terms in the right hand side of (15). Theorem 2.1 is proved. □

Remark 2.1. The assumptions in Theorem 2.1 can not be weaken. But for sequential weak LLN

$$\left\|\frac{[ns]}{n}\left(\Lambda_{[ns]}-\Lambda\right)f\right\|_T\xrightarrow[n\to\infty]{p}0$$

only the validity of the assumption $\mathbb{Q}_j\left|f\right|<\infty,\ \ j=0,1,\ \ f\in\mathcal{F}$ is required.

In order to prove that $\mathcal{D}=T\otimes\mathcal{F}$ are uniform variants of the Glivenko-Cantelli theorem and the Donsker theorem we need some notations from bracketing entropy theory. Let $\mathcal{L}_q\left(\mathbb{Q}\right)$ be the space of functions $f:\mathfrak{X}\to\mathbb{R}$ with norm

$$\|f\|_{\mathbb{Q},q}=\left(\mathbb{Q}|f|^q\right)^{1/q}=\left\{\int\limits_{\mathfrak{X}}|f|^q d\mathbb{Q}\right\}^{1/q}.$$

To determine the complexity or entropy of a set of Borel measurable functions $\mathcal{F}$ it is necessary to define a concept of ε-brackets in $\mathcal{L}_q\left(\mathbb{Q}\right)$. So ε-bracket in $\mathcal{L}_q\left(\mathbb{Q}\right)$ is a pairs of functions $\varphi,\psi\in\mathcal{L}_q\left(\mathbb{Q}\right)$ such that $\mathbb{Q}\left(\varphi\left(X\right)\leqslant\psi\left(X\right)\right)=1$ and $\|\psi-\varphi\|_{\mathbb{Q},q}\leqslant\varepsilon$, that is, $\mathbb{Q}(\psi-\varphi)^q\leqslant\varepsilon^q$. Function $f\in\mathcal{F}$ is covered by bracket $[\varphi,\psi]$ if $Q\left(\varphi\left(X\right)\leqslant f\left(X\right)\leqslant\psi\left(X\right)\right)=1$. Note that functions φ and ψ may not belong to the set $\mathcal{F}$ but they must have finite norms. The bracketing number $N_{[]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q\left(\mathbb{Q}\right)\right)$ is the minimum number of ε-brackets in $\mathcal{L}_q\left(\mathbb{Q}\right)$ needed to cover the set $\mathcal{F}$ (see, [3, 4]):

$$N_{[\,]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q\left(\mathbb{Q}\right)\right)=\min\begin{cases}k:\ \text{for some } f_1,\ldots,f_k\in\mathcal{L}_q\left(\mathbb{Q}\right),\\ \mathcal{F}\subset\bigcup\limits_{i,j}\left[f_i,f_j\right]:\|f_j-f_i\|_{\mathbb{Q},q}\leqslant\varepsilon.\end{cases}$$

The number $H_q\left(\varepsilon\right)=\log N_{[]}\left(\varepsilon,\mathcal{F},\mathcal{L}_q\left(\mathbb{Q}\right)\right)$ is called the metric entropy of class $\mathcal{F}$ in $\mathcal{L}_q\left(\mathbb{Q}\right)$. The metric entropies of a class $\mathcal{F}$ in $\mathcal{L}_q\left(\mathbb{Q}_j\right),\ j=0,1$ is we denoted by $H_{jq}\left(\varepsilon\right)=\log N_{j[]}\left(\varepsilon,F,L_q\left(\mathbb{Q}\right)\right)$. Integrals of metric entropies are

$$J_{j[]}^{(q)}\left(\delta\right)=J_{j[]}\left(\delta,\mathcal{F},\mathcal{L}_q\left(\mathbb{Q}_j\right)\right)=\int_0^\delta\left(H_{jq}\left(\varepsilon\right)\right)^{1/2}d\varepsilon,\quad 0<\delta\leqslant1,\ j=0,1.$$

