Journal of Siberian Federal University. Mathematics & Physics 2018, 11(1), 91–96

УДК 517.55

On Holomorphic Continuation Integrable Functions of Along Finite Families of Complex Lines in n-circular Domain

Bayram P. Otemuratov*
Karakalpak State University
Nukus, 230112
Uzbekistan

Received 06.07.2017, received in revised form 16.08.2017, accepted 30.11.2017

The paper consider, a family of complex lines passing through the final $(n+1)$ the number of points lying in the n-circled field D in $\mathbb{C}^n$ and f integrable on the boundary.

Keywords: integrable functions, holomorphic extension, Szegö kernel, Poisson kernel, complex lines.
DOI: 10.17516/1997-1397-2018-11-1-91-96.

This paper contains some results related to a holomorphic extension of integrable functions f, defined on the boundary of $D \subset \mathbb{C}^n$, $n > 1$, into this area. It's about integrable functions with one-dimensional holomorphic extension property along the complex lines.

In the complex plane $\mathbb{C}$ results about functions with one-dimensional holomorphic extension property is trivial. Therefore, our results are significantly multidimensional.

In papers [1–3] considered sufficient conditions for families of holomorphic extension of integrable functions of complex lines, passing through an open set, belonging to the field D, through the germ generating manifolds in the complex hypersurface.

For instance, Globevnik [4] shows that for continuous functions on the boundary of n points is not sufficient for holomorphic continuation. The paper [5] consider, a family of complex lines passing through the final $(n + 1)$ the number of points lying in the n-circular field D in $\mathbb{C}^n$ and f continuous on the boundary. In this paper we generalize this result for integrable functions.

Let D be a complete strictly convex bounded region in $\mathbb{C}^n$ with smooth boundary and with center at zero, ie, together with each point $z^0 = (z_1^0, \dots, z_n^0) \in D$, it contains polydisc

$$\{z \in \mathbb{C}^n : \ |z_k| \leqslant |z_k^0|, \ k = 1, \dots, n\}.$$

We denote $D^+ = \{(|z_1|, \dots, |z_n|) : \ z \in D\}$ image field D in absolute octant

$$\mathbb{R}_n^+ = \{(x_1, \dots, x_n) : \ |x_k| \geqslant 0, \ k = 1, \dots, n\}.$$

Let be $\partial D^+ = \{(|z_1|, \dots, |z_n|) : \ z \in \partial D\}$.

Let us consider finite measure μ on ∂D^+. The measure μ is a massive on the Shilov boundary [6, sec. 11], if for any the set $E \subset \partial D^+$ of zero measure μ satisfies the condition $\overline{\partial D^+ \setminus E} \supset S(D^+)$, where $S(D^+)$ is the image of the Shilov boundary $S(D)$ in absolute octant. In this case, $S(D^+) = \partial D^+$. From Theorem 3.1 [7] follows that Lebesgue measure μ on the boundary of this area is massive. Henceforth we will always assume, that the measure μ is massive.

We define kernel Szegö of field D

$$h(\bar{\zeta}, z) = \sum_{\alpha \geqslant 0} a_\alpha \bar{\zeta}^\alpha z^\alpha, \tag{1}$$

*bayram_utemurato@mail.ru
© Siberian Federal University. All rights reserved

where

$$a_\alpha = \frac{1}{\int\limits_{\partial D^+} |\zeta|^{2\alpha} d\mu} = \frac{1}{\int\limits_{\partial D^+} |\zeta_1|^{2\alpha_1} \cdot \ldots \cdot |\zeta_n|^{2\alpha_n} d\mu}$$

and $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a multi-index such that $\alpha \geqslant 0$ (i.e $\alpha_k \geqslant 0,\ k = 1, \ldots, n$) and $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$.

We recall the definition the functions of the class $\mathcal{H}^p(D)$.

Holomorphic function $f \in \mathcal{H}^p(D)$ $(p > 0)$, if

$$\sup_{\epsilon > 0} \int_{\partial D} |f(\zeta - \epsilon\nu(\zeta))|^p d\sigma < +\infty,$$

where $d\sigma$ is an element of surface ∂D, and $\nu(\zeta)$ is unit vector to ecternal normal to the surface ∂D at point ζ.

