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On a Second Order Linear Parabolic Equation with Variable Coefficients in a Non-Regular Domain of $\mathbb{R}^3$

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This paper is devoted to the study of the following variable-coefficient parabolic equation in non-divergence form

$$\partial_t u - \sum_{i=1}^{2} a_i(t,x_1,x_2)\partial_{ii}u + \sum_{i=1}^{2} b_i(t,x_1,x_2)\partial_i u + c(t,x_1,x_2)u = f(t,x_1,x_2),$$

subject to Cauchy-Dirichlet boundary conditions. The problem is set in a non-regular domain of the form

$$Q = \left\{(t,x_1) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times \left]0,b\right[,$$

where φ_k, $k = 1,2$ are "smooth" functions. One of the main issues of this work is that the domain can possibly be non-regular, for instance, the singular case where φ_1 coincides with φ_2 for $t = 0$ is allowed. The analysis is performed in the framework of anisotropic Sobolev spaces by using the domain decomposition method. This work is an extension of the constant-coefficients case studied in [15].

Keywords: parabolic equations, non-regular domains, variable coefficients, anisotropic Sobolev spaces.
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1. Introduction and main results

This work is devoted to the study of the following two-space dimensional non-divergence parabolic equation

$$\begin{cases} \partial_t u + \mathcal{L}u = f \in L^2(Q), \\ 1\ u|_{\partial Q \setminus \Sigma_T} = 0, \end{cases} \tag{1.1}$$

where

$$\mathcal{L} = -\sum_{i=1}^{2} a_i(t,x_1,x_2)\partial_{ii} + \sum_{i=1}^{2} b_i(t,x_1,x_2)\partial_i + c(t,x_1,x_2),$$

with $\partial_i = \dfrac{\partial}{\partial x_i}$, $\partial_{ii} = \dfrac{\partial^2}{\partial x_i^2}$, $i = 1,2$. $L^2(Q)$ stands for the space of square-integrable functions on Q with the measure $dt\,dx_1dx_2$, ∂Q is the boundary of Q, Σ_T is the part of the boundary of Q where $t = T$ and the coefficients a_i, b_i, $i = 1,2$ and c satisfy non-degeneracy-assumptions (to be made more precise later). Here Q (see, Fig. 1) is the three-dimensional non-cylindrical domain

$$Q = \left\{(t,x_1) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times \left]0,b\right[,$$

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where T and b are positive numbers, φ_1 and φ_2 are two Lipschitz continuous real-valued functions on $[0,T]$ satisfying

$$\varphi(t) := \varphi_2(t) - \varphi_1(t) > 0, \ \forall t \in]0,T] \text{ and } \varphi(0) = 0.$$

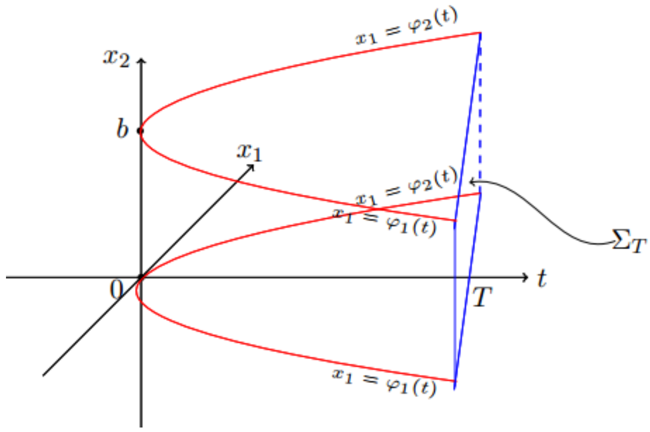

Fig. 1. The non-regular domain Q

Besides being interesting in itself, Problem (1.1) governs, for instance, the concentration of the biological oxygen demand in water in the case of a river with variable width and constant depht, see for example, similar problems in [1] and [31]. Also, the particular form of the operator $\mathcal{L}$ helps us to prove the "energy" type estimate of Proposition 2.1 which is essential in proving the existence of solutions to Problem (1.1).

The difficulty related to this kind of problems (in addition to the presence of variable coefficients) comes from this singular situation for evolution problems, i.e., φ_1 is allowed to coincide with φ_2 for $t = 0$, which prevents the domain Q to be transformed into a regular domain without the appearance of some degenerate terms in the parabolic equation, see for example Sadallah [30]. On the other hand, we cannot recast such problems in semigroups setting like in [6] and [27]. Indeed, since the initial condition is defined on a measure zero set, then the semigroup generating the solution cannot be defined.

It is well known that there are two main approaches for the study of boundary value problems in such non-smooth domains. We can work directly in the non-regular domains and we obtain singular solutions (see, for example [3,16,18] and [20]), or we impose conditions on the non-regular domains (and on the coefficients) to obtain regular solutions (see, for example [2,17] and [30]). It is the second approach that we follow in this work. So, let us consider the anisotropic Sobolev space

$$\mathcal{H}_0^{1,2}(Q) = \left\{ u \in \mathcal{H}^{1,2}(Q) : u|_{\partial Q \setminus \Sigma_T} = 0 \right\},$$

with

$$\mathcal{H}^{1,2}(Q) = \{ u : \partial_t u, \partial^\alpha u \in L^2(Q), |\alpha| \leqslant 2 \},$$

where

$$\alpha = (i_1, i_2) \in \mathbb{N}^2, |\alpha| = i_1 + i_2, \partial^\alpha u = \partial_1^{i_1} \partial_2^{i_2} u.$$

The space $\mathcal{H}^{1,2}(Q)$ is equipped with the natural norm, that is

$$\|u\|_{\mathcal{H}^{1,2}(Q)} = \left(\|\partial_t u\|^2_{L^2(Q)} + \sum_{|\alpha| \leqslant 2} \|\partial^\alpha u\|^2_{L^2(Q)} \right)^{1/2}.$$