Let us recall the important properties of numbers $N_{[]}\left(.\right)$. They tend to $+\infty$ when $\varepsilon\downarrow0$. However, for the Donsker theorems they should converge to $+\infty$ not very fast. This rate of convergence is measured by integrals $J^{(q)}_{j[]}(\delta)$ (for more details, see [3, 4]). Let us prove stronger properties of considered random fields and introduce following normalized empirical processes on $\mathcal{D} = T \otimes \mathcal{F}$:

$$\mathbb{Y}_n(s; f) = \mathbb{Y}_{0n}(s; f) + \mathbb{Y}_{1n}(s; f),$$

$$\mathbb{Z}_n(s; f) = \sqrt{n}\mathbb{Y}_n(s; f) = \mathbb{Z}_{0n}(s; f) + \mathbb{Z}_{1n}(s; f),$$

where for $j = 0, 1$

$$\mathbb{Y}_{jn}(s; f) = \frac{[ns]}{n}\mathbb{Y}_{j[ns]}(s; f),$$

$$\mathbb{Y}_{0n}(s; f) = \frac{[ns]}{n}\sum_{k=1}^{[ns]}((1-\delta_k) f(X_k) - \mathbb{Q}_0 f(X_k)),$$

$$\mathbb{Y}_{1n}(s; f) = \frac{1}{n}\sum_{k=1}^{[ns]}(\delta_k f(X_k) - \mathbb{Q}_1 f(X_k)),$$

$$\mathbb{Z}_{jn}(s; f) = \sqrt{n}\mathbb{Y}_{jn}(s; f) = \sqrt{\frac{[ns]}{n}}G_{j[ns]}f,$$

with $\mathbb{Z}_{jn}(1; f) = G_{jn}f$.

Let $l^\infty(D)$ be a space of all bounded functions on $D = T \otimes \mathcal{F}$ with the supremum norm $\|.\|_{\mathcal{D}}$. In what follows we show that the role of $s \in T$ is negligible in the LLN theorems. Let P^* be the outer probability.

Theorem 2.2. *There exists a universal constant $\mathbb{C}$ such that for every $\varepsilon > 0$*

$$P^*(\|\mathbb{Y}_n(s; f)\|_D > 4\varepsilon) \leqslant 2\,\mathbb{C}\max_{j=0,1} P^*\left(\|\mathbb{Y}_{jn}(1; f)\|_{\mathcal{F}} > \varepsilon\right). \tag{16}$$

Proof. For $(s; f) \in \mathcal{D}$ we have $|\mathbb{Y}_n(s; f)| \leqslant 2\max_{j=0,1}\|\mathbb{Y}_{jn}(s; f)\|_{\mathcal{F}}$. Hence

$$\|\mathbb{Y}_n(s; f)\|_{\mathcal{D}} \leqslant 2\max_{j=0,1}\sup_{0\leqslant s\leqslant 1}\|\mathbb{Y}_{jn}(s; f)\|_{\mathcal{F}}. \tag{17}$$

In the right hand side of (17) the parameter s may take values $\frac{k}{n}$ with $k = 1, \ldots, n$. Because $\mathbb{Y}_{jn}(s; f) = \frac{[ns]}{n}\mathbb{Y}_{j[ns]}(s; f) = \frac{[ns]}{n}\left(\mathbb{Q}_{j[ns]} - \mathbb{Q}_j\right) f, j = 0, 1$, we obtain from (17) that

$$\|\mathbb{Y}_n(s; f)\|_{\mathcal{D}} \leqslant 2\max_{j=0,1}\max_{1\leqslant k\leqslant n}\frac{k}{n}\|(\mathbb{Q}_{jk} - \mathbb{Q}_j) f\|_{\mathcal{F}}. \tag{18}$$

It follows from the Ottaviani inequality A.1.1. [3] that

$$P^*\left(\max_{1\leqslant k\leqslant n}\frac{k}{n}\|(\mathbb{Q}_{jk} - \mathbb{Q}_j) f\|_{\mathcal{F}} > \varepsilon\right) \leqslant \frac{P^*\left(\|(\mathbb{Q}_{jn} - \mathbb{Q}_j) f\|_{\mathcal{F}} > \varepsilon\right)}{1 - \max\limits_{1\leqslant k\leqslant n} P^*\left(\frac{k}{n}\|(\mathbb{Q}_{jk} - \mathbb{Q}_j) f\|_{\mathcal{F}} > \varepsilon\right)}, \; j = 0, 1. \tag{19}$$