It is well known that the normal boundary values of the function $f \in \mathcal{H}^p(D)$ belong to the class $\mathcal{L}^p(\partial D)$ (by measure $d\sigma$).

The existence of Szegö kernels in n-circular domains is given by the following theorem:

Theorem 1. *Let on ∂D^+ given finite measure μ. In order for any function $f \in \mathcal{H}^p(D), (p \geqslant 1)$, existed integral representation Szegö*

$$f(z) = \lim_{r \to 1} \frac{1}{(2\pi i)^n} \int_{\partial D+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta) h(\bar{\zeta}, rz) \frac{d\zeta}{\zeta}, \quad z \in D, \tag{2}$$

where

$$\Delta_{|\zeta|} = \{\zeta :\ \zeta_1 = |\zeta_1| e^{i\theta_1}, \ldots, \zeta_n = |\zeta_n| e^{i\theta_n},\ 0 \leqslant \theta_k \leqslant 2\pi,\ k = 1, \ldots, n,\ |\zeta| \in \partial D^+\},$$

$$\frac{d\zeta}{\zeta} = \frac{d\zeta_1}{\zeta_1} \wedge \cdot \frac{d\zeta_n}{\zeta_n},$$

and Szego kernel $h(\bar{\zeta}, z) = h(\bar{\zeta}_1 z_1, \ldots, \bar{\zeta}_n z_n)$ at fixed $z \in D$ included on $\bar{\zeta}$ в $\mathcal{O}(\overline{D})$, and at fixed $\zeta \in \partial D$ included on z в $\mathcal{O}(D)$, is necessary and sufficient in order to measure μ is massive.

Theorem for continuous functions is given in [6], and in the case of a class of functions $\mathcal{H}^p$ is obtained approximation of functions $f(z)$ functions $f(r\zeta)$ at $r \to 1 - 0, r < 1$, in the metric $\mathcal{H}^p$. Thus a series of (1) by the Theorem 1 converges absolutely for $\zeta \in \overline{D}$ and $z \in D$ and uniformly for $\zeta \in \overline{D}$ and $z \in K$, where K is arbitrary compact from D.

It is clear that the border $\partial D = \bigcup\limits_{|\zeta| \in \partial D^+} \Delta_{|\zeta|}$. It should be noted the obvious property of the Szegö kernel:

$$h(\bar{\zeta}, z) = \overline{h(\zeta, \bar{z})} = h(z, \bar{\zeta}).$$

We introduce a Poisson kernel

$$P(\zeta, z) = \frac{h(\bar{\zeta}, z) h(\zeta, \bar{z})}{h(\bar{z}, z)} = \frac{|h(\bar{\zeta}, z)|^2}{h(\bar{z}, z)}.$$

Note that the the core $P(\zeta, z)$ is defined for $\zeta \in \overline{D}$ and $z \in D$, since $h(\bar{z}, z) > 0$.

Proposition 1. *If $f \in \mathcal{H}^p(D), (p \geqslant 1)$, then the formula is valid*

$$f(z) = \lim_{r \to 1} \frac{1}{(2\pi i)^n} \int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta) P(\zeta, rz) \frac{d\zeta}{\zeta}, \quad z \in D.$$

Proof. By using the formula (2), we have

$$\frac{1}{(2\pi i)^n}\int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta)P(\zeta,z)\frac{d\zeta}{\zeta} = \frac{1}{(2\pi i)^n}\int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta)\frac{h(\bar{\zeta},z)h(\zeta,\bar{z})}{h(\bar{z},z)}\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{(2\pi i)^n}\frac{1}{h(\bar{z},z)}\int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} \Big(f(\zeta)h(\zeta,\bar{z})\Big)h(\bar{\zeta},z)\frac{d\zeta}{\zeta} = \frac{1}{h(\bar{z},z)}f(z)h(z,\bar{z}) = f(z),$$

since the function $f(\zeta)h(\zeta,\bar{z})$ is holomorphic in $\zeta \in \overline{D}$ for a fixed $z \in D$. □

By Lemma 1 from [5], kernel Szegö at $\zeta = z$:

$$h(\bar{z},z) = \sum_{\alpha \geqslant 0} a_\alpha |z|^{2\alpha} > 0$$

in D and $h(\bar{z},z) \to \infty$, if $z \to \partial D$.