In this paper we prove that Problem (1.1) admits a unique solution u in $\mathcal{H}^{1,2}(Q)$, under the following additional assumptions on the smooth differentiable coefficients c, a_i, b_i, $i = 1, 2$ and on the functions of parametrization φ_k, $k = 1, 2$,

$$\varphi_k'(t)\,\varphi(t) \to 0 \quad \text{as} \quad t \to 0, \quad k = 1, 2, \tag{1.2}$$

$$\begin{cases} a_i > 0 \ \ \text{(parabolicity condition)} \\ a_i, b_i, c, \partial_t a_i, \partial_i a_i \in L^\infty(Q), \ i = 1, \ 2, \end{cases} \tag{1.3}$$

with $|a_i| \leqslant c_0$, $|\nabla a_i| \leqslant c_1$, $|b_i| \leqslant c_2$, $|c| \leqslant c_3$, $a_i a_j \geqslant a_0 > 0$ $(i; j = 1, \ 2)$, $b_i^2 \geqslant b_0 > 0$, $c^2 \geqslant d_0 > 0$, where $c_0, c_1, c_2, c_3, a_0, b_0$ and d_0 are positive constants.

Our main result is

Theorem 1.1. *We assume that φ_1 and φ_2 fulfil the condition (1.2), and the coefficients a_i, b_i, $i = 1, 2$, and c fulfil the condition (1.3), then the operator*

$$\mathcal{L} = \partial_t - \sum_{i=1}^{2} a_i(t, x_1, x_2)\partial_{ii} + \sum_{i=1}^{2} b_i(t, x_1, x_2)\partial_i + c(t, x_1, x_2)$$

is an isomorphism from $\mathcal{H}_0^{1,2}(Q)$ into $L^2(Q)$.

The case $a_1 = a_2 = 1$, $b_1 = b_2 = c = 0$, corresponding to the heat operator has been studied in [15] and [17] both in bi-dimensional and multidimensional cases.

Whereas parabolic equations with variables coefficients in cylindrical domains are well studied, the literature concerning such problems in non-cylindrical domains does not seem to be very rich, see [24] for the case of smooth coefficients and [28] for the case of discontinuous coefficients. Concerning parabolic equations in time-varying domains we can find in Fichera [9] and Oleinik [29] solvability results for non-divergence parabolic equations. For the divergence form case, see [5, 14] and [25]. In the case of Hölder spaces functional framework, we can find in Baderko [4] results for non-cylindrical domains of the same kind but which cannot include our domain. In [10], we can find Wiener type criterion in the framework of continuous spaces which cannot include our L^2-case.

Our work is motivated by the interest of researchers for many mathematical questions related to non-regular domains. During the last decades and since many applied problems lead directly to boundary-value problems in "bad" domains, numerous authors studied partial differential equations in non-smooth domains. Among these we can cite [7, 8, 11, 12, 19, 21, 22, 32] and the references therein.

The organization of this paper is as follows. In Section 2, we divide the proof of Theorem 1.1 into three steps:

a) We prove well-posedness results for Problem (1.1) when Q is replaced by the truncated domain

$$Q_\alpha = \left\{(t, x_1) \in \mathbb{R}^2 : \alpha < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times \left]0, b\right[,$$

with $\alpha > 0$, (Theorem 2.1).

b) We approximate Q by a sequence (Q_n), $n \in \mathbb{N}^*$, of such truncated regular domains and we establish a uniform estimate (see Proposition 2.1) of the type

$$\|u_n\|_{\mathcal{H}^{1,2}(Q_n)} \leqslant K \|f\|_{L^2(Q)},$$

where u_n is the solution of Problem (1.1) in Q_n and K is a constant independent of n.

c) We build a solution u of Problem (1.1), by considering $\widetilde{u_n}$ the 0-extension to Q of the solutions u_n (u_n, $n \in \mathbb{N}^*$ exists by Theorem 2.1), and showing (in virtue of Proposition 2.1) that $\widetilde{u_{n_k}} \rightharpoonup u$, weakly in $L^2(Q)$, for a suitable increasing sequence of integers $(n_k)_{k \geqslant 1}$.

Note that this work may be extended at least in the following directions:

1. The function f on the right-hand side of the equation of Problem (1.1), may be taken in $L^p(Q)$, $p \in]1,\infty[$. The domain decomposition method used here does not seem to be appropriate for the space $L^p(Q)$ when $p \neq 2$. An idea for this extension can be found in [13] or in [23].

2. The bi-dimensional case in x, can be naturally extended to an upper dimension in x, such as, for example, the following problem

$$\partial_t u - \sum_{i=1}^{N} a_i(t,x_1,...,x_N)\partial_{ii}u + \sum_{i=1}^{N} b_i(t,x_1,...,x_N)\partial_i u + c(t,x_1,...,x_N)u = f(t,x_1,...,x_N),$$

in the domain

$$\left\{(t,x_1,...,x_N) \in \mathbb{R}^{N+1} : 0 < t < T, 0 \leqslant \sqrt{x_1^2 + ... + x_N^2} < \varphi(t)\right\}, \ N \geqslant 2.$$

These questions will be developed in forthcoming works.

2. Proof of Theorem 1.1

We divide the proof of Theorem 1.1 into three steps.

2.1. Step 1: case of a truncated domain Q_α which can be transformed into a parallelepiped

In this subsection, we replace Q by

$$Q_\alpha = \left\{(t,x_1) \in \mathbb{R}^2 : \alpha < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0,b[,$$

with $\alpha > 0$, (see, Fig. 2). Thus, we have $\varphi(\alpha) > 0$.

