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Boundary Morera Theorem for the Matrix Ball of the Third Type

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In the article we consider a boundary version of Morera's theorem for the matrix ball of the third type.

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1^0. We have the following result of Nagel and Rudin [1] which says that if f is a continuous function on the boundary of the unit ball in $\mathbb{C}^n$ and the integral

$$\int_0^{2\pi} f(\psi(e^{i\varphi}, 0, \dots, 0))e^{i\varphi}d\varphi = 0$$

for all (holomorphic) automorphisms ψ of the ball, the function f extends holomorphically into a ball. For the classical domains, the matrix ball of the first type, and the generalized upper half-plane the boundary analogs of the Morera theorem were obtained in [2–4].

2^0. Let $Z = (Z_1, \dots, Z_n)$ be a vector composed from square matrices Z_j of order m over the field of complex numbers $\mathbb{C}$. We can assume that Z is an element of the space $\mathbb{C}^{m^2 n}$. We introduce on this set of vectors a matrix 'scalar' product according to

$$\langle Z, W\rangle = Z_1W_1^* + \dots + Z_nW_n^*,$$

where W_j^* is a conjugate transpose of the matrix W_j.

The set

$$B_{m,n}^{(1)} = \left\{(Z_1, \dots, Z_n) = Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z\rangle > 0\right\},$$

is called a matrix ball (of the first type); here $\langle Z, Z\rangle = Z_1Z_1^* + Z_2Z_2^* + \dots + Z_nZ_n^*$ is the 'scalar' product, I is the identity $[m \times m]$-matrix, $Z_\nu^* = \overline{Z'}_v$ is the conjugate transpose of Z_ν, $\nu = 1, 2, \dots, n$, [5]. Here $I - \langle Z, Z\rangle > 0$ means that the Hermitian matrix $I - \langle Z, Z\rangle$ is positively defined, i.e. all its eigenvalues are positive.

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3^0. We consider a matrix ball $B^{(3)}_{m,n}$(of the third type) (see [6]:

$$B^{(3)}_{m,n} = \{(Z_1,\dots,Z_n) = Z \in \mathbb{C}^n\,[m\times m] : I + \langle Z, Z\rangle > 0,\ \ Z'_\nu = -Z_\nu\,,\ \nu = 1,\dots,n\}\,.$$

The skeleton (the Shilov boundary) of the matrix ball $B^{(3)}_{m,n}$ is denoted by $X^{(3)}_{m,n}$, i.e.

$$X^{(3)}_{m,n} = \{Z \in \mathbb{C}^n\,[m\times m] : I + \langle Z, Z\rangle = 0,\ \ Z'_\nu = -Z_\nu,\ \nu = 1,\dots,n\}\,.$$

We fix a point $\Lambda^0 \in X^{(3)}_{m,n}$ $(\Lambda^0 = (\Lambda^0_1,\dots,\Lambda^0_n))$ and consider the following embedding of a unit disc Δ in the domain $B^{(3)}_{m,n}$

$$\left\{W \in \mathbb{C}^{m^2 n} :\ W_j = t\Lambda^0_j,\, j = 1,\dots,n,\ |t| < 1\right\}. \tag{1}$$

The boundary T of the disc Δ by this embedding is mapped into a circle lying on $X^{(3)}_{m,n}$. If ψ is an arbitrary (holomorphic) automorphism of $B^{(3)}_{m,n}$, then the set of the form (1) under the action of the automorphism goes into some analytic disc with boundary in $X^{(3)}_{m,n}$.

Theorem 1. *If the function $f \in C(X^{(3)}_{m,n})$ satisfies the following equality*

$$\int_T f(\psi(t\Lambda^0))dt = 0 \tag{2}$$

for all automorphisms ψ of the ball $B^{(3)}_{m,n}$, then f extends holomorphically in $B^{(3)}_{m,n}$ to a function F of class $C(\bar{B}^{(3)}_{m,n})$.

Proof. We parameterize the manifold $X^{(3)}_{m,n}$. For $Z \in X^{(3)}_{m,n}$ we put $Z = e^{i\varphi}U$ where $0 \leqslant \varphi \leqslant 2\pi$, and the element $u^{(1)}_{11}$ in the upper left corner of the U_1 is a positive number. A manifold of such matrices is denoted by X^+. Note that not the whole set $X^{(3)}_{m,n}$ is parameterized in this way, but a set smaller than $X^{(3)}_{m,n}$, differing by a set of measure 0.

