On localization of zeros of an entire function of finite order of growth

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Abstract. The aim of the article is to find conditions on the coefficients of the Taylor expansion of an entire function of finite order of growth in $\mathbb{C}$ that guarantee a specified number of zeros. If the Taylor coefficients for f are real we also give conditions to determine whether the number of real and imaginary zeros is finite or infinite, and whether they are real or imaginary.

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For polynomials the problem of root localization is a classic problem (see, e.g., [1, Ch. 16], [2], [3]) and it has a long history. Root localization of algebraic equations with the use of complex analysis is considered in [4]. For entire functions this problem has not been considered. Nevertheless, there are equations and systems of equations consisting of exponential polynomials (see [5, 6]). This raises the question on the number of roots of such functions, on the number of real or imaginary roots, etc.

1. The absence of zeros

Let function $f = f(z)$ with respect to complex variable z be a holomorphic in a neighborhood of zero in the complex plane $\mathbb{C}$:

$$f(z) = \sum_{k=0}^{\infty} b_k z^k, \quad f(0) = b_0 = 1. \tag{1}$$

Let γ_r be a circle of the form

$$\gamma_r = \{z : \ |z| = r\}, \quad r > 0.$$

Theorem 1. *For function f to be an entire function of finite order of growth which has no zeros, it is necessary and sufficient that for sufficiently small r there exists $k_0 \in \mathbb{N}$ such that*

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$$\int_{\gamma_r} \frac{1}{z^k}\frac{df}{f} = 0 \quad \textit{for all} \quad k \geqslant k_0. \tag{2}$$

In this case the minimum k_0 is equal to the order of function.

Recall that the entire function $f(z)$ has a finite order (of growth) if there exists a positive number A such that

$$f(z) = O(e^{R^A}) \quad \text{for} \quad |z| = R \to +\infty.$$

The infimum of such numbers A is called the *order* of function.

Proof. Let the function f be a function of finite order of growth, which has no zeros in $\mathbb{C}$ then it is well known that it has the form: $f(z) = e^{\varphi(z)}$, where $\varphi(z)$ is a polynomial of some degree k_0 (see, e.g., [7, Ch. 7, Sec. 1.5]). Then

$$\int_{\gamma_r} \frac{1}{z^k}\frac{df}{f} = \int_{\gamma_r} \frac{1}{z^k}\varphi'(z)\,dz = 0 \quad \text{under} \quad k > k_0.$$

Conversely, suppose that condition (2) is fulfilled. Since $f(z)$ is a holomorphic function in a neighborhood of zero and $f(0) \neq 0$ then the values of $f(z)$ lie in a neighborhood of $f(0)$ and this neighborhood does not contain the point 0 for sufficiently small $|z|$. Therefore, the holomorphic function $\varphi(z) = \ln f(z)$ ($\ln 1 = 0$) is defined in the neighborhood of zero.

Let

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_0 = \ln f(0) = \ln b_0.$$

Then, for sufficiently small r we have

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{z^k}\frac{df}{f} = \frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{z^k}\varphi'(z)\,dz = ka_k. \tag{3}$$

When condition (2) is fulfilled we see that $a_k = 0$ under $k > k_0$. Therefore, $\varphi(x)$ is a polynomial of degree k_0. Consequently, $f(z) = e^{\varphi(z)}$ is an entire function of finite order k_0. □

There exists a recursive relationship between coefficients of f and $\varphi(z)$ (see, e.g., [4, §2, Lemma 2.3]).

Lemma 1. *The following relations are true:*

$$a_k = \frac{(-1)^{k-1}}{kb_0^k}\begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix},$$

and

$$b_k = \frac{b_0}{k!}\begin{vmatrix} a_1 & -1 & 0 & \dots & 0 \\ 2a_2 & a_1 & -2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ka_k & (k-1)a_{k-1} & (k-2)a_{k-2} & \dots & a_1 \end{vmatrix}.$$

Therefore, we have the following statement.

Corollary 1. *For function f to be an entire function of finite order k_0 which has no zeros, it is necessary and sufficient that the determinant*

$$\begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix} = 0 \quad \textit{under} \quad k > k_0. \tag{4}$$

where k_0 is the minimum number with this property.

Example 1. Let

$$f(z) = e^z = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!},$$

i.e, $b_0 = 1$, $b_k = \dfrac{1}{k!}$, $k > 1$.

Let us substitute these values into (4). When $k = 1$ the determinant is not equal to zero. For $k > 1$ all determinants are equal to zero since the first two columns are the same. Then the function $f(z)$ is of order 1 and it has no zeros in the complex plane.

2. Auxiliary statements

Let a function $f(z)$ of the form (1) be an entire function of finite order of growth. Zeros of this function are $\alpha_1,\ \alpha_2,\ \dots, \alpha_n, \dots$ (every root appears as many times as its multiplicities). There is a finite or infinite number of zeros. They are indexed in increasing order of magnitude $|\alpha_1| \leqslant |\alpha_2| \leqslant \dots \leqslant |\alpha_n| \leqslant \dots$.

Recall Hadamard decomposition for such functions (see, e.g., [8, Ch. 8, Theorem 8.2.4], [7, Ch. 7, Sec. 2.3]).

Theorem 2. *If $f(z)$ is an entire function of finite order ρ then*

$$f(z) = z^s e^{Q(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \dots + \frac{z^p}{p\alpha_n^p}}, \tag{5}$$

where $Q(z)$ is a polynomial of degree $q \leqslant \rho$, s is the multiplicity of zero of the function f at the point 0, p is some integer number and $p \leqslant \rho$.

The infinite product in (5) converges absolutely and locally uniformly in $\mathbb{C}$. (Recall that a sequence of holomorphic functions converges locally uniformly in an open set U, if it converges uniformly on every compact subset of U.) In what follows we assume for simplicity that $f(0) = 1$. The polynomial $Q(z)$ is of the form

$$Q(z) = \sum_{j=1}^{q} d_j z^j.$$

Here $d_0 = 0$, since $f(0) = 1$.

The expression

$$\Phi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \ldots + \frac{z^p}{p\alpha_n^p}} \tag{6}$$

is called the *canonical product*, and the integer number p is the *genus* of the canonical product depended on f. The genus of entire function $f(z)$ is $\max\{q, p\}$. If ρ' is the order of canonical product (6) then $\rho = \max\{q, \rho'\}$.

Let us consider the following series

$$\sum_{n=1}^{\infty} \frac{1}{|\alpha_n|^{\gamma}}. \tag{7}$$

The infimum of positive γ for which the series (7) converges is called *the rate of convergence* of zeros of the canonical product $\Phi(z)$.

It is well known (see., e.g., [8, Sec. 8, §8.2.5], [7, Ch. 7, Sec. 2.2]) that the rate of convergence of the zeros of the canonical product is equal to its order.

