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On Generation of the Group $\mathbf{PSL_n}(\mathbb{Z}+\mathbf{i}\mathbb{Z})$ by Three Involutions, Two of Which Commute

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It is proved that the projective special linear group $PSL_n(\mathbb{Z}+i\mathbb{Z})$, $n \geqslant 8$, over Gaussian integers $\mathbb{Z}+i\mathbb{Z}$ is generated by three involutions, two of which commute.

Keywords: gaussian intergers, special linear group, generating elements.

Introduction

The main result of the paper is the following theorem.

Theorem 1. *For $n \geqslant 8$ the projective special linear group $PSL_n(\mathbb{Z}+i\mathbb{Z})$ over Gaussian integers $\mathbb{Z}+i\mathbb{Z}$ is generated by three involutions, two of which commute, but for $n = 2, 3$ it is not generated by such three involutions.*

The groups generated by three involutions, two of which commute, will be called (2×2,2)-generated. Here we do not exclude the cases when two or even three involutions are the same. Clearly, if a group has a homomorphic image, which is not (2×2,2)-generated, then it will not be (2×2,2)-generated. Since there exist the homomorphism of $PSL_n(\mathbb{Z}+i\mathbb{Z})$ onto $PSL_n(9)$ then the assertion of the Theorem 1 arises from the fact that the groups $PSL_2(9)$ and $PSL_3(9)$ are not (2×2,2)-generated (see [1]). For $n \geqslant 8$ generating triples of involutions, two of which commute, of the group $PSL_n(\mathbb{Z}+i\mathbb{Z})$ are indicated explicitly. It moreover $n \neq 2(2k+1)$ then we take generating triples of involutions from $SL_n(\mathbb{Z}+i\mathbb{Z})$. Thus, for $n \geqslant 8$ and $n \neq 2(2k+1)$ we have a stronger statement: the group $SL_n(\mathbb{Z}+i\mathbb{Z})$ is (2×2,2)-generated. Earlier, Ya.N.Nuzhin proved that $PSL_n(\mathbb{Z})$ is (2×2,2)-generated if and only if $n \geqslant 5$. In the proof of Theorem 1 the methods of choosing generating triples of involutions developed in [2] are essentially used. Note also that M.C.Tamburini and P.Zucca [3] proved (2×2,2)-generation of the group $SL_n(\mathbb{Z})$ for $n \geqslant 14$.

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1. Notations and Preliminary Results

Througout the paper $\mathbb{Z}$ are integers and $\mathbb{Z}+i\mathbb{Z}$ are Gaussian integers, where $i^2=-1$. The rings $\mathbb{Z}$ and $\mathbb{Z}+i\mathbb{Z}$ are Euclidean rings. Let R be an arbitrary Euclidean ring.

As usually, we will denote by $t_{ij}(k),\ k\in R,\ i\neq j$, the transvections, that is, the matrices E_n+ke_{ij}, where E_n is the identity $(n\times n)$ matrix, and e_{ij} denotes the $(n\times n)$ matrix with (i,j)-entry 1 and all other entries 0. The set $t_{ij}(R)=\{t_{ij}(k),\ k\in R\}$ is a subgroup.

The following lemma is well-known (see, for example, ([4], p.107)).

Lemma 1. *The group $SL_n(R)$ is generated by the subgroups $t_{ij}(R)$, $i,j=1,2,\ldots,n$.*

Let

$$\tau=\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1\\ 0 & 0 & \ldots & 0 & 1 & 0\\ 0 & 0 & \ldots & 1 & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 1 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix},\ \mu=\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1\\ 1 & 0 & \ldots & 0 & 0 & 0\\ 0 & 1 & \ldots & 0 & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

The matrix τ is an involution, and the matrix μ has the order n and acts regularly on the following set of subgroups:

$$M=\{t_{1n}(R),\quad t_{i+1i}(R),\quad i=1,2,\ldots,n-1\}.$$

Commuting among themselves subgroups of the set M, we can get all subgroups $t_{ij}(R)$. Hence, by lemma 1 the group $SL_n(R)$ is generated by the set M. Moreover, the following lemma is true.

Lemma 2. *The group $SL_n(R)$ is generated by one of the subgroups*

$$t_{1n}(R),\quad t_{i+1i}(R),\quad t_{n-1n}(R),\quad t_{ii+1}(R),\quad i=1,2,\ldots,n-1,$$

and the monomial matrix $\eta\mu$ for any diagonal matrix η with $\eta\mu\in SL_n(R)$.

