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On some Sufficient Condition for the Equality of Multi-clone and Super-clone

Nikolay A. Peryazev*
Saint Petersburg State Electrotechnical University
Professor Popov, 5, Saint Petersburg, 197376
Russia

Ivan K. Sharankhaev†
Institute of Mathematics and Informatics
Buryat State University
Smolin, 24a, Ulan-Ude, 670000
Russia

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Multi-clones and super-clones are considered in this paper. They are generalizations of clones. To get a super-clone one need to add to a multi-clone the closure condition with respect to solvability of the simplest equation. The condition for identity of multi-clone and super-clone is proved.

Keywords: multi-operation, multi-clone, super-clone, superposition, operation, substitution.
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Introduction

Clones are studied most actively in the theory of functional systems [1]. Clones are sets of operations that are closed with respect to superposition, and they contain all projection operators. Recently interest in generalizations of clones, namely, hyperclones, multiclones and superclones has been raised [2].

Multi-clone is a set of multi-operations which are closed with respect to superposition, and it contains all complete, empty and projection operations. A super-clone is obtained from a multi-clone by adding the closure condition with respect to solvability of the simplest equation. It is known that super-clones are closely related to clones. Complete Galois connection between them was established [3]. Condition of the equality of multi-clone and super-clone is obtained in this paper.

Let A be an arbitrary finite set, and $B(A)$ be the set of all subsets of A including $\varnothing$.

A mapping from A^n into A is described as an *n-ary operation* on A (the case $n = 0$ is possible). The set of all n-ary operations on A is described as P_A^n, and the set of all operations on A is described as

$$P_A = \bigcup_{n\geqslant 0} P_A^n.$$

A mapping from A^n into $B(A)$ is described as an *n-ary multi-operation* on A (the case $n = 0$ is possible). The set of all n-ary multi-operations on A is described as M_A^n, and the set of all

*nikolai.baikal@gmail.com
†goran5@mail.ru

multi-operations on A is described as

$$M_A = \bigcup_{n \geqslant 0} M_A^n.$$

Multi-operation θ^n of dimension n is described as *empty operation* if for all elements $a_1, \ldots, a_n$ of A relation

$$\theta^n(a_1, \ldots, a_n) = \varnothing$$

is true.

Multi-operation π^n of dimension n is described as a *complete poperation* if for all elements $a_1, \ldots, a_n$ of A relation

$$\pi^n(a_1, \ldots, a_n) = A$$

is true.

Multi-operation e_i^n of dimension n is described as a *projection multi-operation* with respect to i-th argument if for all elements $a_1, \ldots, a_n$ of A relation

$$e_i^n(a_1, \ldots, a_n) = \{a_i\}$$

is true.

Let us note that multi-operation e_i^n can be also considered as operation on A.

Superposition for $f \in M_A^n$ and $f_i \in M_A^m$, $(i = 1, \ldots, n)$, described as

$$(f * f_1, \ldots, f_n),$$

is defined as follows

$$(f * f_1, \ldots, f_n)(a_1, \ldots, a_m) = \bigcup_{b_i \in f_i(a_1, \ldots, a_m)} f(b_1, \ldots, b_n)$$

for all $a_1, \ldots, a_m$ of A.

If $f, f_1, ..., f_n$ are operations then we have definition of operation superposition.

Every subset $K \subseteq P_A$ is described as a *clone* on A if it contains all projection operations, and it is closed with respect to superpositions.

Every subset $R \subseteq M_A$ is described as a *multi-clone* on A if it contains all empty, complete multi-operations, projection multi-operations, and it is closed with respect to superpositions.

1. Super-clones

Solvability with respect to i-th argument for an n-ary multi-operation f is such a multi-operation $(\mu_i f)$ that for all $a_1, \ldots, a_n$ of A relation

$$(\mu_i f)(a_1, \ldots, a_n) = \{a | a_i \in f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)\}$$

is satisfied.

Substitution of a multi-operation g into the place of i-th argument of a multi-operation f is such multi-operation $(f *_i g)$ that relation

$$(f *_i g)(a_1, \ldots, a_{n+m-1}) = \bigcup_{b \in g(a_i, \ldots, a_{i+m-1})} f(a_1, \ldots, a_{i-1}, b, a_{i+m}, \ldots, a_{n+m-1})$$

is satisfied.