Lemma 1 ([5])**.** *The measure*

$$d\sigma = h(U)dtd\sigma^+(U),\ U \in X^+,$$

where $h(U)$ is a smooth positive function, does not depend on t.

Lemma 1 shows that the measure $d\sigma$ can be written as

$$d\sigma = \frac{d\phi}{2\pi}d\sigma_1(U) = \frac{1}{2\pi i}\frac{dt}{t}d\sigma_1(U),$$

where $t = e^{i\phi}$, the measure σ_1 is positive on X^+.

Multiplying equation (2) by $d\sigma_1$ and integrating over X^+, we obtain from (2)

$$\int_{X^{(3)}_{m,n}} f(\psi(Z))z^l_{ks}d\sigma(Z) = 0 \tag{3}$$

where z^l_{ks} are the components of the vector $Z = (Z_1,\dots,Z_n),\ k,s = 1,\dots,m,\ \ l = 1,\dots,n.$

Consider an automorphism $\psi_{B^{(3)}_{m,n}}$ that maps an arbitrary point A from $B^{(3)}_{m,n}$ to 0 [6]. It is defined up to a generalized unitary transformation.

Then, substituting in the condition (3) instead of ψ an automorphism $\psi^{-1}_{B^{(3)}_{m,n}}$ and making the change of variables $W = \psi^{-1}_{B^{(3)}_{m,n}}(Z)$, we obtain

$$\int_{X^{(3)}_{m,n}} f(W)\psi^{A,l}_{ks}(W)d\sigma(\psi_A(W)) = 0, \tag{4}$$

where $\psi^{A,l}_{ks}$ are the components of the automorphism $\psi_{B^{(3)}_{m,n}}$. □

Corollary 1 ([7]). *For any continuous function f defined on the skeleton $X_{m,n}^{(3)}$ the Poisson transformation $F = P[f]$ is a real-analytic function in $\bar{B}_{m,n}^{(3)} \backslash X_{m,n}^{(3)}$ and continuous on $\bar{B}_{m,n}^{(3)}$, and $F = f$ on $X_{m,n}^{(3)}$.*

Corollary 1 shows that $d\sigma(\psi_A(W)) = P(A,W)d\sigma((W),$ where $P(A,W)$ is an invariant Poisson kernel of the domain $B_{m,n}^{(3)}$.

Therefore, from the condition (3) we obtain

$$\int_{X_{m,n}^{(3)}} f(W)\psi_{ks}^{A,l}(W)P(A,W)d\sigma((W)) = 0 \tag{5}$$

for all the points A from $B_{m,n}^{(3)}$ and for all $k,s = 1,\ldots,m,\ \ l = 1,\ldots,n$.

Thus, taking into account Corollary 2 on the properties of the Poisson integral [7] of continuous functions, Theorem 1 follows from the following assertion.

Theorem 2. *If the function $f \in C(X_{m,n}^{(3)})$ and equation (5) holds for all automorphisms $\psi_{B_{m,n}^{(3)}}$ of the domain $B_{m,n}^{(3)}$ transforming a point A from $B_{m,n}^{(3)}$ to 0 and for all $k,s = 1,\ldots,m,\ \ l = 1,\ldots,n$, then the function is the radial boundary value of some function $F \in \sigma(B_{m,n}^{(3)})$.*

Proof. The invariant Poisson kernel for the domain $B_{m,n}^{(3)}$ has the following form (see [7]) for even m

$$P(A,W) = \left[\frac{\left(\det(I^{(m)} + \langle A,A\rangle)\right)}{\left(\det(I^{(m)} + \langle A,W\rangle)\right)^2}\right]^{\frac{(m-1)n}{2}} = \left[\frac{\left(\det(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)\right)}{\left(\det(I^{(m)} + A_1\bar{W}_1 + \cdots + A_n\bar{W}_n)\right)^2}\right]^{\frac{(m-1)n}{2}} =$$

$$= \frac{\left(\det(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)\right)^{\frac{(m-1)n}{2}}}{\left(\det(I^{(m)} + A_1\bar{W}_1 + \cdots + A_n\bar{W}_n)\right)^{\frac{(m-1)n}{2}}\left(\det(I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n)\right)^{\frac{(m-1)n}{2}}},$$