Then sums of zeros in negative power

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{\alpha_n^k}, \quad k \in \mathbb{N},$$

are absolutely convergent series when $k > \rho'$, i.e., when $k > \rho$. It is also known that $\rho' - 1 \leqslant p \leqslant \rho'$ (see, e.g., [8, Sec. 8, §8.2.7])

In what follows we consider the power sums with positive integer exponents k. Let us relate integrals in (2) to power sums σ_k of zeros.

Formula (3) relates integrals in (2) to the expansion coefficients of $\varphi(z) = \ln f(z)$ in the neighborhood of zero. Let us express the integral in terms of the power sums of zeros, using the Hadamard formula. We consider the case of $s = 0$ that is, $f(0) \neq 0$.

In a sufficiently small neighborhood of zero we have (according to the Hadamard formula (5))

$$\varphi(z) = Q(z) + \sum_{n=1}^{\infty} \ln\left[\left(1 - \frac{z}{\alpha_n}\right) e^{P_n(z)}\right],$$

where $P_n(z) = \dfrac{z}{\alpha_n} + \dfrac{z^2}{2\alpha_n^2} + \ldots + \dfrac{z^p}{p\alpha_n^p}$.

The series for $\varphi(z)$ converges absolutely and uniformly in a sufficiently small neighborhood of zero since the zeros α_j are bounded away from zero.

It is obvious that

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{z^k} \cdot dQ(z) = \begin{cases} kd_k & \text{under} \quad 1 \leqslant k \leqslant q, \\ 0 & \text{under} \quad k > q. \end{cases}$$

Let us transform the following expression

$$d\ln\left[\left(1 - \frac{z}{\alpha_n}\right) e^{P_n(z)}\right] = \frac{d\left[\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}\right]}{\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}} =$$

$$= \frac{d\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}} + \left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}} d\left(\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}\right)}{\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}} =$$

$$= \frac{d\left(1 - \frac{z}{\alpha_n}\right)}{\left(1 - \frac{z}{\alpha_n}\right)} + d\left(\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}\right) =$$

$$= \frac{dz}{z - \alpha_n} + \left(\frac{1}{\alpha_n} + \frac{z}{\alpha_n^2} + \cdots + \frac{z^{p-1}}{\alpha_n^p}\right) dz =$$

$$= \frac{dz}{z - \alpha_n} + \frac{1}{\alpha_n}\left[\frac{\left(\frac{z^p}{\alpha_n^p} - 1\right)}{\left(\frac{z}{\alpha_n} - 1\right)}\right] dz = \frac{dz}{z - \alpha_n} + \frac{(z^p - \alpha_n^p)dz}{\alpha_n^{p-1}(z - \alpha_n)} = \frac{z^p dz}{\alpha_n^p(z - \alpha_n)}.$$

Then

$$\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\gamma_r} \frac{d\left[\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}\right]}{z^k\left(1 - \frac{z}{\alpha_n}\right) e^{\frac{z}{\alpha_n} + \frac{z^2}{2\alpha_n^2} + \cdots + \frac{z^p}{p\alpha_n^p}}} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\gamma_r} \frac{z^{p-k}dz}{\alpha_n^p(z - \alpha_n)} =$$

$$= \begin{cases} 0, \quad \text{if} \quad k \leqslant p, \\ -\sum\limits_{n=1}^{\infty} \frac{1}{\alpha_n^k} = -\sigma_k, \quad \text{if} \quad k > p. \end{cases}$$

Thus we have the following stament

Proposition 1. *Let $f(z)$ be an entire function of finite order of growth ρ of the form (5) and $f(0) = 1$. If $q \leqslant p$ then*

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} = \begin{cases} kd_k \quad \textit{under} \quad k \leqslant q, \\ 0 \quad \textit{under} \quad q < k \leqslant p, \\ -\sigma_k \quad \textit{under} \quad k > p. \end{cases}$$

Similarly, we can consider the case of $q > p$. In any case we have

Corollary 2. *The following equality is true*

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{z^k} \frac{df}{f} = -\sigma_k, \quad \textit{if} \quad k > \rho.$$

It follows from (3), Lemma 1 and Corollary 2 that

Corollary 3. *The following relations are true*

$$\sigma_k = -\frac{(-1)^{k-1}}{b_0^k} \begin{vmatrix} b_1 & b_0 & 0 & \dots & 0 \\ 2b_2 & b_1 & b_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kb_k & b_{k-1} & b_{k-2} & \dots & b_1 \end{vmatrix} \quad \textit{under} \quad k > \rho. \tag{8}$$

These formulas connect the power sums σ_k to the coefficients of the function f. In the case when σ_1 is an absolutely convergent series such formulas were considered in [4, §2].

3. **Finite number of zeros**

Consider the entire function of finite order of growth of the form (1). In this section we find conditions for coefficients of function whereby the function has a finite number of zeros. First of all, we need to find the order ρ of function f. To do this, we use the formula (see, e.g., [8, Ch. 8, Sec. 8.3], [7, Ch. 7, §2])

$$\varliminf_{n\to\infty} \frac{\ln(1/|b_n|)}{n \ln n} = \frac{1}{\rho}.$$

If ρ is a fractional number then the function has an infinite number of zeros. In this section we assume that ρ is integer.

We need some results for infinite Hankel matrices. They can be found in [1, Ch. 16, §10] and in [9, Ch. 2].

Consider a sequence of complex numbers $s_0, s_1,\ s_2 \dots$. This sequence defines an infinite Hankel matrix

$$S = \begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & s_4 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{9}$$

The sequent principal minors of the matrix S are designated as $D_0,\ D_1,\ D_2, \dots$:

$$D_p = |s_{j+k}|_0^{p-1}, \quad p = 0, 1, \dots .$$

We also assume that $D_{-1} = 1$.

If for each $p \in \mathbb{N}$ there is a not equal to zero minor of S of order p then the matrix has *infinite rank*. If starting with some p, all minors are equal to zero then the matrix S has *finite rank*. The smallest value of p is called the *rank* of the matrix. Here are two statements about matrices C of finite rank p (see [1, Ch. 16, §10]).

Corollary 4 (Kronecker)**.** *If an infinite Hankel matrix S of the form (9) has finite rank p then $D_{p-1} \neq 0$.*

Converse statement is also true (see [9, §11]).

Corollary 5 (Frobenius)**.** *If minor of infinite Hankel matrix $D_{p-1} \neq 0$ and minors $D_p = D_{p+1} = \dots D_{p+j} = \dots = 0$ then the rank of the matrix S is finite and it is equal to p.*

Theorem 3. *Infinite Hankel matrix has finite rank p if and only if there are p integers $c_1,\ c_2, \dots, c_p$ such that*

$$s_j = \sum_{j=1}^{p} c_j s_{p-j} \quad j > p.$$

This theorem is given in [1, Ch. 16, §10, Theorem 7].