Identification of i and j arguments of a multi-operation f is such a multi-operation $(\Delta_{i,j} f)$ that for all $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ of A relation

$$(\Delta_{i,j} f)(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) = f(a_1, \ldots, a_{j-1}, a_i, a_{j+1}, \ldots, a_n)$$

is satisfied.

Intersection of multi-operations f and g from M_A^n is such a multi-operation $(f \cap g)$ that for all $a_1, \ldots, a_n$ of A relation

$$(f \cap g)(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) \cap g(a_1, \ldots, a_n)$$

is satisfied.

Let us note that by analogy with clones multi-clone can be defined as any subset $R \subseteq M_A$ [4]. A multi-clone contains all empty, complete multi-operations, projection multi-operations, and it is closed with respect to substitutions and identifications.

Lemma 1. *For a set of multi-operations A that contains all empty, complete multi-operations, projection multi-operations the following conditions are equivalent:*
1) A is closed with respect to superpositions and solvabilities;
2) A is closed with respect to substitutions, solvabilities and identifications;
3) A is closed with respect to substitutions, solvabilities and intersections.

Proof. Equivalence of 1) and 2) follows from representation of superposition in terms of substitutions and identifications of arguments, and permutation of arguments i and j of a multi-operation f is expressed as

$$\mu_i(\mu_j(\mu_i f)).$$

Equivalence of 2) and 3) follows from equality

$$\Delta_{i,j} f^n = (((\pi^1 *_1 (e_i^n \cap e_j^n)) \cap f^n) *_j \pi^0).$$

Equivalence of 3) and 1) follows from identity

$$(f \cap g) = (f_\cap * f, g), \text{где} f_\cap = (e_1^2 * e_1^2, (\mu_2 e_1^2)).$$

□

A set is described as a *super-clone* if it satisfies one of the equivalent conditions of Lemma 1.

2. Semi-identity of superposition solvability

To prove the equality of super-clones and multi-clones one should transfer solvability operators through superposition. However, the possibility of such operation is still not proved. In the following lemma we give only the identity inclusion (semi-identity) and show that the identity is not satisfied.

Lemma 2. *The following semi-identity is satisfied:*

$$\mu_i(f^n * g_1^m, \ldots, g_n^m) \subseteq \bigcap_{j=1}^{n} (\mu_i g_j^m * e_1^m, \ldots, e_{i-1}^m, (\mu_j f^n * \pi^m, \ldots, \pi^m, \underbrace{e_i^m}_{j}, \pi^m, \ldots, \pi^m), e_{i+1}^m, \ldots, e_m^m).$$

If f is unary multi-operation then the following identity is satisfied:

$$\mu_i(f * g^m) = (\mu_i g^m * e_1^m, \ldots, e_{i-1}^m, (\mu f * e_i^m), e_{i+1}^m, \ldots, e_m^m).$$

Proof. Let $a \in \mu_i(f^n * g_1^m, \dots, g_n^m)(b_1, \dots, b_m)$. It follows from the definition of solvability that

$$b_i \in (f^n * g_1^m, \dots, g_n^m)(b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_m).$$

Then it follows from the definition of superposition that there are $x_1, \dots, x_n$ such that

$$b_i \in f^n(x_1, \dots, x_n),$$
$$x_j \in g_j^m(b_1, \dots, b_{i-1}, a, b_{i+1}, \dots, b_m),$$

for $j = 1, \dots, n$.

Using solvabilities with respect to various arguments, we obtain

$$a \in \mu_i g_j^m(b_1, \dots, b_{i-1}, x_j, b_{i+1}, \dots, b_m),$$

$$x_j \in \mu_j f^n(x_1, \dots, x_{j-1}, b_i, x_{j+1}, \dots, x_n),$$

for $j = 1, \dots, n$.

Then

$$a \in (\mu_i g_j^m * e_1^m, \dots, e_{i-1}^m, (\mu_j f^n * \pi^m, \dots, \pi^m, \underbrace{e_i^m}_{j}, \pi^m, \dots, \pi^m), e_{i+1}^m, \dots, e_m^m)(b_1, \dots, b_m),$$

for every $j = 1, \dots, n$.