and for odd m

$$P(A,W) = \left[\frac{\left(\det(I^{(m)} + \langle A,A\rangle)\right)}{\left(\det(I^{(m)} + \langle A,W\rangle)\right)^2}\right]^{\frac{mn}{2}} = \left[\frac{\left(\det(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)\right)}{\left(\det(I^{(m)} + A_1\bar{W}_1 + \cdots + A_n\bar{W}_n)\right)^2}\right]^{\frac{mn}{2}} =$$

$$= \frac{\left(\det(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)\right)^{\frac{mn}{2}}}{\left(\det(I^{(m)} + A_1\bar{W}_1 + \cdots + A_n\bar{W}_n)\right)^{\frac{mn}{2}}\left(\det(I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n)\right)^{\frac{mn}{2}}}.$$

Let

$$\begin{aligned}
&A = (A_1, ..., A_n) = \\
&\quad = \left(0, a_{12}^1, \ldots, a_{1m}^1; a_{m1}^1, \ldots, a_{m(m-1)}^1, 0; \ldots; 0, a_{12}^n, \ldots, a_{1m}^n; \ldots; a_{m1}^n, \ldots, a_{m(m-1)}^n, 0\right) = \\
&\qquad\qquad = \left(\|a_{sp}^1, \ldots, a_{sp}^n\|\right), \\
&W = (W_1, ..., W_n) = \\
&\quad = (0, w_{12}^1, \ldots, w_{1m}^1; w_{m1}^1, \ldots, w_{m(m-1)}^1, 0; \ldots; 0, w_{12}^n, \ldots, w_{1m}^n; \ldots; w_{m1}^n, \ldots, w_{m(m-1)}^n, 0) = \\
&\qquad\qquad = \left(\|w_{sp}^1, \ldots, w_{sp}^n\|\right),
\end{aligned}$$

where $\|a_{sp}^l\| = \|-a_{ps}^l\|$, $\|w_{sp}^l\| = \|-w_{ps}^l\|$, $(s,p = 1,\ldots,m)$, $l = 1,\ldots,n$.

We find the expression

$$\sum_{s,p=1}^{m}\sum_{l=1}^{n}\bar{b}_{sp}^l\frac{\partial P(A,W)}{\partial \bar{a}_{sp}^1}. \tag{6}$$

Denote

$$I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n = \|\alpha_{qj}\| \quad (q, j = 1, \dots, m),$$

where $\alpha_{qj} = \delta_{qj} + \sum_{k=1}^{m}\sum_{l=1}^{n} w_{qk}^l \overline{a}_{jk}^l,\ a_{sp}^l = -a_{ps}^l,\ w_{sp}^l = -w_{ps}^l,\ q, j = 1, \dots, m$, and δ_{qj} is the Kronecker delta.

Using the usual rule for differentiating a determinant for any $s{=}1,\dots,m$ we obtain

$$\sum_{p=1}^{m}\sum_{l=1}^{n}\overline{a}_{sp}^l \frac{\partial \det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)}{\partial \overline{a}_{sp}^l} =$$

$$= \det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n) - \det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)[s, s],$$

where $\det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)[s, s]$ denotes the algebraic complement to the element α_{ss} in the matrix $\det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)$.

Then

$$\sum_{s,p=1}^{m}\sum_{l-1}^{n}\overline{a}_{sp}^l \frac{\partial \det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)}{\partial \overline{a}_{sp}^l} =$$

$$= m\det(I^{(m)}{+}W_1\bar{A}_1 + \dots + W_n\bar{A}_n) - \sum_{s=1}^{m}\det(I^{(m)} + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)[s, s].$$

Similarly,

$$\sum_{s,p=1}^{m}\sum_{l-1}^{n}\overline{a}_{sp}^l \frac{\partial \det(I^{(m)} + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)}{\partial \overline{a}_{sp}^l} =$$

$$= m\det(I^{(m)}{+}A_1\bar{A}_1 + \dots + A_n\bar{A}_n) - \sum_{s=1}^{m}\det(I^{(m)} + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)[s, s].$$