For elements of $PSL_n(R)$ we will be also using matrix representation, assuming that two element are equal if they only differ by multiplication with a scalar matrix of $SL_n(R)$. In the next sections for elements of the groups $SL_n(R)$ and $PSL_n(R)$ we will also be using the teminology of Chevalley groups, consedering $SL_n(R)$ and $PSL_n(R)$ as universal and adjoint Chevalley group respectivelly.

Let Φ be a root system of type A_l with the basis

$$\Pi=\{r_1,r_2,\ldots,r_l\},$$

where

$$l=n-1.$$

The Chevalley group $A_l(R)$ (universal and adjoint) of type A_l over ring R is generated of root subgroups

$$X_r=\{x_r(t),\ t\in R\},\ r\in\Phi,$$

where $x_r(t)$ are root elements.

For any $r \in \Phi$ and $t \neq 0$ we set

$$n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t), \quad n_r = n_r(1), \quad h_r(-1) = n_r^2.$$

The map

$$t_{i+1i}(t) \to x_{r_i}(t), \quad i = 1, 2, \dots, l, \quad t \in R,$$

is extended up to isomorphism of the group $SL_n(R)$ onto universal Chevalley group $A_l(R)$. The monomial matrices τ and μ indicated above are preimages of the elements w_0 and w respectively from the Weyl group W under natural homomorphism of the monomial subgroup N onto W, where $w_0(r) \in \Phi^-$ for any $r \in \Phi^+$ and

$$w = w_{r_1} w_{r_2} \dots w_{r_l}.$$

Here Φ^+ are the positive roots and Φ^- are negative roots. We can reformulate of Lemma 2 in terms of Chevalley groups.

Lemma 3. *The Chevalley group $A_l(R)$ is generated by any root subgroup*

$$X_{\pm r_i}, \quad r_i \in \Pi, \quad X_{\pm(r_1+\dots+r_l)}$$

and the monomial element n_w if $w = w_{r_1} w_{r_2} \dots w_{r_l}$.

Througout the paper we use the notation $a^b = bab^{-1}, \ \ [a,b] = aba^{-1}b^{-1}$.

2. Generating Triples of Involutions

Let τ and μ be as in the first paragraph. The matrices τ and

$$\tau\mu = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

are involutions, but they do not necessarity belong to $SL_n(\mathbb{Z})$ (this depends on their size). We select diagonal matrices η_1 and η_2 with elements ± 1 such that the matrices $\eta_1\tau$ and $\eta_2\tau\mu$ belong to $SL_n(\mathbb{Z})$ and their images in $PSL_n(\mathbb{Z})$ are involutions. We take η_1, η_2 to be the following matrices:

for $n = 4k + 1 \ \ (= 5, 9, \dots)$

$$\eta_1 = \eta_2 = E_n;$$

for $n = 2(2k + 1) + 1 \ \ (= 7, 11, \dots)$

$$\eta_1 = -E_n, \quad \eta_2 = E_n;$$

for $n = 4k \ \ (= 8, 12, \dots)$

$$\eta_1 = E_n, \qquad \eta_2 = \mathrm{diag}(E_{n-1}, -1);$$

for $n = 2(2k+1) \ \ (= 6, 10, \dots)$

$$\eta_1 = \mathrm{diag}(-E_{2k+1}, E_{2k+1}), \qquad \eta_2 = E_n.$$

Further for the group $PSL_n(\mathbb{Z}+i\mathbb{Z})$ we explicitly write down triples of generating involutions α, β, γ, two of which commute. For odd $n \geqslant 7$ by definition

$\alpha = t_{21}(i)t_{n-1n}(i)\,\mathrm{diag}(1, -1, -1, E_{n-6}, -1, -1, 1),$

$\beta = \eta_1\tau,$

$\gamma = \eta_2\tau\mu.$

For even $n \geqslant 6$ by definition

$\alpha = t_{21}(1)t_{n-1n}(-1)\,\mathrm{diag}(1, -1, -1, E_{n-6}, -1, -1, 1),$

$\beta = \mathrm{diag}(i, -i, 1, \dots, 1)\eta_1\tau\,\mathrm{diag}(-i, i, 1, \dots, 1),$

$\gamma = \eta_2\tau\mu.$

The next lemma is verified by direct calculation.

Lemma 4. *Let α, β, γ are search as above. Then:*
1) $\alpha\beta = \beta\alpha$;
2) α, γ are involutions from $SL_n(\mathbb{Z}+i\mathbb{Z})$ (and hence in $PSL_n(\mathbb{Z}+i\mathbb{Z})$);
3) β is involution from $SL_n(\mathbb{Z}+i\mathbb{Z})$ if $n \neq 2(2k+1)$;
4) if $n = 2(2k+1)$, then $\beta^2 = -E_n$ and hence image β is involution in $PSL_n(\mathbb{Z}+i\mathbb{Z})$.