Thus, the numerator of (19) converges to zero as $n \to \infty$ on condition that $\mathcal{F}$ is a weak Glivenko-Cantelli class. The term

$$\max_{1\leqslant k\leqslant n} P^*\left(\frac{k}{n}\|(Q_{jk} - Q_j) f\|_{\mathcal{F}} > \varepsilon\right)$$

indexed by $k \leqslant n$ can be controlled with the help of inequality

$$k\|(\mathbb{Q}_{jk} - \mathbb{Q}_j) f\|_{\mathcal{F}} \leqslant 2\sum_{k=1}^{n_0} F(X_k) + 2n_0 P^* F, \; j = 0, 1 \tag{20}$$

for an envelope function F of the class $\mathcal{F}$. For sufficiently large n_0 the terms indexed by $k > n_0$ are bounded away from 1 by the uniform weak LLN for Q_{jn}, $j = 0, 1$. Moreover, the denominator in (19) is bounded away from zero. Using inequalities (19) and (20) twice, we obtain (16) from (17) and (18). Theorem 2.2 is proved. □

Let us introduce some definitions of uniform weak and strong LLN [2] and adapt them to our processes.

Definition 2.1. *A class of measurable functions $\mathcal{F}$ is a sequential weak Glivenko-Cantelli class if*

$$\|\mathbb{Y}_n(s; f)\|^*_{\mathcal{D}} \xrightarrow[n\to\infty]{P} 0.$$

Definition 2.2. *A class of measurable functions $\mathcal{F}$ is a weak Glivenko-Cantelli class if*

$$\mathbb{Y}_n(1; \cdot)^* \xrightarrow[n\to\infty]{P} 0,$$

where $\mathbb{Y}_n(1; \cdot)^*$ is the measurable cover function of $\mathbb{Y}_n(1; \cdot)$.

Definition 2.3. *A class of measurable functions $\mathcal{F}$ is a sequential strong Glivenko-Cantelli class if*

$$\|\mathbb{Y}_n(s; f)\|^*_{\mathcal{D}} \xrightarrow[n\to\infty]{a.s.} 0.$$

Definition 2.4. *A class of measurable functions $\mathcal{F}$ is a strong Glivenko-Cantelli class if*

$$\mathbb{Y}_n(1; \cdot)^* \xrightarrow[n\to\infty]{a.s.} 0.$$

Because $\|\mathbb{Y}_n(1; \cdot)\|_{\mathcal{F}} \leqslant \|\mathbb{Y}_n(s; f)\|_{\mathcal{D}}$ then by Theorem 2.2 for every $\varepsilon > 0$ we have

$$P^*\left(\|\mathbb{Y}_n(1; \cdot)\|_{\mathcal{F}} > 2\varepsilon\right) \leqslant P^*\left(\|\mathbb{Y}_n(s; f)\|_{\mathcal{D}} > 2\varepsilon\right) \leqslant CP^*\left(\|\mathbb{Y}_n(1; \cdot)\|_{\mathcal{F}} > \varepsilon\right). \tag{21}$$

Taking into account (21), we have

Corollary 2.1. *A class $\mathcal{F}$ is a sequential weak (or strong) Glivenko-Cantelli class if and only if it is a weak (or strong) Glivenko-Cantelli class.*

Consider singleton set of measurable functions $\{f\}$. If $Q|f| < \infty$ then by weak LLN

$$\|\mathbb{Y}_n(1; \cdot)\|_{\{f\}} = (\mathbb{Q}_n - \mathbb{Q}) f \xrightarrow[n\to\infty]{P} 0,$$

and by Corollary 2.1 the singleton set $\{f\}$ is a sequential weak Glivenko-Cantelli class.