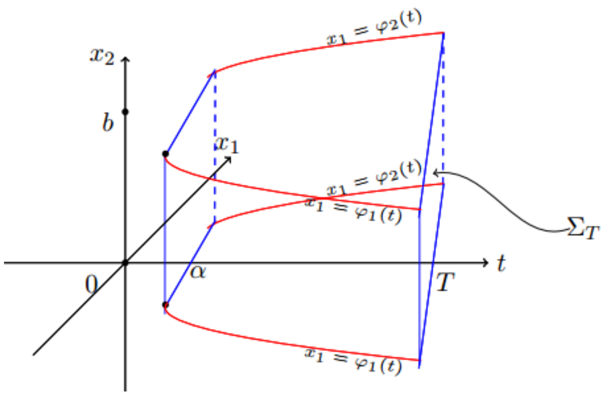

Fig. 2. The truncated domain Q_α

We can find a change of variable ψ mapping Q_α into the parallelepiped

$$P_\alpha =]\alpha, T[\times]0,1[\times]0,b[,$$

which leaves the variable t unchanged. ψ is defined as follows:

$$\begin{aligned} \psi: \quad & Q_\alpha \longrightarrow P_\alpha, \\ (t,x_1,x_2) & \longmapsto \psi(t,x_1,x_2) = (t,y_1,y_2) = \left(t, \frac{x_1 - \varphi_1(t)}{\varphi(t)}, x_2\right). \end{aligned}$$

The mapping ψ transforms the parabolic equation in the domain Q_α into a variable-coefficient parabolic equation in the parallelepiped P_α. Indeed, the equation

$$\partial_t u - \sum_{i=1}^{2} a_i(t,x_1,x_2)\partial_{ii}u + \sum_{i=1}^{2} b_i(t,x_1,x_2)\partial_i u + c(t,x_1,x_2)u = f(t,x_1,x_2)$$

in Q_α is equivalent to the following

$$\partial_t v - \sum_{i=1}^{2} a_i\,\widetilde{(t,y_1,y_2)}\partial_{ii}v + \sum_{i=1}^{2} b_i\,\widetilde{(t,y_1,y_2)}\partial_i v + c\,\widetilde{(t,y_1,y_2)}v = g(t,y_1,y_2)$$

in P_α, where $a_i\,\widetilde{(t,y_1,y_2)}$, $b_i\,\widetilde{(t,y_1,y_2)}$ and $c\,\widetilde{(t,y_1,y_2)}$ are defined by

$$a_1\,\widetilde{(t,y_1,y_2)} = \frac{a_1\left(t,\varphi(t)\,y_1+\varphi_1(t),y_2\right)}{\varphi^2(t)},\quad a_2\,\widetilde{(t,y_1,y_2)} = a_2\left(t,\varphi(t)\,y_1+\varphi_1(t),y_2\right),$$

$$b_1\,\widetilde{(t,y_1,y_2)} = \frac{b_1\left(t,\varphi(t)\,y_1+\varphi_1(t),y_2\right)}{\varphi(t)}\left[1-\varphi'(t)\,y_1-\varphi_1'(t)\right],$$

$$b_2\,\widetilde{(t,y_1,y_2)} = b_2\left(t,\varphi(t)\,y_1+\varphi_1(t),y_2\right),\qquad c\,\widetilde{(t,y_1,y_2)} = c\left(t,\varphi(t)\,y_1+\varphi_1(t),y_2\right),$$

and

$$g\left(t,y_1,y_2\right) = f\left(t,x_1,x_2\right),\ \ v\left(t,y_1,y_2\right) = u\left(t,x_1,x_2\right).$$

Since the functions a_i, b_i, $i=1,2,$ c and φ are bounded, and using the fact that the mapping ψ is tri-Lipschitz, then, it is easy to check the following

Lemma 2.1. *$u \in \mathcal{H}^{1,2}(Q_\alpha)$ if and only if $v \in \mathcal{H}^{1,2}(P_\alpha)$.*

The boundary conditions on v which correspond to the boundary conditions on u are the following

$$v|_{\partial P_\alpha \smallsetminus \Gamma_T} = 0,$$

where Γ_T is the part of the boundary of P_α where $t=T$. In the sequel, the variables (t,y_1,y_2) will be denoted again by (t,x_1,x_2).

Theorem 2.1. *The operator*

$$\mathcal{L}' = \partial_t - \sum_{i=1}^{2} a_i\,\widetilde{(t,x_1,x_2)}\partial_{ii} + \sum_{i=1}^{2} b_i\,\widetilde{(t,x_1,x_2)}\partial_i + c\,\widetilde{(t,x_1,x_2)}$$

is an isomorphism from $\mathcal{H}_0^{1,2}(P_\alpha)$ into $L^2(P_\alpha)$, with

$$\mathcal{H}_0^{1,2}(P_\alpha) = \left\{v \in \mathcal{H}^{1,2}(P_\alpha) : v|_{\partial P_\alpha \smallsetminus \Gamma_T} = 0\right\}.$$

Proof. Since the differentiable coefficients $a_i\,\widetilde{(t,x_1,x_2)}, b_i\,\widetilde{(t,x_1,x_2)}$, $i=1,2$ and $c\,\widetilde{(t,x_1,x_2)}$ are bounded in $\overline{P_\alpha}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [24]. □

We shall need the following result in order to justify all the calculus of the next subsection.

Lemma 2.2. *The space*

$$\left\{v \in H^4(P_\alpha) : v|_{\partial_p P_\alpha} = 0\right\}$$

is dense in the space

$$\left\{v \in \mathcal{H}^{1,2}(P_\alpha) : v|_{\partial_p P_\alpha} = 0\right\}.$$

Here, $\partial_p P_\alpha$ is the parabolic boundary of P_α and H^4 stands for the usual Sobolev space defined, for instance, in Lions-Magenes [26].

The proof of the above lemma may be found in [15].

Remark 2.1. *In Lemma 2.2, we can replace P_α by Q_α with the help of the change of variable ψ defined above.*

2.2. Step 2: uniform estimate

We denote $u_n \in \mathcal{H}^{1,2}(Q_n), n \in \mathbb{N}^*$, the solution of Problem (1.1) corresponding to a second member $f_n = f|_{Q_n} \in L^2(Q_n)$ in

$$Q_n = \left\{(t,x_1) \in \mathbb{R}^2 : \frac{1}{n} < t < T, \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0,b[\,.$$

Proposition 2.1. *There exists a constant K_1 independent of n such that*

$$\|u_n\|_{\mathcal{H}^{1,2}(Q_n)} \leqslant K_1 \|f_n\|_{L^2(Q_n)} \leqslant K_1 \|f\|_{L^2(Q)}\,.$$

In order to prove Proposition 2.1, we need some preliminary results.