In Sections 3 and 4 we prove that involutions α, β, γ generate the group $PSL_n(\mathbb{Z}+i\mathbb{Z})$, $n \geqslant 8$, for odd and even n respectively. Further the next remark will be useful. By the construction

$$\beta\gamma = \eta_3\mu$$

for some diagonal element $\eta_3 \in PSL_n(\mathbb{Z}+i\mathbb{Z})$. Therefore by Lemma 2 the proof of Theorem 1 can be reduced to verification of the hypothesis of the next lemma.

Lemma 5. *If a group is generated by involutions α, β, γ and contains one of the subgroups*

$$t_{1n}(\mathbb{Z}+i\mathbb{Z}), \quad t_{i+1i}(\mathbb{Z}+i\mathbb{Z}), \quad t_{n-1n}(\mathbb{Z}+i\mathbb{Z}), \quad t_{ii+1}(\mathbb{Z}+i\mathbb{Z}), \quad i = 1, 2, \dots, n-1,$$

(in terminology of Chevalley groups one of root subgroups $X_{\pm r_i}$, $r_i \in \Pi$, $X_{\pm(r_1+\dots+r_l)}$,) then it coincides with the group $PSL_n(\mathbb{Z}+i\mathbb{Z})$.

3. Proof of Theorem 1 for Odd n $\geqslant$ 9

Let $\alpha, \beta, \gamma, \tau, \mu, \eta_1, \eta_2, \eta_3$ be as in sections 1 and 2, $n \geqslant 9$ and $l = n-1$. In terminology of Chevalley groups

$\alpha = x_{r_1}(i)x_{-r_l}(i)h_{r_2}(-1)h_{r_{l-1}}(-1)$,

$\beta = \eta_1\tau = n_{w_0}$,

$\gamma = \eta_2\tau\mu = n_{w_0}n_w$,

$\eta \equiv \beta\gamma = n_w$,

where $w = w_{r_1}w_{r_2}\dots w_{r_l}$.

Direct calculations give that

$$\alpha^\eta = x_{r_2}(\pm i)x_{r_1+\dots+r_l}(\pm i)h_{r_3}(-1)h_{r_l}(-1),$$

$$\alpha^{\eta^2} = x_{r_3}(\pm i)x_{-r_1}(\pm i)h_{r_4}(-1)h_{r_1+\dots+r_l}(-1),$$

$$[\alpha, \alpha^\eta] = x_{r_1+r_2}(\pm 1)x_{r_1+\dots+r_{l-1}}(\pm 1),$$

$$([\alpha, \alpha^\eta]\alpha^{\eta^2})^2 = x_{r_1+r_2+r_3}(\pm i)x_{r_2}(\pm i)x_{r_2+r_3}(\pm 1)x_{r_2+\dots+r_{l-1}}(\pm i),$$

$$\theta \equiv (([\alpha, \alpha^\eta]\alpha^{\eta^2})^2)^\eta = x_{r_2+r_3+r_4}(\pm i)x_{r_3}(\pm i)x_{r_3+r_4}(\pm 1)x_{r_3+\dots+r_l}(\pm i),$$

$$[\theta, [\alpha, \alpha^\eta]] = x_{r_1+r_2+r_3}(\pm i)x_{r_1+r_2+r_3+r_4}(\pm 1)x_{r_1+\dots+r_l}(\pm i),$$

$$[\alpha, [\theta, [\alpha, \alpha^\eta]]] = x_{r_1+\dots+r_{l-1}}(\pm 1),$$

$$[\alpha, [\theta, [\alpha, \alpha^\eta]]]^\beta = x_{-r_2-\dots-r_l}(\pm 1),$$

$$[[\theta, [\alpha, \alpha^\eta]], [\alpha, [\theta, [\alpha, \alpha^\eta]]]^\beta] = x_{r_1}(\pm i).$$

Taking sequentially $(l-1)$-commutator of the elements

$$x_{r_1}(\pm i),\ x_{r_1}(\pm i)^\eta = x_{r_2}(\pm i),\ x_{r_2}(\pm i)^\eta = x_{r_3}(\pm i),\ \dots,\ x_{r_{l-1}}(\pm i)^\eta = x_{r_l}(\pm i),$$

we get the element $x_{r_1+\dots+r_l}(\pm 1)$. On the other hand, $(x_{r_l}(\pm i)^\eta)^\beta = x_{r_1+\dots+r_l}(\pm i)$.