Theorem 4. *Matrix S has finite rank p if and only if the sum of the series*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots \tag{10}$$

is a rational function with respect to z. In this case, the rank of the matrix S coincides with the number of poles of $R(z)$. Each pole is considered with regard to its multiplicity.

This statement is given in [1, Ch. 16, §10].

Consider again an entire function $f(z)$ with integer order ρ. By the properties of an entire function the power sums σ_k are absolutely convergent series when $k > \rho$. We introduce $s_j = \sigma_{2k_0+j}$, $2k_0 > \rho + 1$, $j = 0, 1, \ldots$. Consider an infinite Hankel matrix S of the form (9).

Theorem 5. *Function f has a finite number of zeros if and only if the rank of the matrix S is finite. The number of distinct zeros of function f is equal to the rank of the matrix S.*

Proof. Let us assume that $\alpha_1, \ldots, \alpha_p$ are zeros of function f. The number of zeros is finite (each root is considered with regard to its multiplicity). Then

$$\sigma_k = \sum_{j=1}^{p} \frac{1}{\alpha_j^k}, \quad k \geqslant k_0 > \rho, \tag{11}$$

and $s_j = \sigma_{2k_0+j}$, $j \geqslant 0$.

Let us consider a polynomial $P(x)$ of degree p with the roots $1/\alpha_1, \ldots,$ $1/\alpha_p$ and with the coefficient at the highest degree equal to 1:

$$P(z) = z^p + c_1 z^{p-1} + \ldots + c_{n-1} z + c_p.$$

The coefficients of the polynomial can be found with the use of the classical Newton formulas

$$\sigma_j + c_1 \sigma_{j-1} + \ldots + c_{j-1} \sigma_1 + j c_j, \quad 1 \leqslant j \leqslant p.$$

When $j > p$ they have the form

$$\sigma_j + c_1 \sigma_{j-1} + \ldots + c_p \sigma_{j-p} = 0$$

or

$$\sigma_j = -c_1 \sigma_{j-1} - \ldots \sigma_{j-p} c_p.$$

Taking sums σ_{2k_0+j} for s_j, we obtain

$$s_j = -s_{j-1} c_1 - \ldots - s_{j-p} c_p.$$

Thus, Theorem 3 shows that the rank of S is finite and it does not exceed p.

Suppose now that the rank of S is finite and it is equal to q. According to Theorem 4, this rank is the number of poles of the rational function $R(z)$ (formula (10)). Each pole is considered with regard to its multiplicity.

Series $R(z)$ converges (absolutely and uniformly) for z lying outside of a disk centered at the origin. The disk contains all poles (as yet

unknown) of function $R(z)$. Let us transform $R(z)$, assuming that $|\alpha_m z| > 1$ for all α_m:

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \ldots = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k}\left(\sum_{m=1}^{\infty}\frac{1}{\alpha_m^{2k_0+k}}\right) =$$

$$= \frac{1}{z}\sum_{m=1}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{\alpha_m^{2k_0}}\cdot\frac{1}{\alpha_m^k z^k}\right) = \frac{1}{z}\sum_{m=1}^{\infty}\frac{1}{\alpha_m^{2k_0}}\left(\sum_{k=0}^{\infty}\frac{1}{(\alpha_m z)^k}\right) =$$

$$= \frac{1}{z}\sum_{m=1}^{\infty}\frac{1}{\alpha_m^{2k_0}}\cdot\frac{\alpha_m z}{\alpha_m z - 1} = \sum_{m=1}^{\infty}\frac{1}{\alpha_m^{2k_0}}\cdot\frac{\alpha_m}{\alpha_m z - 1}.$$

Changing the order of summation of series is justified because they converge absolutely. By hypothesis, $R(z)$ is rational function. Let us show that this series contains only a finite number of terms. Consider the following function

$$R^*(w) = R\left(\frac{1}{w}\right) = \sum_{m=1}^{\infty}\frac{1}{\alpha_m^{2k_0}}\cdot\frac{\alpha_m}{\frac{\alpha_m}{w} - 1} = \sum_{m=1}^{\infty}\frac{1}{\alpha_m^{2k_0-1}}\cdot\frac{w}{\alpha_m - w}.$$

Let us analyze this series to find the convergence domain. Roots are arranged in increasing order of magnitude. Let $|w| \leqslant r < |\alpha_1|$ then

$$\left|1 - \frac{w}{\alpha_m}\right| \geqslant 1 - \frac{|w|}{|\alpha_m|} \geqslant 1 - \frac{r}{|\alpha_1|} = c.$$

That is $\dfrac{|w|}{|\alpha_m|} \leqslant \dfrac{r}{|\alpha_1|}$ and we have

$$\sum_{m=1}^{\infty}\frac{1}{|\alpha_m|^{2k_0-1}}\cdot\frac{1}{\left|1 - \frac{w}{\alpha_m}\right|} \leqslant c\sum_{m=1}^{\infty}\frac{1}{|\alpha_m|^{2k_0-1}}.$$

The last series converges by the choice of k_0. Thus series $R^*(w)$ converges absolutely and uniformly inside the disk $\{|w| < |\alpha_1|\}$. Separating out the roots with the same magnitude $|\alpha_1| = \ldots = |\alpha_n|$, we obtain

$$R^*(w) = \sum_{m=1}^{n}\frac{1}{\alpha_m^{2k_0-1}}\cdot\frac{1}{\frac{\alpha_m}{w} - 1} + \sum_{m=n+1}^{\infty}\frac{1}{\alpha_m^{2k_0-1}}\cdot\frac{1}{\frac{\alpha_m}{w} - 1}. \tag{12}$$

The first sum is a finite sum of fractions with poles $\alpha_1, \ldots, \alpha_n$. The second sum is the series that defines a holomorphic function for $|w| < |\alpha_{n+1}|$ (it is due to the consideration given above). Because $R^*(w)$

is rational function then the second series in (12) is also a rational function.) Let us consider the following function

$$R^*(w) = \frac{P(w)}{Q(w)},$$

where $P(w)$ and $Q(w)$ are polynomials. Then $P(w) = Q(w)R^*(w)$. Since left hand side of this expression is a polynomial then $Q(\alpha_1) = \ldots = Q(\alpha_n) = 0$. If s is the degree of Q then $Q(\alpha_{n+1}) = \ldots = Q(\alpha_s) = 0$. Therefore, series $R^*(w)$ has finite number of fractions and it is equal to the number q of distinct roots α_j. So rank S is q. If the number of all roots (with regard to multiplicity) is equal to p then the first part of the theorem shows that $p \geqslant q$. □

Note that by Corollary 3 power sums s_j are expressed in terms of the Taylor coefficients of the function f.

Let us assume that entire function f has real coefficients then it has either real or complex conjugate zeros. Note that in this case all the power sums σ_k and, accordingly, the numbers s_j are real.