Assume that the region D satisfies the condition (A):
$h(\bar{\zeta},rz)$ is uniformly bounded in z outside any neighborhood of ζ at $\zeta, z \in \partial D$ and $\zeta \neq z$, $r \to 1$.

Theorem 2. *If the area D satisfies the condition (A) and $f \in \mathcal{L}^p(\partial D)$, then the Poisson integral*

$$F(z) = P[f](z) = \lim_{r\to 1}\frac{1}{(2\pi i)^n}\int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta)P(\zeta,rz)\frac{d\zeta}{\zeta}$$

is a real-analytic function in the region D and its boundary values in the metric L^p coincides with f on the ∂D.

Proof. Real-analytic $F(z)$ follows from the real-analyticity of the Szegö kernel and Poisson kernel. From the condition (A) and Lemma 1 [5] follows that $P(\zeta,rz)$ tends uniformly to zero outside an arbitrary neighborhood of the point ζ for $\zeta, z \in \partial D$, $\zeta \neq z$ and $r \to 1$. Moreover $P(\zeta,z) > 0$ и $P[1](\zeta) = 1$. Hence the Poisson kernel $P(\zeta,z)$ is an approximate identity [8, Th. 1.9]. □

We use the notation $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, $d\bar{\zeta}[k] = d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \ldots \wedge d\bar{\zeta}_n$.

The denominator of kernel $\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n) \neq 0$ at $\zeta, z \in \partial D$ и $\zeta \neq z$. Really, $\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n) = 0$ determines the complex tangent plane to ∂D at the point ζ. If the area D is strictly convex, then the tangent plane intersects the boundary of the domain only at the point ζ. The Szegö kernel area of the D is expressed by Leray kernel according to Corollary 26.13 [6] and the view the denominator Szegö kernel thus does not vary, so such fields satisfy the condition (A).

Differential form of

$$\omega = c\sum_{k=1}^{n}(-1)^{k-1}\bar{\zeta}_k\, d\bar{\zeta}[k]\, d\zeta,$$

where $c = \dfrac{(n-1)!}{(2\pi i)^n}$. We find this form of narrowing on the∂D for the field of view

$$D = \{z \in \mathbb{C}^n : \ \rho\big(|z_1|^2, \ldots, |z_n|^2\big) < 0\},$$

where $\rho(z)$ is twice smooth function $\operatorname{grad}\rho = \Big(\dfrac{\partial\rho}{\partial z_1}, \ldots, \dfrac{\partial\rho}{\partial z_n}\Big) \neq 0$ on ∂D.

We denote $|z_k|^2 = t_k$, $k = 1, \ldots, n$. Then

$$\operatorname{grad}\rho = \Big(\frac{\partial\rho}{\partial t_1}\bar{z}_1, \ldots, \frac{\partial\rho}{\partial t_n}\bar{z}_n\Big) \neq 0.$$

Function ρ can be chosen such that $|\operatorname{grad}\rho|\big|_{\partial D}= 1$. Let be $\nu = \omega\big|_{\partial D}$, then, as is easily verified (look at example, [9, Lemma 3.5]),

$$\nu = c\sum_{k=1}^{n}\bar{\zeta}_k\frac{\partial\rho}{\partial\bar{\zeta}_k}d\sigma = c\sum_{k=1}^{n}t_k\frac{\partial\rho}{\partial t_k}d\sigma,$$

where $d\sigma$ is the measure of Lebesgue on ∂D. In the case of n-circular areas $d\sigma = d\sigma_+ \cdot d\sigma'$, where $d\sigma'$ is measure defined by the form

$$\frac{1}{(2\pi i)^n}\frac{d\zeta_1}{\zeta_1}\wedge\ldots\wedge\frac{d\zeta_n}{\zeta_n},$$

and $d\sigma_+$ is the measure of Lebesgue on ∂D^+. Therefore

$$\nu = c\sum_{k=1}^{n}t_k\frac{\partial\rho}{\partial t_k}\,d\sigma_+\cdot d\sigma'.$$

We denote

$$\mu = c\sum_{k=1}^{n}t_k\frac{\partial\rho}{\partial t_k}\,d\sigma_+. \tag{3}$$

Lemma 1. *If D is complete n-circular domain, then μ is a measure on the ∂D^+.*

The proof of Lemma given in [5].