Definition 2.5. *A class of measurable functions $\mathcal{F}$ is a sequential complete Glivenko-Cantelli class if*

$$\sum_{n=1}^{\infty} P\left(\|\mathbb{Y}_n(s; f)\|^*_{\mathcal{D}} > 1\right) < \infty \tag{22}$$

and $\|\mathbb{Y}_n(s; f)\|^*_{\mathcal{D}} \xrightarrow[n\to\infty]{C} 0.$

Definition 2.6. *A class of measurable functions $\mathcal{F}$ is a complete Glivenko-Cantelli class if*

$$\|\mathbb{Y}_n(1; \cdot)\|^*_{\mathcal{F}} \xrightarrow[n\to\infty]{C} 0.$$

By introducing summation in each side of inequality (21) we have

Corollary 2.2. *A class of measurable functions $\mathcal{F}$ is a sequential complete Glivenko-Cantelli class if only if it is a complete Glivenko-Cantelli class.*

The sequential SLLN was proved in Theorem 2.1 in terms of the second moment condition. But such results can be established by bracketing entropy.

Theorem 2.3. *Let us assume that*

$$\mathcal{F} \subset \mathcal{L}_2(\mathbb{Q}_j) \ \ \text{and} \ \ J^{(2)}_{j[]}(1) < \infty, \ j = 0, 1. \tag{23}$$

Then $\mathcal{F}$ is a sequential strong Glivenko-Cantelli class, that is,

$$\left\| \frac{[ns]}{n} \left(\Lambda_{[ns]} - \Lambda\right) f \right\|^*_{\mathcal{D}} \xrightarrow[n\to\infty]{a.s.} 0. \tag{24}$$

Proof. Let us obtain almost sure convergence (24) in terms of the complete convergence

$$\left\| \frac{[ns]}{n} \left(\Lambda_{[ns]} - \Lambda\right) f \right\|^*_{\mathcal{D}} \xrightarrow[n\to\infty]{C} 0. \tag{25}$$

Consider Corollary 2.2. In order to prove (25) it is enough to prove

$$\|(\Lambda_n - \Lambda) f\|^*_{\mathcal{F}} \xrightarrow[n\to\infty]{C} 0. \tag{26}$$

Taking into acount (14), we have

$$(\Lambda_n - \Lambda) f = (1 - p_n) U_{1n}(f) - p_n U_{0n}(f) - (p_n - p) \mathbb{Q} f, \tag{27}$$

where $U_{jn}(f) = \int_{\mathfrak{X}} f d(\mathbb{Q}_{jn} - \mathbb{Q}_j)$, $j = 0, 1$. Using Proposition 3.3 [2] with the condition $\mathbb{Q}_j f^2 < \infty$, $j = 0, 1$, we obtain

$$U_{jn}(f) \xrightarrow[n\to\infty]{C} 0, j = 0, 1 \tag{28}$$

Using the Berstein inequality [5],

$$\sum_{n=1}^{\infty} P\left(|p_n - p| > \varepsilon\right) \leqslant 2 \sum_{n=1}^{\infty} \exp\left(-\frac{n\varepsilon^2}{4}\right) < \infty, \quad \varepsilon > 0,$$

we obtain

$$p_n \xrightarrow[n\to\infty]{C} p. \tag{29}$$

Statements (26) and (25) follow from (27)–(29). This completes the proof of (24) and Theorem 2.3.

3. Sequential uniform central limit theorem

Let us consider the sequential specially normalized empirical $\mathcal{D} = T \otimes \mathcal{F}$ — indexed random fields defined by relation (12). It was proved under the mild conditions [1] that

$$\Delta_n(1; f) \underset{n\to\infty}{\Rightarrow} \Delta f \ \text{ in } \ l^\infty(\mathcal{F}), \tag{30}$$

where $\{\Delta f, f \in \mathcal{F}\}$ is a Gaussian fields with zero mean and subject to hypothesis $\mathcal{H}$ that it coincides with the $\mathbb{Q}$-Brownian bridge with covariance (8). Here we extend convergence (30) to the sequential field (12). To begin with we prove that two-dimensional vector-field

$$\{(\mathbb{Z}_n(s; f), \mathbb{Z}_{1n}(t; g)), (s; f), (t; g) \in \mathcal{D}\} \tag{31}$$

weakly converges to corresponding Gaussian field uniformly with respect to semimetric of product space $l^\infty(\mathcal{D}) \otimes l^\infty(\mathcal{D})$ for every Donsker class of measurable functions $\mathcal{F}$.