Now we raise the question of the number of real and complex zeros. Because function $f(z)$ has a finite number of distinct zeros which is equal to the rank of Hankel matrix S, the solution of this problem is reduced to the classic problem of finding the number of distinct real roots of a polynomial (see, e.g., [1, Ch. 16, §9]).

Consider an infinite Hankel matrix S of the form (9) with $s_j = \sigma_{2k_0+j}$. The rank of the matrix is p. We introduce the truncated matrix S_p:

$$S_p = \begin{pmatrix} s_0 & s_1 & s_2 & \ldots & s_{p-1} \\ s_1 & s_2 & s_3 & \ldots & s_p \\ s_2 & s_3 & s_4 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ s_{p-1} & s_p & s_{p+1} & \ldots & s_{2p-2} \end{pmatrix}, \tag{13}$$

and truncated Hankel quadratic form

$$S_p(x,x) = \sum_{k,j=0}^{p-1} s_{j+k} x_j x_k. \tag{14}$$

Distinct zeros of function f are $\beta_1, \ldots, \beta_p$ with multiplicities $n_1, \ldots, n_p$, respectively.

Because $s_j = \sigma_{2k_0+j}$ then

$$S_p(x,x) = \sum_{k,j=0}^{p-1} \sum_{m=1}^{p} \frac{n_m}{\beta_m^{j+k+2k_0}} x_j x_k =$$

$$= \sum_{m=1}^{p} \frac{n_m}{\beta_m^{2k_0}} \left(x_0 + \frac{x_1}{\beta_m} + \ldots + \frac{x_{p-1}}{\beta_m^{p-1}} \right)^2 . \tag{15}$$

Linear forms

$$Z_m = \frac{1}{\beta_m^{2k_0}} \cdot \left(x_0 + \frac{x_1}{\beta_m} + \ldots + \frac{x_{p-1}}{\beta_m^{p-1}} \right), \quad 1 \leqslant m \leqslant p,$$

are linearly independent because the determinant composed of their coefficients is the Vandermonde determinant and it is distinct from zero. If the forms Z_m and Z_k are complex conjugate then we can consider $\frac{1}{2}(Z_m + Z_k))$ and $\frac{1}{2i}(Z_m - Z_k)$ instead. Wherein, these forms are linearly independent and real.

In relation (15) each real root corresponds to a squared number and conjugate root corresponds to the difference of squared numbers. Next we use the Frobenius theorem on rank and signature of Hankel form $S_p(x, x)$ (see, e.g., [1, Ch. 10, §10]). If we have $D_{m-1} \neq 0$ for $m = 0, \ldots, p$.

Suppose that for some $h < k$ minors $D_h \neq 0$, $D_k \neq 0$ and all intermediate minors are equal to zero, i.e., $D_{h+1} = \ldots = D_{k-1} = 0$. Sign is assigned to these zero determinants (see [1, Ch. 10, §10, Theorem 24])

$$\operatorname{sign} D_{h+j} = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign} D_h, \quad 1 \leqslant j \leqslant k-1. \tag{16}$$

Theorem 6. *The number of different real zeros of f with real coefficients is equal to the difference between the number of constant signs and the number of sign changes in the series $D_{-1},\ D_0,\ D_1, \ldots, D_{p-1}$.*

Example 2. Let us consider the function $f(z) = (1 - z)e^z$, $f(0) = 1$. The order of growth is $\rho = 1$. Then

$$\ln f(z) = z + \ln(1 - z) = 2z + \sum_{k=2}^{\infty} \frac{z^k}{k}.$$

Using Corollary 3, we obtain that $\sigma_k = 1$ for $k \geqslant 2$, i.e., rank of Hankel matrix S is equal to 1. Then by Theorem 5 the number of roots is equal to 1 and by Theorem 6 this root is real.

Remark 1. The above statements show that in the case of a finite number of zeros the study of entire functions reduces to the study of polynomials. To study other features related to the root localization in the context of [1, 2] one need to factorize function $f(z)$, i.e., to extract polynomial from this function (see [10]). Roots of this polynomial coincide with the roots of function f.

4. **Infinite number of zeros**

From the previous section we obtain

Corollary 6. *Function $f(z)$ has an infinite number of zeros if and only if the rank of the matrix S of the form (9) is infinite, where $s_j = \sigma_{2k_0+j}$.*

Conditions of infinite rank means by Corollary 5 that there is a strictly increasing sequence of positive integers j_k, $k = 1, 2, \ldots$, such that all $D_{j_k} \neq 0$.

In what follows we need some properties of infinite Hankel matrices of infinite rank.

Consider a sequence of complex numbers $s_0,\ s_1,\ \ldots, s_k, \ldots$ such that the series

$$\sum_{k=1}^{\infty} |s_k| = C < \infty. \tag{17}$$

Suppose that indices $0 \leqslant i_1 < \ldots < i_k$, and $0 \leqslant j_1 < \ldots < j_k$. Let us introduce minor

$$S\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix}.$$

It consists of the elements of S, standing at the intersection of the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. In particular,

$$D_k = S\begin{pmatrix} 0 & \ldots & k-1 \\ 0 & \ldots & k-1 \end{pmatrix}.$$

Lemma 2. *If condition (17) is fulfilled then the following inequalities are true*

$$M_k = \left| S\begin{pmatrix} 0 & \ldots & k-1 \\ j_1 & \ldots & j_k \end{pmatrix} \right| \leqslant C^k \tag{18}$$

for all $j_1, \ldots, j_k$. In particular, if $C = 1$ then $M_k \leqslant 1$. If $C < 1$ then

$$M_k \to 0 \quad \textit{under} \quad k \to \infty.$$

Proof. We prove Lemma 2 by induction with respect to k. When $k = 1$ condition (18) obviously holds. We proceed from k to $k+1$. Expanding the determinant

$$S\begin{pmatrix} 0 & \ldots & k \\ j_1 & \ldots & j_{k+1} \end{pmatrix}$$

in the last row, we have

$$\left| S\begin{pmatrix} 0 & \ldots & k \\ j_1 & \ldots & j_{k+1} \end{pmatrix} \right| = \left| \sum_{m=1}^{k+1} s_{k-1+j_m} S\begin{pmatrix} 0 & \ldots & \ldots & \ldots & k-1 \\ j_1 & \ldots & [j_m] & \ldots & j_{k+1} \end{pmatrix} \right| \leqslant$$

$$\leqslant \sum_{m=1}^{k+1} |s_{k-1+j_m}| C^k \leqslant C^{k+1},$$

taking into account that

$$\sum_{m=1}^{k+1} |s_{k-1+j_m}| \leqslant \sum_{k=1}^{\infty} |s_k| = C.$$

Symbol $[j_m]$ means that the determinant has no column with the number j_m. □

Consider an infinite Hankel form

$$S(x,x) = \sum_{j,k=0}^{\infty} s_{j+k} x_j x_k. \tag{19}$$

This double series converges absolutely, for example, when $|x_j| \leqslant j^{-2}$.