Corollary 1. *If the area D is complete n-circular strictly convex domain, then the measure μ is a massive measure on ∂D^+.*

We consider *a modified Poisson kernel*

$$Q(\zeta, z, w) = \frac{h(\bar{\zeta}, z)h(\zeta, w)}{h(w, z)}.$$

Then at $w = \bar{\zeta}$ we get $Q(\zeta, z, \bar{z}) = P(\zeta, z)$ и $h(\bar{z}, z) > 0$. Therefore, there exists a neighborhood U diagonals $w = \bar{z}$ in $D_z \times D_w$, wherein $h(w, z) \neq 0$.

We consider the function

$$\Phi(z, w) = c\int_{\partial D} f(\zeta)Q(\zeta, z, w)d\nu = c\int_{\partial D^+} d\mu\int_{\Delta_{|\zeta|}} f(\zeta)Q(\zeta, z, w)\frac{d\zeta}{\zeta}, \quad (z, w) \in D \times D.$$

This function is a holomorphic on $(z, w) \in U$, and at $w = \bar{z}$ function $\Phi(z, w) = F(z)$ and

$$\frac{\partial^{\delta+\gamma}\Phi(z, w)}{\partial z^\delta\partial w^\gamma}\Big|_{w=\bar{z}} = \frac{\partial^{\delta+\gamma}F(z)}{\partial z^\delta\partial\bar{z}^\gamma}, \tag{4}$$

where

$$\frac{\partial^{\delta+\gamma}\Phi(z, w)}{\partial z^\delta\partial w^\gamma} = \frac{\partial^{\delta_1+\ldots+\delta_n+\gamma_1+\ldots\gamma_n}\Phi(z, w)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\partial w_1^{\gamma_1}\cdots\partial w_n^{\gamma_n}},$$

$$\frac{\partial^{\delta+\gamma}F(z)}{\partial z^\delta\partial\bar{z}^\gamma} = \frac{\partial^{\delta_1+\ldots+\delta_n+\gamma_1+\ldots\gamma_n}F(z)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\partial\bar{z}_1^{\gamma_1}\cdots\partial\bar{z}_n^{\gamma_n}},$$

and $\delta = (\delta_1, \ldots, \delta_n),\ \gamma = (\gamma_1, \ldots, \gamma_n)$

Let be $\zeta = bt$, $b \in \mathbb{CP}^{n-1}$, $t \in \mathbb{C}$, whereas as shown in [10] (look also, [9, §15])

$$\omega = c\frac{dt}{t}\wedge\lambda(b), \tag{5}$$

where $\lambda(b)$ is a differential form the type $(n-1, n-1)$, not depend on the t.

In following we shall assume that there is a direction $b^0 \neq 0$ such that

$$\langle b^0, \bar{\zeta} \rangle \neq 0 \text{ для всех } \zeta \in \overline{D}. \tag{6}$$

We denote by $\mathfrak{L}_\Gamma$ the set of complex lines of the form

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \ \zeta_j = z_j + b_j t, \ j = 1, \dots, n, \ t \in \mathbb{C}\}, \tag{7}$$

passing through a point $z \in \Gamma$ in the direction of the vector $b \in \mathbb{CP}^{n-1}$ (direction b is defined up to multiplication by complex number $\lambda \neq 0$).

By Sard's theorem for almost all $z \in \mathbb{C}^n$ and for a fixed $b \in \mathbb{CP}^{n-1}$ intersection $l_{z,b} \cap \partial D$ represents a set of finite number of piecewise smooth curves (except for the degenerate case where $\partial D \cap l_{z,b} = \varnothing$).

It is known that if $f \in \mathcal{L}^p(\partial D)$, $p \geqslant 1$, then for almost all $z \in D$ and almost all $b \in \mathbb{CP}^{n-1}$ function $f \in \mathcal{L}^p(\partial D \cap l_{z,b})$ (see [1]).

We say that the function $f \in \mathcal{L}^p(\partial D)$ possesses *dimensional holomorphic extension property along the complex lines* $l_{z,b} \in \mathfrak{L}_\Gamma$ of the type (7), if almost all lines $l_{z,b}$ such that $\partial D \cap l_{z,b} \neq \varnothing$ there exists a function f_l with the following properties

1) $f_l \in \mathcal{H}^p(D \cap l_{z,b})$,

2) normal boundary values in the metric $\mathcal{H}^p$ of function f_l coincides with f on the set $\partial D \cap l_{z,b}$ almost everywhere.