Thus we have

$$a \in \bigcap_{j=1}^{n} (\mu_i g_j^m * e_1^m, \dots, e_{i-1}^m, (\mu_j f^n * \pi^m, \dots, \pi^m, \underbrace{e_i^m}_{j}, \pi^m, \dots, \pi^m), e_{i+1}^m, \dots, e_m^m)(b_1, \dots, b_m).$$

Obviously, when $n = 1$ all reverse consequences are satisfied, and hence the identity holds. □

The following example shows that the reverse inclusion is not always true. Let us use vector representation of multi-operations [3].
Let $f^2 = (501042013)$, $g_1^1 = (465)$, $g_2^1 = (736)$. Then

$$\mu_1(f^2 * g_1^1, g_2^1) = (752),$$

$$(\mu_1 g_1^1 * (\mu_1 f^2 * e_1^1, \pi^1)) \cap (\mu_1 g_2^1 * (\mu_2 f^2 *, \pi^1, e_1^1)) = (756).$$

We obtain

$$\mu_1(f^2 * g_1^1, g_2^1) \subset (\mu_1 g_1^1 * (\mu_1 f^2 * e_1^1, \pi^1)) \cap (\mu_1 g_2^1 * (\mu_2 f^2 *, \pi^1, e_1^1)).$$

3. Equality of multi-clone and super-clone

Below identities for the transfer of the solvability operator inside the term are found. This is appropriate for terms over intersection and substitution. In what follows brackets that are uniquely recovered are removed.

Lemma 3. *The following identities are satisfied:*
1) $\mu_i(f \cap g) = \mu_i f \cap \mu_i g$;
2) $\mu_i(f^n *_j g^m) = (\mu_i f^n *_j g^m)$ *for* $i \in \{1, ..., j-1, j+m, ..., n+m-1\}$;
3) $\mu_i(f^n *_j g^m) = \alpha^{n+m-1}(\mu_i g^m *_i \mu_j f^n)$ *for* $i \in \{j, ..., j+m-1\}$, *where* α^{n+m-1} *is some transposition of arguments.*

Proof. 1) Let for all $a_1, \dots, a_n$ relation

$$a \in \mu_i(f \cap g)(a_1, \dots, a_n)$$

is satisfied. According to the definition of μ_i, there is a relationship

$$a_i \in (f \cap g)(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n)$$

Then we have $a_i \in f(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n)$ and $a_i \in g(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n)$, and also $a \in \mu_i f(a_1, \dots, a_n)$ and $a \in \mu_i g(a_1, \dots, a_n)$. Thus, we obtaine the following condition

$$a \in \mu_i f(a_1, \dots, a_n) \cap \mu_i g(a_1, \dots, a_n).$$

This condition is equivalent to the original condition, and equality 1) is proved.

2) Let for all $a_1, \dots, a_{n+m-1}$ relation

$$a \in \mu_i(f^n *_j g^m)(a_1, \dots, a_{n+m-1})$$

is satisfied, where $i \in \{1, \dots, j-1, j+m, \dots, n+m-1\}$. According to the definition of μ_i, relation

$$a_i \in (f^n *_j g^m)(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_{n+m-1})$$

is also satisfied. According to the definition of $*_j$, there is an element a_0 such that $a_0 \in g^m(a_j, \dots, a_{j+m-1})$ and $a_i \in f^n(a_1, \dots, a_{j-1}, a_0, a_{j+m}, \dots, a_{n+m-1})$, where $i \in \{1, \dots, j-1, j+m, \dots, n+m-1\}$. These conditions are equivalent to the following conditions:

$$a_0 \in g^m(a_j, \dots, a_{j+m-1}) \text{ and } a \in \mu_i f^n(a_1, \dots, a_{j-1}, a_0, a_{j+m}, \dots, a_{n+m-1}).$$

Thus, we obtain

$$a \in (\mu_i f^n *_j g^m)(a_1, \dots, a_{n+m-1}).$$

This condition is equivalent to the original condition. Equality 2) is proved.