Indeed, taking into account condition (17), we have

$$|S(x,x)| \leqslant C \sum_{j,k=0}^{\infty} |x_j||x_k| = C \left(\sum_{j=0}^{\infty} |x_j| \right)^2.$$

In what follows we assume that all $D_j \neq 0$. Examples of such matrices we present later.

Consider the truncated Hankel matrix S_p of the form (13) and truncated Hankel form $S_p(x,x)$ of the form (14).

Reduction of quadratic forms to the sum of the squares gives (see [1, Ch. 10, §3])

$$S_p(x,x) = \sum_{k=0}^{p-1} \frac{1}{D_{k-1} D_k} (X_k^{(p)})^2, \tag{20}$$

where

$$X_k^{(p)} = \sum_{q=k}^{p-1} S \begin{pmatrix} 0 & \dots & k-1 & k \\ 0 & \dots & k-1 & q \end{pmatrix} x_q.$$

Lemma 2 implies that

$$|X_k^{(p)}| \leqslant C^k \sum_{q=k}^{p-1} |x_q|.$$

Setting

$$|x_q| \leqslant \frac{1}{q^2},$$

we find that for such x_q there exists a limit

$$\lim_{p\to\infty} X_k^{(p)} = X_k = \sum_{q=k}^{\infty} S\begin{pmatrix} 0 & \dots & k-1 & k \\ 0 & \dots & k-1 & q \end{pmatrix} x_q.$$

On the other hand, under the same x_q there exists a limit

$$\lim_{p\to\infty} S_p(x,x) = S(x,x).$$

Besides

$$|X_k| \leqslant C_1 C^k,$$

where

$$C_1 = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Therefore we have

$$S(x,x) = \sum_{k=0}^{\infty} \frac{1}{D_{k-1}D_k} X_k^2. \tag{21}$$

Note that in view of identity (20) when squaring X_k and substituting the result into (21), each product $x_j x_k$ in $S(x,x)$ contains only a finite number of terms.

Thus, we obtain the following statement.

Proposition 2. *If Hankel matrix S of the form (9) satisfies condition (17) and all $D_p \neq 0$ then relation (21) holds, where series*

$$X_k = \sum_{q=k}^{\infty} S\begin{pmatrix} 0 & \dots & k-1 & k \\ 0 & \dots & k-1 & q \end{pmatrix} x_q, \quad k = 0, \dots .$$

is absolutely convergent when $|x_q| \leqslant \frac{1}{q^2}$.

Let us write equality (21) in another form

$$S(x,x) = \sum_{k=0}^{\infty} \frac{D_k}{D_{k-1}} Y_k^2,$$

where

$$Y_k = \frac{1}{D_k} \sum_{q=k}^{\infty} S\begin{pmatrix} 0 & \dots & k-1 & k \\ 0 & \dots & k-1 & q \end{pmatrix} x_q. \tag{22}$$

Let us treat system (22) as an infinite system of equations with respect to x_p, $p = 0, 1, \dots$. Let us denote the infinite matrix of this system as A. Elements of the matrix are denoted as $a_{j,k}$. The matrix is upper triangular with unit main diagonal.

Consider cofactors $A(j,k)$ to elements $a_{j,k}$ of matrix A. These cofactors are well defined, since below a certain line there is unit on the main diagonal. Then the Laplace formula shows that all $A(j,k)$ are determinants of a finite matrix. It is clear that $A(j,j)=1$, $A(j,k)=0$ for $j>k$. Let us consider an infinite matrix B that consists of elements $A(k,j)$. This is also an upper triangular matrix. Multiplying matrix A by B according to the rule of matrix multiplication, we find that sums

$$\sum_{j=1}^{\infty} a_{j,k} A(j,s)$$

are finite, they are equal to 1 if $s=k$ and they are equal to 0 if $s \neq k$. This follows from the rule for finding the finite inverse matrix with unit determinant. Therefore

$$AB = BA = I,$$

where I is the infinite identity matrix.

Multiplying equality (22) by B, we obtain expressions for x_q in term of infinite series with respect to Y_k. These series obviously converge by Proposition 2.

Let us consider the entire function of finite order of growth ρ of the form (1) with an infinite number of zeros $\alpha_1, \ldots, \alpha_n, \ldots$, power sums σ_k of the form (11) and $s_j = \sigma_{2k_0+j}$. Let us check whether condition (17) is fulfilled for s_j. We have

$$\sum_{k=1}^{\infty} |s_k| = \sum_{k=0}^{\infty} \left| \sum_{j=1}^{\infty} \frac{1}{\alpha_j^{2k_0+k}} \right| \leqslant \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{|\alpha_j|^{2k_0+k}} =$$

$$= \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{|\alpha_j|^{2k_0+k}} = \sum_{j=1}^{\infty} \frac{1}{|\alpha_k|^{2k_0}} \sum_{k=1}^{\infty} \frac{1}{|\alpha_j|^{k}} = \sum_{j=1}^{\infty} \frac{1}{|\alpha_j|^{2k_0}} \cdot \frac{1}{|\alpha_j|-1},$$

if all $|\alpha_j| > 1$.

The following inequality holds

$$\frac{1}{|\alpha_j|-1} \leqslant \frac{1}{|\alpha_1|-1}$$

by virtue of monotonically increasing sequence of absolute values $|\alpha_j|$. Therefore, series (17) converges and

$$\sum_{k=1}^{\infty} |s_k| \leqslant \frac{1}{|\alpha_1|-1} \cdot \sum_{j=1}^{\infty} \frac{1}{|\alpha_j|^{2k_0}} = C.$$

Remark 2. If some of the roots $|\alpha_j| \leqslant 1$ then we can consider the function $f(rz)$, where $r>0$. Zeros of this function are α_j/r. Therefore,

for sufficiently small r, they are greater than 1 in absolute value. Then sums σ_k are multiplied by r^k and minors d_p are multiplied by $r^{2k_0+p(p-1)}$. As this takes place, determinants d_p are not equal to zero and their signs are not changed. Following statements are true for arbitrary f.