All root subgroups X_r are commutative and, evidently, the elements 1 and i generate additively all ring $\mathbb{Z}+i\mathbb{Z}$. Therefore, if a subgroup is generated by involutions α, β, γ, then it contains the root subgroup $X_{r_1+\dots+r_l}$ and hence by Lemma 5 it coincides with the subgroup $PSL_n(\mathbb{Z}+i\mathbb{Z})$.

Note, that for even n these generating involutions α,β,γ do not generate $PSL_n(\mathbb{Z}+i\mathbb{Z})$, since by conjugating by diagonal element

$$\mathrm{diag}(1,i,1,1,i,1,1,\ldots,i,1,1,i)$$

the subgroup generated by these involutions, we only get the group $PSL_n(\mathbb{Z})$.

4. Proof of Theorem 1 for Even $\mathbf{n \geqslant 8}$

Let $\alpha,\beta,\gamma,\tau,\mu,\eta_1,\eta_2,\eta_3$ be as in paragraphs 1 and 2, $n\geqslant 8$ and $l=n-1$. In terminology of Chevalley groups

$\alpha = x_{r_1}(1)x_{-r_l}(-1)h_{r_2}(-1)h_{r_{l-1}}(-1)$,

$\beta = \mathrm{diag}(i,-i,1,\ldots,1)\eta_1\tau\,\mathrm{diag}(-i,i,1,\ldots,1) = h_{r_1}(i)n_{w_0}h_{r_1}(-i) = h_{r_1}(i)h_{r_l}(i)n_{w_0}$,

$\gamma = \eta_2\tau\mu = n_{w_0}n_w$,

$\eta \equiv \beta\gamma = h_{r_1}(i)h_{r_l}(i)n_w$,

где $w = w_{r_1}w_{r_2}\ldots w_{r_l}$.

Direct calculations give that

$$\alpha^{\eta} = x_{r_2}(\pm i)x_{r_1+\cdots+r_l}(\pm 1)h_{r_3}(-1)h_{r_l}(-1),$$

$$\alpha^{\eta^2} = x_{r_3}(\pm i)x_{-r_1}(\pm 1)h_{r_4}(-1)h_{r_1+\cdots+r_l}(-1),$$

$$[\alpha,\alpha^{\eta}] = x_{r_1+r_2}(\pm i)x_{r_1+\cdots+r_{l-1}}(\pm 1),$$

$$([\alpha,\alpha^{\eta}]\alpha^{\eta^2})^2 = x_{r_1+r_2+r_3}(\pm 1)x_{r_2}(\pm i)x_{r_2+r_3}(\pm 1)x_{r_2+\cdots+r_{l-1}}(\pm 1),$$

$$\theta \equiv (([\alpha,\alpha^{\eta}]\alpha^{\eta^2})^2)^{\eta} = x_{r_2+r_3+r_4}(\pm i)x_{r_3}(\pm i)x_{r_3+r_4}(\pm 1)x_{r_3+\cdots+r_l}(\pm i),$$

$$[\theta,[\alpha,\alpha^{\eta}]] = x_{r_1+r_2+r_3}(\pm 1)x_{r_1+r_2+r_3+r_4}(\pm i)x_{r_1+\cdots+r_l}(\pm 1),$$

$$[\alpha,[\theta,[\alpha,\alpha^{\eta}]]] = x_{r_1+\cdots+r_{l-1}}(\pm 1),$$

$$[\alpha,[\theta,[\alpha,\alpha^{\eta}]]]^{\beta} = x_{-r_2-\cdots-r_l}(\pm 1),$$

$$[[\theta,[\alpha,\alpha^{\eta}]],[\alpha,[\theta,[\alpha,\alpha^{\eta}]]]^{\beta}] = x_{r_1}(\pm 1).$$

Taking sequentially $(l-1)$-commutator of the elements

$$x_{r_1}(\pm 1),\ x_{r_1}(\pm 1)^\eta = x_{r_2}(\pm i),\ x_{r_2}(\pm i)^\eta = x_{r_3}(\pm i),\ \ldots,\ x_{r_{l-3}}(\pm i)^\eta = x_{r_{l-2}}(\pm i),$$

$$x_{r_{l-2}}(\pm i)^\eta = x_{r_{l-1}}(\pm 1),\ x_{r_{l-1}}(\pm 1)^\eta = x_{r_l}(\pm 1),$$

we get the element $x_{r_1+\cdots+r_l}(\pm i)$. On the other hand, $(x_{r_l}(\pm 1)^\eta)^\beta = x_{r_1+\cdots+r_l}(\pm 1)$.

Therefore, if a subgroup is generated by the involutions α, β, γ, then it contains the root subgroup $X_{r_1+\cdots+r_l}$ and hence by Lemma 5 it coincides with subgroup $PSL_n(\mathbb{Z}+i\mathbb{Z})$.

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