Theorem 3.1. *Let us consider the class $\mathcal{F}$ such that*

$$\mathcal{F} \subset \mathcal{L}_2(\mathbb{Q}_j) \quad and \quad J^{(2)}_{j[\,]}(1) < \infty,\ j = 0, 1. \tag{32}$$

Then for $n \to \infty$ sequence of random vector-fields (31) weakly converge in $l^\infty(\mathcal{D}) \otimes l^\infty(\mathcal{D})$ to the Kiefer-Müller-type Gaussian field $\{(\mathbb{Z}(s; f), \mathbb{Z}_1(t; g)), (s; f), (t; g) \in \mathcal{D}\}$ with zero mean and covariance structure

$$\begin{aligned} &\operatorname{cov}(\mathbb{Z}(s; f), \mathbb{Z}(t; g)) = \min(s; t)\{\mathbb{Q}fg - \mathbb{Q}f\mathbb{Q}g\}, \\ &\operatorname{cov}(\mathbb{Z}_1(s; f), \mathbb{Z}_1(t; g)) = \min(s; t)\{\mathbb{Q}_1 fg - \mathbb{Q}_1 f\mathbb{Q}_1 g\}, \\ &\operatorname{cov}(\mathbb{Z}(s; f), \mathbb{Z}_1(t; g)) = \min(s; t)\{\mathbb{Q}_1 fg - \mathbb{Q}f\mathbb{Q}_1 g\}. \end{aligned} \tag{33}$$

Proof. Consider the first condition in (32). Then for the fixed $f \in \mathcal{F}$ it follows that $\mathbb{Q}_j f^2 < \infty$, $j = 0, 1$ and hence $\mathbb{Q}f^2 = \mathbb{Q}_0 f^2 + \mathbb{Q}_1 f^2 < \infty$. For every such Donsker class $\mathcal{F}$ with the second condition in (32) the sequences $\mathbb{Z}_n(s; f)$ and $\mathbb{Z}_{1n}(t; g)$ are asymptotically tight (see, Lemma 1.3.8 in [3]). There exists a tight Borel measurable version of Gaussian processes $\mathbb{Z}(s; f)$ and $\mathbb{Z}_1(t; g)$, that is, the Kiefer-Müller processes with zero mean and jointly covariances (32). Tightness and measurability of limiting processes $\mathbb{Z}(\cdot, \cdot)$ and $\mathbb{Z}_1(\cdot, \cdot)$ are equivalent to the existence of versions of all sample paths $(s; f) \mapsto \mathbb{Z}(s; f)$, $(t; g) \mapsto \mathbb{Z}_1(t; g)$ uniformly bounded and uniformly continuous with respect to the corresponding semimetrics with squares given by (see, [3], p. 226)

$$E(\mathbb{Z}(s; f) - \mathbb{Z}(t; g))^2 = |s - t|\left[\sigma^2_{\mathbb{Q}}(f)\, I(s > t) + \sigma^2_{\mathbb{Q}}(g)\, I(s \leqslant t)\right] + \min(s; t)\, \sigma^2_{\mathbb{Q}}(f - g),$$

$$E(\mathbb{Z}_1(s; f) - \mathbb{Z}_1(t; g))^2 = |s - t|\left[\sigma^2_{\mathbb{Q}_1}(f)\, I(s > t) + \sigma^2_{\mathbb{Q}_1}(g)\, I(s \leqslant t)\right] + \min(s; t)\, \sigma^2_{\mathbb{Q}_1}(f - g),$$

where $\sigma^2_{\mathbb{Q}}(f) = \mathbb{Q}(f - \mathbb{Q}f)^2$, $\sigma^2_{\mathbb{Q}_1}(f) = \mathbb{Q}_1(f - \mathbb{Q}_1 f)^2$.