Lemma 2.3. *Let $]\alpha,\beta[\subset \mathbb{R}$. There exists a constant K_2 (independent of α and β)such that*

$$\left\|w^{(j)}\right\|^2_{L^2(]\alpha,\beta[)} \leqslant K_2(\beta-\alpha)^{2(2-j)}\left\|w^{(2)}\right\|^2_{L^2(]\alpha,\beta[)}, \quad j=0,1,$$

for every $w \in H^2(]\alpha,\beta[) \cap H_0^1(]\alpha,\beta[)$, where $w^{(j)}$, $j = 1,\ 2$, denotes the derivative of order j of w on $]\alpha,\beta[$ and $w^{(0)} = w$.

Lemma 2.4. *For every $\epsilon > 0$, chosen such that $\varphi(t) \leqslant \epsilon$, there exists a constant C_1 independent of n such that for $i = 1, 2$*

$$\left\|\partial_i^j u_n\right\|^2_{L^2(Q_n)} \leqslant C_1 \epsilon^{2(2-j)} \|\partial_{ii} u_n\|^2_{L^2(Q_n)}, \quad j = 0,\ 1,$$

where $\partial_i^1 u_n = \partial_i u_n$ and $\partial_i^0 u_n = u_n$.

Proof. Replacing in Lemma 2.3 w by u_n and $]\alpha,\beta[$ by $]\varphi_1(t),\varphi_2(t)[$, for a fixed t, we obtain

$$\begin{aligned}\int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_i^j u_n\right)^2 dx_1 &\leqslant K_2(\varphi(t))^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_{ii}u_n)^2\, dx_1 \leqslant \\ &\leqslant K_2 \epsilon^{2(2-j)} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_{ii}u_n)^2\, dx_1,\end{aligned}$$

with $i = 1, 2$ and $j = 0, 1$. Integrating in the previous inequality with respect to t, then with respect to x_2, we get the desired result with $C_1 = K_2$. □

Proof of Proposition 2.1. Let us denote the inner product in $L^2(Q_n)$ by $\langle .,. \rangle$, then we have

$$\begin{aligned}&\|f_n\|^2_{L^2(Q_n)} = \left\|\partial_t u_n - \sum_{i=1}^{2} a_i(t,x_1,x_2)\partial_{ii}u_n + \sum_{i=1}^{2} b_i(t,x_1,x_2)\partial_i u_n + c(t,x_1,x_2)u_n\right\|^2_{L^2(Q_n)} = \\ &= \|\partial_t u_n\|^2_{L^2(Q_n)} + \sum_{i=1}^{2}\|a_i\partial_{ii}u_n\|^2_{L^2(Q_n)} + \sum_{i=1}^{2}\|b_i\partial_i u_n\|^2_{L^2(Q_n)} + \|cu_n\|^2_{L^2(Q_n)} - \\ &-2\sum_{i=1}^{2}\langle\partial_t u_n, a_i\partial_{ii}u_n\rangle + 2\sum_{i=1}^{2}\langle\partial_t u_n, b_i\partial_i u_n\rangle + 2\langle\partial_t u_n, cu_n\rangle - 2\sum_{i=1}^{2}\langle a_i\partial_{ii}u_n, b_1\partial_1 u_n\rangle - \\ &-2\sum_{i=1}^{2}\langle a_i\partial_{ii}u_n, b_2\partial_2 u_n\rangle - 2\sum_{i=1}^{2}\langle a_i\partial_{ii}u_n, cu_n\rangle + 2\sum_{i=1}^{2}\langle b_i\partial_i u_n, cu_n\rangle + \\ &+2\langle a_{11}\partial_{11}u_n, a_{22}\partial_{22}u_n\rangle - 2\langle b_1\partial_1 u_n, b_2\partial_2 u_n\rangle.\end{aligned}$$

1) Estimation of $-2\langle \partial_t u_n, a_i \partial_{ii} u_n \rangle,\ i = 1,\ 2$: We have

$$\partial_t u_n \partial_{ii} u_n = \partial_i \left(\partial_t u_n \partial_i u_n \right) - \frac{1}{2} \partial_t \left(\partial_i u_n \right)^2 .$$

Then

$$\begin{aligned} -2\langle \partial_t u_n, a_i \partial_{ii} u_n \rangle &= -2 \int_{Q_n} a_i \partial_t u_n \partial_{ii} u_n dt\ dx_1 dx_2 = \\ &= \int_{Q_n} a_i \left[-2\partial_i \left(\partial_t u_n \partial_i u_n \right) + \partial_t \left(\partial_i u_n \right)^2 \right] dt\ dx_1 dx_2 = \\ &= \int_{\partial Q_n} a_i \left[\left(\partial_i u_n \right)^2 \nu_t - 2\partial_t u_n \partial_i u_n \nu_i \right] d\sigma + \\ &\quad + \int_{Q_n} \left[2\partial_i a_{ii} \left(\partial_t u_n \partial_i u_n \right) - \partial_t a_{ii} \left(\partial_i u_n \right)^2 \right] dt\ dx_1 dx_2, \end{aligned}$$

where $\nu_t, \nu_i,\ i = 1, 2$ are the components of the unit outward normal vector at ∂Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t = \dfrac{1}{n}$, $x_2 = 0$ and $x_2 = b$ we have $u_n = 0$ and consequently $\partial_i u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_i = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_0^b \int_{\varphi_1(T)}^{\varphi_2(T)} a_i(T, x_1, x_2) \left(\partial_i u_n \right)^2 dx_1 dx_2$$

is nonnegative, since $a_i(T, x_1, x_2) > 0$. On the part of the boundary where $x_1 = \varphi_k(t),\ k = 1, 2$, we have

$$\nu_1 = \frac{(-1)^k}{\sqrt{1 + (\varphi_k')^2 (t)}}, \quad \nu_t = \frac{(-1)^{k+1} \varphi_k'(t)}{\sqrt{1 + (\varphi_k')^2 (t)}} \quad \text{and} \quad \nu_2 = 0.$$