In what follows we assume that all $|\alpha_j| > 1$. Therefore, for such f Proposition 2 is valid. Let us recall the Binet-Cauchy formula for the product of rectangular matrices. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix}.$$

Matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mm} \end{pmatrix},$$

is $C = A \cdot B$, then the next formula of Binet-Cauchy is true (see, e.g., [1, Ch. 1, §2])

(23)

$$\det C = \sum_{1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant n} A \begin{pmatrix} 1 & 2 & \dots & m \\ k_1 & k_2 & \dots & k_m \end{pmatrix} \cdot B \begin{pmatrix} k_1 & k_2 & \dots & k_m \\ 1 & 2 & \dots & m \end{pmatrix}.$$

Suppose that A and B are infinite rectangular matrices of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \dots \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Lemma 3. *If series*

$$\sum_{n=1}^{\infty} a_{k_1 n} \cdot b_{n k_2} \tag{24}$$

converges absolutely for all $1 \leqslant k_1, k_2 \leqslant m$*, then we have the formula*

(25)

$$\det C = \sum_{1 \leqslant k_1 < k_2 < \ldots < k_m} A \begin{pmatrix} 1 & 2 & \dots & m \\ k_1 & k_2 & \dots & k_m \end{pmatrix} \cdot B \begin{pmatrix} k_1 & k_2 & \dots & k_m \\ 1 & 2 & \dots & m \end{pmatrix}$$

and the resulting series converges absolutely.

To prove equality (25) we apply the Binet-Cauchy formula (23) to finite submatrices of matrix A of order $(m \times n)$ and to finite submatrices of matrix B of order $(n \times m)$ and then we take the limit $n \to \infty$. The convergence of the resulting series ensures the convergence of series (24). $\square$

For function f we introduce infinite matrix Δ with elements

$$\delta_{kj} = \frac{1}{\alpha_j^{2k_0+k}}, \quad k, j = 0, 1, \dots k \dots ,$$

and Δ' is the transpose of matrix Δ.

Suppose that matrix Δ_m consists of the first m rows of the matrix Δ.

Proposition 3. *Determinants D_m are represented in the form*

$$D_m = \sum_{1 \leqslant k_1 < k_2 < \dots < k_m} \left[\Delta_m \begin{pmatrix} 1 & 2 & \dots & m \\ k_1 & k_2 & \dots & k_m \end{pmatrix} \right]^2 \tag{26}$$

and this series converges absolutely.

Proof. We apply formula (25) to the matrices Δ_m and Δ'_m. We use the following facts: matrix and its transpose have the same determinant and

$$s_k = \sigma_{2k_0+k} = \sum_{j=1}^{\infty} \frac{1}{\alpha_j^{2k_0+k}}.$$

Therefore, s_{j+k} is infinite inner product of infinite rows of the matrix Δ_m and the infinite columns of the matrix Δ'_m.

□

Theorem 7. *Let us assume that function $f(z)$ has real coefficients. All zeros of function f are real if and only if $D_m > 0$, $m = 0, 1, \dots$.*

Proof. Suppose that all zeros of $f(z)$ are real. Using formula (26), we obtain that D_m is the sum of non-negative minors, some of them are strictly positive, since they are Vandermonde determinants with different columns.

Let all $D_m > 0$. Consider an infinite Hankel form $S(x, x)$ of the form (19) with real variables x_j. We previously obtained that this form is absolutely convergent double series when $|x_k| \leqslant k^{-2}$, $k \geqslant 1$.

Assume that $\beta_1,\ \beta_2, \dots, \beta_k \dots$ are distinct zeros of function f with multiplicities $n_1,\ n_2, \dots, n_k, \dots$, respectively.

Let us transform the form $S(x, x)$:

$$S(x, x) = \sum_{j,k=0}^{\infty} s_{j+k} x_j x_k = \sum_{j,k=0}^{\infty} \sum_{m=1}^{\infty} \frac{n_m}{\beta_m^{2k_0+j+k}} x_j x_k =$$

$$= \sum_{m=1}^{\infty} \sum_{j,k=0}^{\infty} \frac{n_m}{\beta_m^{2k_0+j+k}} x_j x_k = \sum_{m=1}^{\infty} \frac{n_m}{\beta_m^{2k_0}} \left(x_0 + \frac{x_1}{\beta_m} + \dots + \frac{x_k}{\beta_m^k} + \dots \right)^2.$$

Permutation of the order of summation is justified since corresponding series converge.

Let us denote

$$Z_m = \frac{1}{\beta_m^{k_0}} \left(x_0 + \frac{x_1}{\beta_m} + \ldots + \frac{x_k}{\beta_m^k} + \ldots \right),$$

then we find that

$$S(x,x) = \sum_{m=1}^{\infty} n_m Z_m^2. \tag{27}$$

If a zero β_m is real then $Z_m^2 > 0$. If a zero is complex then the sum of squares $Z_m^2 + \overline{Z}_m^2 = P_m^2 - Q_m^2$, where $P_m = \operatorname{Re} Z_m$ and $Q_m = \operatorname{Im} Z_m$. This means that a positive square in representation (27) corresponds to a real zero and difference of squares corresponds a complex zero. Relation (27) can be written as

$$S(x,x) = \sum_{m=1}^{\infty} r_m F_m^2, \tag{28}$$

where r_m is 1 or -1 and linear infinite forms F_m are real.

One should show that if all $D_m > 0$, then all $r_m = 1$ in (28). Suppose that $r_{m_0} = -1$ for some m_0. Consider the system of equations

(29)

$$F_0 = 0, \ldots F_{m_0-1} = 0, F_{m_0+1} = 0, \ldots F_k = 0 \text{ for sufficiently large } k.$$

All equations in this system are different.

Let us rearrange all x_j with $j > k$ to the right side of system (29). Then we have some convergent series in the right side of the system. Coefficients at $x_0, \ldots, x_{k-1}$ form the Vandermonde matrix (or its real or imaginary part). Therefore, we express $x_0, \ldots, x_{k-1}$ from system (29) in the form of convergent series of other variables. Substituting $x_0, \ldots, x_{k-1}$ into system (29), we obtain that the right side is equal to zero.

It is clear that on substitution these solutions into F_{m_0} this form cannot be identically equal to zero because this equation is not a consequence of the system of equations (29). Then there are exist $x_k, \ldots$, such that $F_{m_0} \neq 0$.

Let us recall that the form F_m follows from form Z_m and all variables $|x_j|$ are bounded by a constant c_1. Then we obtain

$$|Z_m| \leqslant \frac{n_m}{|\beta_m|^{2k_0}} \left(|x_0| + \left| \frac{x_1}{\beta_m} \right| + \ldots + \left| \frac{x_k}{\beta_m^k} \right| + \ldots \right) \leqslant$$

$$\leqslant \frac{c_1 n_m}{|\beta_m|^{2k_0}} \left(1 + \left|\frac{1}{\beta_m}\right| + \ldots + \left|\frac{1}{\beta_m^k}\right| + \ldots\right) = \frac{c_1}{|\beta_m|^{2k_0}} \frac{n_m}{|\beta_m| - 1}.$$

We assume that $|\beta_m| > 1$ for all m. Therefore,

$$|Z_m| \leqslant \frac{c_1 n_m}{|\beta_m|^{2k_0+1}}.$$

Then

$$\sum_{m=p}^{\infty} |Z_m| \leqslant c_1 \sum_{m=p}^{\infty} \frac{1}{|\alpha_m|^{2k_0+1}} = \delta. \tag{30}$$

The last expression is the remainder of the convergent series, so it can be made arbitrarily small uniformly with respect to x_j. Choosing sufficiently large k such that $\delta < |F_{m_0}|$, we find that $S(x, x)$ has a negative value in this case.