On the other hand, the considered vector-field is the normalized sequential sum of independent and identically distributed random vectors

$$(\mathbb{Z}_n(s; f), \mathbb{Z}_{1n}(t; g)) = n^{-1/2} \sum_{k=1}^{[n(s \wedge t)]} (f(X_k) - \mathbb{Q}f, \delta_k g(X_k) - \mathbb{Q}_1 g). \tag{34}$$

Then by the multivariate CLT the marginals of the sequence of vector-fields converge to the marginals of a Gaussian vector-valued field with zero mean and covariance matrix defined by structure (33). Vector-field (34) is element of $l^\infty(\mathcal{D}) \otimes l^\infty(\mathcal{D})$, and it also induces tight sequences of distributions in product space by Lemma 1.4.3 [3].

Covariance structure of vector (34) has the form

$$\begin{aligned} &\operatorname{cov}(\mathbb{Z}_n(s; f), \mathbb{Z}_n(t; g)) = \frac{\min([ns], [n, t])}{n}\{\mathbb{Q}fg - \mathbb{Q}f\mathbb{Q}g\}, \\ &\operatorname{cov}(\mathbb{Z}_{1n}(s; f), \mathbb{Z}_{1n}(t; g)) = \frac{\min([ns], [n, t])}{n}\{\mathbb{Q}_1 fg - \mathbb{Q}_1 f\mathbb{Q}_1 g\}, \\ &\operatorname{cov}(\mathbb{Z}_n(s; f), \mathbb{Z}_{1n}(t; g)) = \frac{\min([ns], [n, t])}{n}\{\mathbb{Q}_1 fg - \mathbb{Q}f\mathbb{Q}_1 g\}, \end{aligned} \tag{35}$$

and we see that (33) is the limiting value of (35). These arguments complete the proof of Theorem 3.1. □

Remark 3.1. Consider relation (34). At $g \equiv 1$ for $s, t \in T$ and $f \in \mathcal{F}$ we have $\mathbb{Q}_1 1 \equiv p$ and hence

$$\operatorname{cov}(\mathbb{Z}(s, f), \mathbb{Z}_1(t; 1) = \min(s, t)\{\mathbb{Q}_1 f - p\mathbb{Q}f\} = \min(s, t) \cdot \Lambda f. \tag{36}$$

Because covariance (36) is zero for any $s,t \in T$ and $f \in \mathcal{F}$ under hypothesis $\mathcal{H}$ then Kiefer-Müller field $\{\mathbb{Z}(s;f),\ (s,f) \in \mathcal{D}\}$ and rescaled Wiener process $\{\mathbb{Z}_1(t;1),\ t \in T\}$ with covariance $\min(s,t)\,p(1-p)$ are independent. We use this fact in the following theorem. Now we consider the intermediate random field

$$\left\{\Delta_n^*(s;f) = \frac{[ns]}{n^{1/2}} \cdot \left(\Lambda_{[ns]} - \Lambda\right) f,\ \ (s;f) \in \mathcal{D}\right\}, \tag{37}$$

connected by $\Delta_n(s;f)$ in terms of $\Delta_n^*(s;f) = (p_n(1-p_n))^{1/2} \cdot \Delta_n(s;f)$. Process (37) plays a supporting role in the study of basic process (12) which property of weak convergence to a corresponding Gaussian process is contained in the following statement.