Consequently, the corresponding boundary integral is

$$I_{n,i} = \sum_{k=1}^{2} (-1)^{k+i+1} \int_0^b \int_{\frac{1}{n}}^{T} a_i(t, \varphi_k(t), x_2) \varphi_k'(t) \left[\partial_i u_n \left(t, \varphi_k(t), x_2 \right) \right]^2\ dt\ dx_2.$$

Furthermore,

$$\left| \int_{Q_n} \partial_t a_i \left(\partial_i u_n \right)^2 dt\ dx_1 dx_2 \right| \leqslant c_1 \left\| \partial_i u_n \right\|^2_{L^2(Q_n)},$$

and for every $\epsilon > 0$

$$\begin{aligned} \left| \int_{Q_n} \partial_i a_i \left(\partial_t u_n \partial_i u_n \right) dt\ dx_1 dx_2 \right| &\leqslant c_1 \int_{Q_n} \left| \partial_t u_n \right| \left| \partial_i u_n \right| dt\ dx_1 dx_2 \leqslant \\ &\leqslant c_1 \frac{\epsilon}{2} \left\| \partial_t u_n \right\|^2_{L^2(Q_n)} + \frac{c_1}{2\epsilon} \left\| \partial_i u_n \right\|^2_{L^2(Q_n)}. \end{aligned}$$

Then for $i = 1, 2$ we have

$$-2\langle \partial_t u_n, \partial_i u_n \rangle \geqslant - \left| I_{n,1,i} \right| - \left| I_{n,2,i} \right| - c_1 \left\| \partial_i u_n \right\|^2_{L^2(Q_n)} - c_1 \epsilon \left\| \partial_t u_n \right\|^2_{L^2(Q_n)} - \frac{c_1}{\epsilon} \left\| \partial_i u_n \right\|^2_{L^2(Q_n)} \quad (2.1)$$

where

$$I_{n,k,i} = (-1)^{k+i+1} \int_0^b \int_{\frac{1}{n}}^{T} a_i(t, \varphi_k(t), x_2) \varphi_k'(t) \left[\partial_i u_n \left(t, \varphi_k(t), x_2 \right) \right]^2\ dt\ dx_2,\ k = 1, 2.$$

Lemma 2.5. *There exists a positive constant K_4 independent of n such that*

$$|I_{n,k,1}| \leqslant K_4\epsilon \left\|\partial_{11}u_n\right\|^2_{L^2(Q_n)}, \ k=1,\ 2,$$
$$|I_{n,k,2}| \leqslant K_4\epsilon \left\|\partial_{22}u_n\right\|^2_{L^2(Q_n)} + c_0\epsilon \left\|\partial_{12}u_n\right\|^2_{L^2(Q_n)}, \ k=1,\ 2,$$

where $\partial_{12}u_n = \dfrac{\partial^2 u_n}{\partial x_1 \partial x_2}$.

Proof. We convert the boundary integral $I_{n,1,1}$ into a surface integral by setting

$$\begin{aligned}\left[\partial_1 u_n\left(t,\varphi_1\left(t\right),x_2\right)\right]^2 &= -\left.\frac{\varphi_2\left(t\right)-x_1}{\varphi\left(t\right)}\left[\partial_1 u_n\left(t,x_1,x_2\right)\right]^2\right|_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)} = \\ &= -\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_1\left\{\frac{\varphi_2\left(t\right)-x_1}{\varphi\left(t\right)}\left[\partial_1 u_n\right]^2\right\} dx_1 = \\ &= -2\int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\varphi_2\left(t\right)-x_1}{\varphi\left(t\right)}\partial_1 u_n.\partial_{11}u_n\ dx_1 + \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{1}{\varphi\left(t\right)}\left[\partial_1 u_n\right]^2 dx_1.\end{aligned}$$

Then, we have

$$\begin{aligned}I_{n,1,1} &= -\int_0^b \int_{\frac{1}{n}}^T a_1(t,\varphi_1\left(t\right),x_2)\varphi_1'\left(t\right)\left[\partial_1 u_n\left(t,\varphi_1\left(t\right),x_2\right)\right]^2\ dt\ dx_2 = \\ &= -\int_{Q_n} a_1(t,\varphi_1\left(t\right),x_2)\frac{\varphi_1'\left(t\right)}{\varphi\left(t\right)}\left(\partial_1 u_n\right)^2 dt\ dx_1 dx_2 + \\ &\quad + 2\int_{Q_n} a_1(t,\varphi_1\left(t\right),x_2)\frac{\varphi_2\left(t\right)-x_1}{\varphi\left(t\right)}\varphi_1'\left(t\right)\left(\partial_1 u_n\right)\left(\partial_{11}u_n\right)\ dt\ dx_1 dx_2.\end{aligned}$$

Thanks to Lemma 2.4, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_1 u_n\right]^2 dx_1 \leqslant K_2\varphi\left(t\right)^2 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{11}u_n\right]^2 dx_1.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_1 u_n\right]^2 \frac{|\varphi_1'|}{\varphi} dx_1 \leqslant K_2\left|\varphi_1'\right|\varphi \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{11}u_n\right]^2 dx_1,$$

consequently

$$|I_{n,1,1}| \leqslant K_2\int_{Q_n} c_0\left|\varphi_1'\right|\varphi\left(\partial_{11}u_n\right)^2 dt\ dx_1 dx_2 + 2\int_{Q_n} c_0\left|\varphi_1'\right|\left|\partial_1 u_n\right|\left|\partial_{11}u_n\right|\ dt\ dx_1 dx_2,$$

since $\left|\dfrac{\varphi_2\left(t\right)-x_1}{\varphi\left(t\right)}\right| \leqslant 1$. Using the inequality

$$2\left|\varphi_1'\partial_1 u_n\right|\left|\partial_{11}u_n\right| \leqslant \epsilon\left(\partial_{11}u_n\right)^2 + \frac{1}{\epsilon}\left(\varphi_1'\right)^2\left(\partial_1 u_n\right)^2$$

for all $\epsilon > 0$, we obtain

$$|I_{n,1,1}| \leqslant K_2\int_{Q_n}\left[c_0\left|\varphi_1'\right|\varphi + c_0\epsilon\right]\left(\partial_{11}u_n\right)^2 dt\ dx_1 dx_2 + \frac{c_0}{\epsilon}\int_{Q_n}\left(\varphi_1'\right)^2\left(\partial_1 u_n\right)^2 dt\ dx_1 dx_2.$$