At the same time we have representation (21) for $S(x, x)$. It shows that by the conditions of the theorem it is non-negative. This contradicts our assumption that $r_{m_0} = -1$. □

Example 3. Let us consider the following entire function $f(z)$ of order $\rho = \dfrac{1}{2}$:

$$f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n+1)!}.$$

It is known (see, e.g., [11, Ch. 12, Issue 449]) that

$$\ln \frac{\sin\sqrt{z}}{\sqrt{z}} = -\sum_{n=1}^{\infty} \frac{2^{2n-1} B_n}{(2n)!} \cdot \frac{z^n}{n}, \tag{31}$$

where B_n are Bernoulli numbers (see, e.g., [11, Ch. 12, Issue 449]). They are all positive.

It follows from (3) and Corollary 2 that

$$\sigma_n = \frac{2^{2n-1} B_n}{(2n)!} \quad \text{when} \quad n \geqslant 1.$$

Let us set $s_n = \sigma_{n+1}$, D_m is defined as

$$D_m = \begin{vmatrix} s_0 & s_1 & \ldots & s_{m-1} \\ s_1 & s_2 & \ldots & s_m \\ \ldots & \ldots & \ldots & \ldots \\ s_{m-1} & s_m & \ldots & s_{2m-1} \end{vmatrix},$$

and $B_1 = \dfrac{1}{6}$, $B_2 = \dfrac{1}{30}$, $B_3 = \dfrac{1}{42}$, $B_4 = \dfrac{1}{30}$, $B_5 = \dfrac{5}{66}, \ldots$.

We factor out terms to the power of 2 from determinant D_m and find that the sign of D_m coincides with the sign of the determinant

$$\Delta'_m = \begin{vmatrix} \frac{B_1}{2!} & \frac{B_2}{4!} & \dots & \frac{B_m}{(2m)!} \\ \frac{B_2}{4!} & \frac{B_3}{6!} & \dots & \frac{B_{m+1}}{(2m+2)!} \\ \dots & \dots & \dots & \dots \\ \frac{B_m}{(2m)!} & \frac{B_{m+1}}{(2m+2)!} & \dots & \frac{B_{2m}}{(4m)!} \end{vmatrix}$$

Therefore, $\Delta'_1 = s_0 = \dfrac{1}{12} > 0$ and

$$\Delta'_2 = \begin{vmatrix} \frac{1}{12} & \frac{1}{30\cdot 24} \\ \frac{1}{30\cdot 24} & \frac{1}{42\cdot 720} \end{vmatrix} = \frac{1}{12\cdot 720} \begin{vmatrix} 1 & \frac{1}{60} \\ 1 & \frac{1}{42} \end{vmatrix} > 0.$$

Similarly we have

$$\Delta'_3 = \begin{vmatrix} \frac{1}{12} & \frac{1}{30\cdot 24} & \frac{1}{42\cdot 720} \\ \frac{1}{30\cdot 24} & \frac{1}{42\cdot 720} & \frac{1}{30\cdot 8!} \\ \frac{1}{42\cdot 720} & \frac{1}{30\cdot 8!} & \frac{5}{66\cdot (10)!} \end{vmatrix} = c \begin{vmatrix} 1 & \frac{1}{30} & \frac{1}{210} \\ 1 & \frac{1}{21} & \frac{1}{140} \\ 1 & \frac{1}{20} & \frac{1}{99} \end{vmatrix} > 0,$$

where $c > 0$.

This is fully consistent with Theorem 7. Moreover, Theorem 7 shows that all D_m composed of Bernoulli numbers are positive.

Let us consider the case when function $f(z)$ has only imaginary zeros. In the case of polynomials the condition can be found in terms of inners wherein they have imaginary roots [2, §2.4].

Theorem 8. *Let an entire function $f(z)$ of the form (1) with real Taylor coefficients has the order of growth $\rho < 2$. For function f has only imaginary roots it is necessary and sufficient that the determinants*

$$\Delta_m = \begin{vmatrix} s_0 & s_2 & \dots & s_{2m-2} \\ s_2 & s_4 & \dots & s_{2m} \\ \dots & \dots & \dots & \dots \\ s_{2m-2} & s_{2m} & \dots & s_{4m-2} \end{vmatrix}, \tag{32}$$

are positive for all m and conditions

$$\sum_{m=0}^{n} \frac{b_{n-m}(-b_1)^m}{m!} \begin{cases} = 0 & \textit{for} \quad \textit{even} \quad n, \\ \geqslant 0 & \textit{for} \quad \textit{odd} \quad n. \end{cases} \tag{33}$$

are satisfied.

Proof. Let us assume that function f has imaginary zeros $\pm i\gamma_j$, $\gamma_j \in \mathbb{R}$. Then the Taylor decomposition of the canonical product of

f contains only even powers of z. Indeed, the product in (5) has the form

$$\left(1-\frac{z}{i\gamma_n}\right)e^{\frac{z}{i\gamma_n}+\frac{z^2}{2(i\gamma_n)^2}+\ldots+\frac{z^p}{p(i\gamma_n)^p}}\cdot\left(1+\frac{z}{i\gamma_n}\right)e^{\frac{z}{-i\gamma_n}+\frac{z^2}{2(-i\gamma_n)^2}+\ldots+\frac{z^p}{p(-i\gamma_n)^p}}=$$

$$=\left(1+\frac{z^2}{\gamma_n^2}\right)e^{\frac{z^2}{-\gamma_n^2}+\ldots+\left(\frac{z^p}{p(i\gamma_n)^p}+\frac{(-z)^p}{p(i\gamma_n^p)}\right)}.$$

So this product depends on z^2. Therefore, the canonical product (6) that corresponds to function f, takes the form:

$$\Phi(z)=\sum_{j=0}^{\infty}c_jz^j=\sum_{j=0}^{\infty}c_{2j}z^{2j}.$$

We introduce the function

$$H(w)=\sum_{j=0}^{\infty}c_{2j}w^j.$$

Zeros of this function are only numbers $-\gamma_n^2$. Therefore, $H(w)$ has only real zeros. Function $H(w)$ is also an entire function of finite order of growth and the order is hulf of the order of the canonical product $\Phi(z)$. So in our case the order of growth of the function $H(w)$ less than one hulf.

Let us consider the power sums for the function $H(w)$:

$$\sum_{n=1}^{\infty}\left(-\gamma_n^{-2}\right)^k.$$

It is clear that this power sum is equal to the sum of σ_{2k} of function $f(z)$.

Since by Theorem 2 $p\leqslant\rho$ and $q\leqslant\rho$ then in our case $p=0$ and $q=0$.