Theorem 3.2. *Under conditions of Theorem 3.1 for $n \to \infty$ we have*

$$\Delta_n(s;f) \underset{n\to\infty}{\Rightarrow} \Delta(s;f) \ \ in\ l^\infty(\mathcal{D}), \tag{38}$$

where $\{\Delta(s;f),\ (s;f) \in \mathcal{D}\}$ is a Gaussian field with zero mean and hypothesis H is valid. For $s,t \in T$ and $f,g \in \mathcal{F}$ it coincides with Kiefer-Müller random field with covariance

$$\operatorname{cov}(\Delta(s,t)\,\Delta(t;g)) = \min(s,t)\,(\mathbb{Q}fg - \mathbb{Q}f\mathbb{Q}g). \tag{39}$$

Proof. Let us consider process (37) and represent it in the form of linear functional of sequential subempirical processes

$$\Delta_n^*(s;f) = \left(\frac{[ns]}{n}\right)^{1/2} \cdot \left(G_{1[ns]}f - pG_{[ns]}f - \mathbb{Q}fG_{1[ns]}1\right) + R_n(s;t) = \Delta_n^0(s;f) + R_n(s;f), \tag{40}$$

where $R_n(s;f) = n^{-1/2}[ns]\left(p_{[ns]} - p\right)\left(\mathbb{Q}_{[ns]}f - \mathbb{Q}f\right)$ and hence

$$\|R_n(s;f)\|_{\mathcal{D}} = o_p(1),\ \ n \to \infty. \tag{41}$$

We consider only $\Delta_n^0(s;f)$. It is not difficult to see that $\Delta_n^0(s;f)$ have zero mean and for $s,t \in T,\ f,g \in F$ its covariance is

$$\operatorname{cov}\left(\Delta_n^{(0)}(s;f),\Delta_n^{(0)}(t;g)\right) = \frac{\min([ns],[nt])}{n}\sum_{j=1}^{9} C_j, \tag{42}$$

where

$$\begin{aligned}
&C_1 = \mathbb{Q}_1fg - \mathbb{Q}_1f\mathbb{Q}_1g, && C_2 = -p(\mathbb{Q}_1fg - \mathbb{Q}f\mathbb{Q}_1g), && C_3 = -(1-p)\,\mathbb{Q}f\mathbb{Q}_1g,\\
&C_4 = -p(\mathbb{Q}_1fg - \mathbb{Q}g\mathbb{Q}_1f), && C_5 = p^2(\mathbb{Q}fg - \mathbb{Q}f\mathbb{Q}g), && C_6 = p\mathbb{Q}f(\mathbb{Q}_1g - p\mathbb{Q}g),\\
&C_7 = -(1-p)\,\mathbb{Q}g\mathbb{Q}_1f, && C_8 = p\mathbb{Q}g(\mathbb{Q}_1f - p\mathbb{Q}f), && C_9 = p(1-p)\,\mathbb{Q}f\mathbb{Q}g.
\end{aligned} \tag{43}$$

Taking into account Theorem 3.1, we have

$$\Delta_n^0(s;f) \Rightarrow \Delta^0(s;f) \ \text{ in } l^\infty(\mathcal{D}), \tag{44}$$

where $\Delta^0(\cdot;\cdot)$ is a mean zero Gaussian process and accordingly to (42) its covariance is

$$\operatorname{cov}\left(\Delta^0(s;f),\Delta^0(t;g)\right) = \min(s,t)\sum_{j=1}^{9} C_j, \tag{45}$$

where C_j are defined in (43). Assuming that hypothesis $\mathcal{H}$ is valid and taking into account Remark 3.1, it is easy to obtain that

$$\operatorname{cov}\left(\Delta^0(s;f),\Delta^0(t;g)\right) = p\,(1-p)\min(s,t)\,(\mathbb{Q}fg - \mathbb{Q}f\mathbb{Q}g)\,, \tag{46}$$

and

$$(p\,(1-p))^{-1/2}\cdot\Delta_n^0\,(s;f) \underset{n\to\infty}{\Rightarrow} \Delta\,(s;f) \ \text{ in } \ l^\infty\,(\mathcal{D})\,. \tag{47}$$

Now relation (38) follows from (39)–(47). Theorem 3.2 is proved. □

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Последовательные эмпирические процессы независимости

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Мы доказываем равномерные усиленные законы больших чисел и центральную предельную теорему для специальных последовательных эмпирических процессов независимости для специальных классов измеримых функций.

Ключевые слова: последовательные эмпирические процессы, метрическая энтропия, теоремы Гливенко-Кантелли и Донскера.