Lemma 2.4 yields

$$\frac{1}{\epsilon}\int_{Q_n}(\varphi_1')^2(\partial_1 u_n)^2\,dt\ dx_1dx_2 \leqslant K_2\frac{1}{\epsilon}\int_{Q_n}(\varphi_1')^2\varphi^2(\partial_{11}u_n)^2\,dt\ dx_1dx_2.$$

Thus,

$$|I_{n,1,1}| \leqslant K_2\int_{Q_n}c_0\left[|\varphi_1'|\,|\varphi|+\frac{1}{\epsilon}(\varphi_1')^2|\varphi|^2\right](\partial_{11}u_n)^2\,dt\ dx_1dx_2+\int_{Q_n}c_0\epsilon\,(\partial_{11}u_n)^2\,dt\ dx_1dx_2 \leqslant$$
$$\leqslant (2K_2+1)\,c_0\epsilon\int_{Q_n}(\partial_{11}u_n)^2\,dt\ dx_1dx_2,$$

since $|\varphi_1'\varphi|\leqslant\epsilon$. Finally, taking $K_4=(2K_2+1)\,c_0$, we obtain

$$|I_{n,1,1}|\ \leqslant\ K_4\epsilon\,\|\partial_{11}u_n\|^2_{L^2(Q_n)}.$$

The inequalities

$$|I_{n,2,1}|\ \leqslant\ K_4\epsilon\,\|\partial_{11}u_n\|^2_{L^2(Q_n)},$$

and

$$|I_{n,k,2}|\ \leqslant\ K_4\epsilon\,\|\partial_{22}u_n\|^2_{L^2(Q_n)}+c_0\epsilon\,\|\partial_{12}u_n\|^2_{L^2(Q_n)},\quad k=1,\ 2$$

can be proved by a similar method. This ends the proof of Lemma 2.5.

2) Estimation of $2\langle a_1\partial_{11}u_n, a_2\partial_{22}u_n\rangle$: We have

$$\partial_{11}u_n.\partial_{22}u_n=\ \partial_1(\partial_1u_n.\partial_{22}u_n)-\partial_2(\partial_1u_n.\partial_{12}u_n)+(\partial_{12}u_n)^2.$$

Then

$$\begin{aligned}2\langle a_1\partial_{11}u_n, a_2\partial_{22}u_n\rangle &= 2\int_{Q_n}a_1a_2\partial_{11}u_n.\partial_{22}u_n\,dt\,dx_1dx_2=\\
&=2\int_{Q_n}a_1a_2\left[\partial_1(\partial_1u_n.\partial_{22}u_n)-\partial_2(\partial_1u_n.\partial_{12}u_n)+(\partial_{12}u_n)^2\right]dt\,dx_1dx_2=\\
&=2\int_{\partial Q_n}a_1a_2\left[\partial_1u_n.\partial_{22}u_n\nu_1-\partial_1u_n.\partial_{12}u_n\nu_2\right]d\sigma+\\
&+\ 2\int_{Q_n}a_1a_2(\partial_{12}u_n)^2\,dt\,dx_1dx_2-\\
&-\ 2\int_{Q_n}\partial_1(a_1a_2)\,.\,(\partial_1u_n.\partial_{22}u_n)\,dt\,dx_1dx_2+\\
&+\ 2\int_{Q_n}\partial_2(a_1a_2)(\partial_1u_n.\partial_{12}u_n)\,dt\,dx_1dx_2,\end{aligned}$$

where $\nu_t,\nu_i,\ i=1,2$ are the components of the unit outward normal vector at ∂Q_n. We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t=\frac{1}{n}$, $x_2=0$ and $x_2=b$ we have $u_n=0$ and consequently $\partial_1u_n=0$. The corresponding boundary integral vanishes. On the part of the boundary where $t=T$, we have $\nu_1=\nu_2=0$. Accordingly the corresponding boundary integral vanishes. On the part of the boundary where $x_1=\varphi_k(t)$, $k=1,2$, we have $\nu_2=0,\ u_n=0$ and consequently $\partial_{22}u_n=0$. The corresponding boundary integral vanishes. So,

$$2\int_{\partial Q_n}a_1a_2\left[\partial_1u_n.\partial_{22}u_n\nu_1-\partial_1u_n.\partial_{12}u_n\nu_2\right]d\sigma=0.$$

Furthermore,

$$2\int_{Q_n} a_1 a_2 \left(\partial_{12}u_n\right)^2 dt\ dx_1dx_2 \geqslant 2a_0 \left\|\partial_{12}u_n\right\|^2_{L^2(Q_n)},$$

and for every $\epsilon > 0$

$$-2\int_{Q_n} \partial_1\left(a_1a_2\right).\left(\partial_1u_n.\partial_{22}u_n\right) dt\ dx_1dx_2 \geqslant -\beta_1\epsilon\left\|\partial_{22}u_n\right\|^2_{L^2(Q_n)} - \frac{\beta_1}{\epsilon}\left\|\partial_1u_n\right\|^2_{L^2(Q_n)},$$

$$+2\int_{Q_n} \partial_2\left(a_1a_2\right)\left(\partial_1u_n.\partial_{12}u_n\right) dt\ dx_1dx_2 \geqslant -\beta_1\epsilon\left\|\partial_{12}u_n\right\|^2_{L^2(Q_n)} - \frac{\beta_1}{\epsilon}\left\|\partial_1u_n\right\|^2_{L^2(Q_n)},$$

with β_1 is a positive constant. Then, we have

$$2\langle a_1\partial_{11}u_n, a_2\partial_{22}u_n\rangle \geqslant (2a_0-\beta_1\epsilon)\left\|\partial_{12}u_n\right\|^2_{L^2(Q_n)} - \beta_1\epsilon\left\|\partial_{22}u_n\right\|^2_{L^2(Q_n)} - \frac{2\beta_1}{\epsilon}\left\|\partial_1u_n\right\|^2_{L^2(Q_n)}. \quad (2.2)$$

It is easy to establish the following estimates.