3) Let for all $a_1, \dots, a_{n+m-1}$ relation

$$a \in \mu_i(f^n *_j g^m)(a_1, \dots, a_{n+m-1})$$

is satisfied, where $i \in \{j, \dots, j+m-1\}$. According to the definition of μ_i, relation

$$a_i \in (f^n *_j g^m)(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_{n+m-1})$$

is also satisfied. According to the definition of $*_j$, there is an element a_0 such that $a_0 \in g^m(a_j, \dots, \underset{i}{a}, \dots, a_{j+m-1})$ and

$$a_i \in f^n(a_1, \dots, a_{j-1}, a_0, a_{j+m}, \dots, a_{n+m-1}).$$

Hence we have $a \in \mu_i g^m(a_j, \dots, \underset{i}{a_0}, \dots, a_{j+m-1})$ and

$$a_0 \in \mu_j f^n(a_1, \dots, a_{j-1}, a_i, a_{j+m}, \dots, a_{n+m-1}).$$

Then

$$a \in (\mu_i g^m *_i \mu_j f^n)(a_j, \dots, a_{i-1}, a_1, \dots, a_{j-1}, a_i, a_{j+m}, \dots, a_{n+m-1}, a_{i+1}, \dots, a_{j+m-1}).$$

Thus, we obtain

$$a \in (\mu_i g^m *_i \mu_j f^n)(a_j, \dots, a_{i-1}, a_1, \dots, a_{j-1}, a_i, a_{j+m}, \dots, a_{n+m-1}, a_{i+1}, \dots, a_{j+m-1}).$$

This condition is equivalent to the original condition for transposition of elements $a_1, \dots, a_{n+m-1}$. Equality 3) is proved. □

Theorem. *Let us assume that a set of multi-operations R contains multi-operation e_1^2 and it is closed with respect to solvabilities. If multi-clone and super-clone are generated by R then they are equal.*

Proof. In what follows we use standard concept of term over the set $\{*_i, \mu_i, \cap\}$. The notation $\Phi[f_1, \dots, f_k]$ means that term Φ depends on $f_1, \dots, f_k$.

Let us assume that an arbitrary multi-operation g is represented by term $\Phi[f_1, \dots, f_k]$ in a super-clone, where $f_s \in R$, $s = 1, \dots, k$. Using identities of Lemma 3, we transform term Φ into term $\Psi[h_1, \dots, h_r]$ in which μ_i can occur only for h_j, where $h_j \in R$, $j = 1, \dots, r$. According to conditions of the theorem, $\mu_i h_j \in R$. Since $(\mu_2 e_1^2) \in R$ the intersection is expressed by a term because $(f \cap g) = (f_\cap * f, g)$, where $f_\cap = (e_1^2 * e_1^2, (\mu_2 e_1^2))$. Thus, we obtain representation of g by the term over R without the use of the condition of closure with respect to solvability. □

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References

[1] D.Lau, Function Algebras on Finite Sets, Springer-Verlag, Berlin YeideWater Resourses Research, 2006.

[2] N.A.Peryazev, Clones, co-clones, hyperclones and superclones, *Uchenie Zapiski Kazan. Univer. Ser. Fiz. Matem. Nauki*, **151**(2009), no. 2, 120–125 (in Russian).

[3] N.A.Peryazev, I.K.Sharankhaev, Galois theory for clones and superclones, *Diskr. mat.*, **27**(2015), no 4, 79–93 (in Russian).

[4] A.I.Maltsev, Iterative algebras and Post varieties, *Algebra i Logika*, **5**(1966), no. 2, 5–24 (in Russian).

Об одном достаточном условии равенства мультиклона и суперклона

Николай А. Перязев
Санкт-Петербургский государственный электротехнический университет
Попова, 5, Санкт-Петербург, 197376
Россия
Иван К. Шаранхаев
Институт математики и информатики
Бурятский государственный университет
Смолина, 24а, Улан-Удэ, Респ. Бурятия, 670000
Россия

Рассматриваются мультиклоны и суперклоны, которые являются обобщениями таких стандартных объектов, как клоны. Суперклон получается из мультиклона добавлением условия замкнутости относительно разрешимости простейшего уравнения. В статье доказано условие, при котором мультиклон и суперклон совпадают.

Ключевые слова: мультиоперация, мультиклон, суперклон, суперпозиция, операция, подстановка.