Let us consider the kernel of the Bochner-Martinelli

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} \, d\bar{\zeta}[k] \wedge d\zeta,$$

where $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, and $d\bar{\zeta}[k]$ is obtained from $d\bar{\zeta}$ by ejection differential $d\bar{\zeta}_k$.

For a function $f \in \mathcal{L}^p(\partial D)$ we define the Bochner-Martinelli integral as follows:

$$F(z) = \int_{\partial D_\zeta} f(\zeta) U(\zeta, z), \quad z \notin \partial D. \tag{8}$$

Function $F(z)$ is a harmonic outside the border domain and converges to zero as $|z| \to \infty$.

We call *set* $\mathfrak{L}_\Gamma$ *sufficient for holomorphic continuation*, if the fact that $f \in \mathcal{L}^p(\partial D)$ has a one-dimensional holomorphic extension property along for almost all complex lines of the family $\mathfrak{L}_\Gamma$ follows that the function f holomorphically continued into D up to class functions $\mathcal{H}^p$.

Theorem 3. *Let be D the bounded n-circular a strictly convex domain and function $f \in \mathcal{L}^p(\partial D)$ has a one-dimensional holomorphic extension property along the complex lines passing through the origin, then $\Phi(0, w) = \text{const}$ u $\left.\frac{\partial^\delta \Phi(z, w)}{\partial z^\delta}\right|_{z=0}$ is a polynomial in w degree not higher $\|\delta\|$.*

For continuous functions 3 is proved in [5].

References

[1] B.P.Otemuratov, On functions of class L^p with a property of one-dimensional holomorphic extension, *Vestnik KrasGU. Ser. fiz. mat. nauki,* Krasnoyarsk, (2006), no. 9, 95-100 (in Russian).

[2] B.P.Otemuratov, On multidimensional theorems of Morera for integrable functions, *Uzb. mat. journal*, Tashkent, (2009), no. 2, 112–119 (in Russian).

[3] B.P.Otemuratov, Some sets of complex lines of minimal dimension sufficient for holomorphic continuation of integrable functions, *Journal SFU. Mathematics and physics. Krasnoyarsk*, **5**(2012), no. 1, 97–106 (in Russian).

[4] J.Globevnik, Small families of complex lines for testing holomorphic extendibility, *Amer. J. of Math.*, (2012), no 6, 1473–1490.

[5] A.M.Kytmanov, S.G.Myslivets, Holomorphic extension of functions along finite families of complex straight lines in an n-circular domain, *Sib. Math. J.*, **57**(2016), no. 4, 618–631.

[6] L.A.Aizenberg, A.P.Yuzhakov, Integral representations and residues in multidimensional complex analysis, Novosibirsk, Nauka, 1979 (in Russian).

[7] G.M.Henkin, Method of integral representations in complex analysis, Results of science technology. Modern problems of mathematics., vol. 7, Moscow, 1985 (in Russian).

[8] I.Stein, G.Weis, Introduction to Fourier analysis on Euclidean spaces, Moscow, Mir, 1974 (in Russian).

[9] A.M.Kytmanov, Bochner-Martinelli integral and its applications, Novosibirsk, Nauka, 1992 (in Russian).

[10] A.M.Kytmanov, S.G.Myslivets, Higher-dimensional boundary analogs of the Morera theorem in problems of analytic continuation of functions, *J. Math. Sci.*, **120**(2004), no. 6, 1842–1867.

О голоморфном продолжении интегрируемых функций вдоль конечных семейств комплексных прямых в n-круговой области

Байрам П. Отемуратов
Каракалпакский государственный университет
Ч.Абдирова, 1, Нукус, 230112,
Узбекистан

В работе рассмотрены семейства комплексных прямых, проходящих через конечное $(n+1)$ число точек, лежащих в n-круговой области D в $\mathbb{C}^n$ и f интегрируемых на границе.

Ключевые слова: интегрируемые функции, голоморфное продолжение, ядро Сеге, ядро Пуассона, комплексные прямые.