Lemma 2.6. *Set $c_4 = c_0c_2$, $c_5 = c_0c_3$ and $c_6 = c_2c_3$. Then, for every $\epsilon > 0$ we have*

$$\begin{aligned}
2\langle \partial_tu_n, b_i\partial_iu_n\rangle &\geqslant -\epsilon c_2\left\|\partial_tu_n\right\|^2_{L^2(Q_n)} - \frac{c_2}{\epsilon}\left\|\partial_iu_n\right\|^2_{L^2(Q_n)}, i=1,2,\\
2\langle \partial_tu_n, cu_n\rangle &\geqslant -\epsilon c_3\left\|\partial_tu_n\right\|^2_{L^2(Q_n)} - \frac{c_3}{\epsilon}\left\|u_n\right\|^2_{L^2(Q_n)},\\
-2\langle a_i\partial_{ii}u_n, b_k\partial_ku_n\rangle &\geqslant -c_4\epsilon\left\|\partial_{ii}u_n\right\|^2_{L^2(Q_n)} - \frac{c_4}{\epsilon}\left\|\partial_ku_n\right\|^2_{L^2(Q_n)},\ i=1,2;\ k=1,2,\\
-2\langle a_i\partial_{ii}u_n, cu_n\rangle &\geqslant -c_5\epsilon\left\|\partial_{ii}u_n\right\|^2_{L^2(Q_n)} - \frac{c_5}{\epsilon}\left\|u_n\right\|^2_{L^2(Q_n)},\ i=1,2,\\
2\langle b_i\partial_iu_n, cu_n\rangle &\geqslant -c_6\epsilon\left\|u_n\right\|^2_{L^2(Q_n)} - \frac{c_6}{\epsilon}\left\|\partial_iu_n\right\|^2_{L^2(Q_n)},\\
2\langle b_1\partial_1u_n, b_2\partial_2u_n\rangle &\geqslant -b_0\epsilon\left\|\partial_1u_n\right\|^2_{L^2(Q_n)} - \frac{b_0}{\epsilon}\left\|\partial_2u_n\right\|^2_{L^2(Q_n)}.
\end{aligned}$$

Now, summing up the estimates (2.1), (2.2) and making use of Lemma 2.5 and Lemma 2.6 then we obtain

$$\begin{aligned}
\left\|f_n\right\|^2_{L^2(Q_n)} \geqslant{}& (1-\alpha\epsilon)\left\|\partial_tu_n\right\|^2_{L^2(Q_n)} + d_0\left\|u_n\right\|^2_{L^2(Q_n)} - \alpha\left(\epsilon+\frac{1}{\epsilon}\right)\left\|u_n\right\|^2_{L^2(Q_n)} +\\
&+ b_0\big(\left\|\partial_1u_n\right\|^2_{L^2(Q_n)} + \left\|\partial_2u_n\right\|^2_{L^2(Q_n)}\big) - \alpha\left(\epsilon+\frac{1}{\epsilon}\right)\big(\left\|\partial_1u_n\right\|^2_{L^2(Q_n)} + \left\|\partial_2u_n\right\|^2_{L^2(Q_n)}\big) +\\
&+ (a_0-\alpha\epsilon)\sum_{i=1}^{2}\left\|\partial_{ii}u_n\right\|^2_{L^2(Q_n)} + (2a_0-\beta_1\epsilon-c_0\epsilon)\left\|\partial_{12}u_n\right\|^2_{L^2(Q_n)},
\end{aligned}$$

where α is a positive constant independent of n. Thanks to Lemma 2.4, it follows that for $i = 1, 2$

$$-\alpha\left(\epsilon+\frac{1}{\epsilon}\right)\left\|\partial_iu_n\right\|^2_{L^2(Q_n)} \geqslant -\alpha\left(\epsilon+\frac{1}{\epsilon}\right)C_1\epsilon^2\left\|\partial_{ii}u_n\right\|^2_{L^2(Q_n)}$$

and

$$-\alpha\left(\epsilon+\frac{1}{\epsilon}\right)\left\|u_n\right\|^2_{L^2(Q_n)} \geqslant -\alpha\left(\epsilon+\frac{1}{\epsilon}\right)C_1\epsilon^4\left\|\partial_{ii}u_n\right\|^2_{L^2(Q_n)}.$$

Therefore,

$$\|f_n\|^2_{L^2(Q_n)} \geqslant (1-\alpha\epsilon)\|\partial_t u_n\|^2_{L^2(Q_n)} + d_0\|u_n\|^2_{L^2(Q_n)} -$$
$$-\alpha\left(\epsilon+\frac{2}{\epsilon}\right)C_1\epsilon^4\left(\|\partial_{11}u_n\|^2_{L^2(Q_n)} + \|\partial_{22}u_n\|^2_{L^2(Q_n)}\right) + b_0\left(\|\partial_1 u_n\|^2_{L^2(Q_n)} + \|\partial_2 u_n\|^2_{L^2(Q_n)}\right) -$$
$$-\alpha\left(\epsilon+\frac{1}{\epsilon}\right)C_1\epsilon^2\left(\|\partial_1^2 u_n\|^2_{L^2(Q_n)} + \|\partial_2 u_n\|^2_{L^2(Q_n)}\right) +$$
$$+(a_0-\alpha\epsilon)\sum_{i=1}^{2}\|\partial_{ii}u_n\|^2_{L^2(Q_n)} + (2a_0-\beta_1\epsilon-c_0\epsilon)\|\partial_{12}u_n\|^2_{L^2(Q_n)}, \quad (2.3)$$

which implies

$$\|f_n\|^2_{L^2(Q_n)} \geqslant (1-\alpha\epsilon)\|\partial_t u_n\|^2_{L^2(Q_n)} + d_0\|u_n\|^2_{L^2(Q_n)} + b_0\left(\|\partial_1 u_n\|^2_{L^2(Q_n)} + \|\partial_2 u_n\|^2_{L^2(Q_n)}\right) +$$
$$+\left(a_0-\alpha\epsilon-\alpha C_1(\epsilon^2+\epsilon)-\alpha C_1(\epsilon^5+\epsilon^3)\right)\left(\|\partial_{11}u_n\|^2_{L^2(Q_n)} + \|\partial_{22}u_n\|^2_{L^2(Q_n)}\right) +$$
$$+(2a_0-\beta_1\epsilon-c_0\epsilon)\|\partial_{12}u_n\|^2_{L^2(Q_n)}.$$