Therefore, Hadamard decomposition of the function $H(w)$ has the form

$$H(w)=\prod_{n=1}^{\infty}\left(1-\frac{w}{\beta_n}\right).$$

It is clear that if all zeros of this function are real they are negative if and only if that the Taylor expansion of $H(w)$ has non-negative coefficients c_{2j}. These coefficients are the coefficients of the Taylor expansion for the canonical product $\Phi(z)$.

The necessary condition that all zeros are imaginary is that the Taylor coefficients of the canonical product $\Phi(z)$ have non-negative values and odd-numbered coefficients are equal to 0.

Let us find condition on the Taylor coefficients of the function $f(z)$ which guarantees non-negativity of the coefficients of the canonical product. Because the order of growth of the function is $\rho < 2$ then its Hadamard expansion takes the form:

$$f(z) = e^{d_1 z} \cdot \Phi(z),$$

where $d_1 < 2$ and may be equal to zero.

Using Proposition 2, Lemma 1 and (3), we find that $d_1 = b_1$.

Then $\Phi(z) = e^{-b_1 z} \cdot f(z)$. Therefore,

$$\Phi(z) = \sum_{m=0}^{\infty} \frac{(-b_1 z)^m}{m!} \cdot \sum_{k=0} b_k z^k = \sum_{n=0}^{\infty} z^n \sum_{m+k}^{n} \frac{b_k(-b_1)^m}{m!} =$$

$$= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} \frac{b_{n-m}(-b_1)^m}{m!}.$$

So the Taylor coefficients of the canonical product $\Phi(z)$ are

$$c_n = \sum_{m=0}^{n} \frac{b_{n-m}(-b_1)^m}{m!}.$$

It follows from the last relation and Theorem 7 that the theorem is proved. □

Remark 3. If $b_1 = 0$ then function $f(z)$ coincides with its canonical product and condition (33) is equivalent to

$$b_{2n} \geqslant 0, \quad b_{2n-1} = 0.$$

Recall the definition of a type of entire function.

Let $f(z)$ have a finite order ρ. If there exists a positive number K such, that

$$f(z) = O(e^{KR^\rho}) \quad \text{for} \quad |z| = R \to +\infty$$

then the function f has a finite type.

The infimum of such numbers K is called the *type* of function and denotes κ.

If function $f(z)$ is an entire function either of order $\rho < 1$ or of order $\rho = 1$ and $\kappa = 0$ then its Hadamard expansion has the form (see, e.g., [7, Ch. 7, §2])

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right).$$

Therefore, the following statement is true

Corollary 7. *Let a function f have real coefficients and let it be an entire function either of order $\rho < 1$ or of order $\rho = 1$ and $\kappa = 0$. All zeros of function f are imaginary if and only if all minors of the form (32) are positive, the even-numbered Taylor coefficients of function f are non-negative and the odd-numbered coefficients are zero.*

Proof. In this case the entire function coincides with its canonical product, the previous considerations include this statement. □

Example 4. Let us consider the following function

$$f(z) = \frac{\sinh z}{z} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k+1)!}.$$

This function has the order of growth $\rho = 1$ and it coincides with its canonical product. The even-numbered coefficients of the Taylor expansion of $f(z)$ are positive and the odd-numbered coefficients are zero. Therefore, the necessary condition of Theorem 8 is satisfied.

Consider the determinants of Theorem 8. Formula (31) gives

$$\ln \frac{\sinh z}{z} = \ln \frac{\sin iz}{iz} = -\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} B_n}{(2n)!} \cdot \frac{z^{2n}}{2n}.$$

Therefore, the power sums $a(x)$ with even numbers are

$$\sigma_{2n} = \frac{(-1)^n 2^{2n-1} B_n}{(2n)!} \quad \text{under} \quad n \geqslant 1,$$

and $s_{sn} = \sigma_{2n+2}$. Determinant Δ_m in Example 3 takes the form

$$\Delta'_m = \begin{vmatrix} \frac{B_1}{2!} & \frac{-B_2}{4!} & \dots & \frac{(-1)^m B_m}{(2m)!} \\ \frac{-B_2}{4!} & \frac{B_3}{6!} & \dots & \frac{B_{m+1}}{(2m+2)!} \\ \dots & \dots & \dots & \dots \\ \frac{(-1)^m B_m}{(2m)!} & \frac{(-1)^{m+1} B_{m+1}}{(2m+2)!} & \dots & \frac{B_{2m}}{(4m)!} \end{vmatrix}.$$

The element in this determinant that stands at the intersection of j-th row and s-th column is negative if and only if $j + s$ is odd. Then this determinant is equal to the determinant Δ_m from Example 3. In fact, the product of elements taken one by one from each row and each column contains an even number of elements with odd sums of row number and column number.

Therefore, this example is reduced to Example 3.

Now consider the case when D_m are positive or negative. Suppose, as before, that all $D_m \neq 0$. We introduce the sequence

$$D_{-1}D_0,\ D_0D_1, \dots, D_mD_{m+1}, \dots, \tag{34}$$

Recall that $D_{-1} = 1$.

Theorem 9. *Let us assume that function f has real coefficients. If sequence (34) contains exactly m negative numbers then function f has m distinct pairs of complex conjugate zeros and an infinite number of real zeros. If sequence (34) has an infinite number of negative numbers then function $f(z)$ has an infinite number of complex conjugate zeros.*

The proof essentially repeats the proof of Theorem 7 and the arguments from [1, Ch. 10, §1]. If sequence (34) contains exactly m negative numbers then we choose sufficiently big k and $k > m$. Suppose that the sequence $Z_1, \ldots, Z_k$ contains s squares with negative sign and $s > m$. Then we set to zero in expression (21) those X_k which are included in (21) with negative sign. There are exactly m such equations. We set to zero squares with positive sign in relation (28). There are exactly $k - s$ such equations and $(k - s) + m < k$. Then there is no zero solution of these equations. Using inequality (30), we see that expression (21) is non-negative and expression (28) is strictly less than zero. This is impossible. The case $s < m$ is treated similarly. It is clear that if there is an infinite number of negative numbers in sequence (34) then the number of complex zeros is infinite. □

Remark 4. Sequence (34) has always an infinite number of positive elements, while the number of real zeros of function $f(z)$ can be finite or infinite. Therefore, this sequence cannot be used to obtain condition of existence of a finite number of real zeros.

Remark 5. In Remark 2 we note that if the condition $|\alpha j| > 1$ is not satisfied for all j then the transformation $z = rw$ does not change the signs of all D_m. In addition, under such transformation real roots remain real and complex roots remain complex so Theorems 7–9 remain true.

Remark 6. If sequence (34) has zero elements then theorem 9 remains true if signs of sequence elements are arranged according to the Frobenius rule (16). We also note that we have excluded the case where D_m are equal to zero, starting from a certain number, since the number of function zeros is infinite.

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