Then, it is sufficient to choose ϵ verifying

$$(1-\alpha\epsilon)>0, \quad (2a_0-\beta_1\epsilon-c_0\epsilon)>0 \text{ and } \left(a_0-\alpha\epsilon-\alpha C_1(\epsilon^2+\epsilon)-\alpha C_1(\epsilon^5+\epsilon^3)\right)>0$$

to get a constant $K_0>0$ independent of n such that

$$\|f_n\|_{L^2(Q_n)} \geqslant K_0\|u_n\|_{\mathcal{H}^{1,2}(Q_n)},$$

and since

$$\|f_n\|_{L^2(Q_n)} \leqslant \|f\|_{L^2(Q)},$$

there exists a constant $K_1>0$, independent of n satisfying

$$\|u_n\|_{\mathcal{H}^{1,2}(Q_n)} \leqslant K_1\|f_n\|_{L^2(Q_n)} \leqslant K_1\|f\|_{L^2(Q)}.$$

This completes the proof of Proposition 2.1. □

2.3. Step 3: passage to the limit

Choose a sequence Q_n $n=1,2,\dots$ of reference domains (see the above subsection) such that $Q_n \subseteq Q$. Then we have $Q_n \to Q$, as $n\to\infty$. Consider the solution $u_n \in \mathcal{H}^{1,2}(Q_n)$ of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_n - \sum\limits_{i=1}^{2} a_i(t,x_1,x_2)\partial_{ii}u_n + \sum\limits_{i=1}^{2} b_i(t,x_1,x_2)\partial_i u_n + c(t,x_1,x_2)u_n = f & \text{in } Q_n, \\ u_n|_{\partial Q_n \setminus \Sigma_T} = 0. \end{cases}$$

Such a solution u_n exists by Theorem 2.1. Let $\widetilde{u_n}$ the 0-extension of u_n to Q. In virtue of Proposition 2.1, we know that there exists a constant C such that

$$\|\widetilde{u_n}\|_{\mathcal{H}^{1,2}(Q_n)} C\|f\|_{L^2(Q)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial^\alpha u_n}$ for $1\leqslant|\alpha|\leqslant 2$ are bounded functions in $L^2(Q)$. So, for a suitable increasing sequence of integers n_k, $k=1,2,\dots$, there exist functions

$$u,\ v \text{ and } v_\alpha,\ 1\leqslant|\alpha|\leqslant 2$$

in $L^2(Q)$ such that

$$\widetilde{u_{n_k}} \rightharpoonup u \text{ weakly in } L^2(Q), \ k \to \infty,$$
$$\widetilde{\partial_t u_{n_k}} \rightharpoonup v \text{ weakly in } L^2(Q), \ k \to \infty,$$
$$\widetilde{\partial^\alpha u_{n_k}} \rightharpoonup v_\alpha \text{ weakly in } L^2(Q), \ k \to \infty,$$

$1 \leqslant |\alpha| \leqslant 2$. Clearly,

$$v = \partial_t u, \ v_\alpha = \partial^\alpha u, \ 1 \leqslant |\alpha| \leqslant 2$$

in the sense of distributions in Q, then in $L^2(Q)$. So, $u \in \mathcal{H}^{1,2}(Q)$ and

$$\partial_t u - \sum_{i=1}^{2} a_i(t,x_1,x_2)\partial_{ii}u + \sum_{i=1}^{2} b_i(t,x_1,x_2)\partial_i u + c(t,x_1,x_2)u = f \text{ in } Q.$$

On the other hand, the solution u satisfies the boundary conditions $u|_{\partial Q \setminus \Sigma_T} = 0$, since

$$\forall n \in \mathbb{N}^*, \ u|_{Q_n} = u_n.$$

This proves the existence of a solution to Problem (1.1). Notice that we have the estimate

$$\|u\|_{\mathcal{H}^{1.2}(Q)} \leqslant K \|f\|_{L^2(Q)},$$

which implies the uniqueness of the solution.

Remark 2.2. *If $\varphi_1(0) < \varphi_2(0)$ and $\varphi_1(T) = \varphi_2(T)$, then the result given in Theorem 1.1 holds true under the assumption*

$$\varphi_k'(t)\,\varphi(t) \to 0 \quad as \quad t \to T, \quad k = 1,2$$

instead of hypothesis (1.2).

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О линейном параболическом уравнении второго порядка с переменными коэффициентами в нерегулярной области $\mathbb{R}^3$

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Настоящая работа посвящена изучению следующего параболического уравнения с переменными коэффициентами в недивергентной форме:

$$\partial_t u - \sum_{i=1}^{2} a_i(t, x_1, x_2)\partial_{ii}u + \sum_{i=1}^{2} b_i(t, x_1, x_2)\partial_i u + c(t, x_1, x_2)u = f(t, x_1, x_2),$$

с учетом граничных условий Коши-Дирихле. Задача задана в нерегулярной области вида

$$Q = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T, \varphi_1(t) < x_1 < \varphi_2(t)\right\} \times]0, b[\,,$$

где φ_k, $k = 1, 2$ являются гладкими функциями. Одной из основных задач этой работы служит то, что область может быть нерегулярной, например, допускается особый случай, когда φ_1 совпадает с φ_2 при $t = 0$. Анализ проводится в рамках анизотропных пространств Соболева с использованием метода декомпозиции областей. Эта работа является обобщением случая постоянных коэффициентов, изучаемого в [15].

Ключевые слова: параболические уравнения, нерегулярные области, переменные коэффициенты, анизотропные пространства Соболева.