Hence for even m we have the following equality

$$\frac{m\,(m-1)\,n}{2}P(A, W)\cdot\left[\frac{\sum_{s=1}^{m}\det(I + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)[s, s]}{\det(I + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)} - \right.$$
$$\left. - \frac{\sum_{s=1}^{m}\det(I + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)[s, s]}{\det(I + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)}\right] =$$
$$= \frac{m\,(m-1)\,n}{2}P(A, W)[Sp(I + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)^{-1} - Sp(I + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)^{-1}]. \quad (7)$$

for odd m the following equality

$$\frac{m\,(m+1)\,n}{2}P(A, W)\cdot\left[\frac{\sum_{s=1}^{m}\det(I + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)[s, s]}{\det(I + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)} - \right.$$
$$\left. - \frac{\sum_{s=1}^{m}\det(I + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)[s, s]}{\det(I + A_1\bar{A}_1 + \dots + A_n\bar{A}_n)}\right] =$$
$$= \frac{m\,(m+1)\,n}{2}P(A, W)[Sp(I{+}\,W_1\bar{A}_1{+}\dots + W_n\bar{A}_n)^{-1} - Sp(I{+}\,A_1\bar{A}_1{+}\dots + A_n\bar{A}_n)^{-1}]. \quad (7')$$

Here SpW denotes the trace of W.

The mapping of the form [6]

$$\psi_A(W) = \overline{Q}^{-1}\left((I + W_1\bar{A}_1 + \dots + W_n\bar{A}_n)\right)^{-1}\sum_{s=1}^{n}(W_s - A_s)\,Q_{sk}, \quad k = 1, \dots, n,$$

transforming a point A to the origin, is an automorphism of the matrix ball $B^{(3)}_{m,n}$, where Q is the block matrix $\overline{Q}(I + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)Q' = I$.

If the condition (5) holds for the components of the map $\psi_A(W)$, the same condition holds for the components of the map

$$\psi_A(W) = \left(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n\right)^{-1}\left(I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n\right)^{-1}\sum_{s=1}^{n}(W_s - A_s),$$

because the matrices Q and $I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n$ are non-degenerate and depend only on A.

Then from (5) we get

$$\int_{X^{(3)}_{m,n}} f(W)\psi^{A,l}_{ks}(W)P(A,W)d\sigma((W)) = 0,$$

where $\psi^{A,l}_{ks}(W)$ are components of the map $\psi_A(W)$, $(s,p = 1,\ldots,m)$, $\nu = 1,\ldots,n$. Consider the sum

$$\sum_{s,p=1}^{m}\sum_{l=1}^{n}\overline{a}^{l}_{sp}\psi^{A,l}_{sp}.$$

Obviously, this expression is equal to $Sp\left\langle\psi_A(W), A\right\rangle$, as [8–10]

$$\begin{aligned}\sum_{s,p=1}^{m}\sum_{l=1}^{n}\overline{a}^{l}_{sp}\psi^{A,l}_{sp} = Sp\Big[&(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)^{-1}(I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n)^{-1}\times\\ &\times\left(A_1\bar{A}_1 + \cdots + A_n\bar{A}_n - W_1\bar{A}_1 - \cdots - W_n\bar{A}_n\right)\Big] =\\ = Sp\Big[&(I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)^{-1}(I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n)^{-1}\times\\ &\times\left(\left(I + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n\right) - \left(I + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n\right)\right)\Big] =\\ = Sp\Big[&(I^{(m)} + W_1\bar{A}_1 + \cdots + W_n\bar{A}_n)^{-1} - (I^{(m)} + A_1\bar{A}_1 + \cdots + A_n\bar{A}_n)^{-1}\Big].\end{aligned} \tag{8}$$

Comparing formulas (7) and (8), from the hypothesis of the theorem we obtain

$$\sum_{s,p=1}^{m}\sum_{l=1}^{n}\overline{a}^{l}_{sp}\frac{\partial F(A)}{\partial\overline{b}^{l}_{sp}} = 0, \tag{9}$$

where $F(A) = \int_{X^{(3)}_{m,n}} f(W)P(A,W)d\sigma(W)$ is the Poisson integral of the function f. □

4^0. The proof of this theorem shows that it remains valid if the condition (5) holds only for automorphisms $\psi_{B^{(3)}_{m,n}}$, for which the point A lies in an open set $V \subset B^{(3)}_{m,n}$.
As Theorem 1, Theorem 2 can be generalized.

Theorem 3. *Let the function $f \in C(X^{(3)}_{m,n})$ and the condition (2) holds for all automorphisms ψ that transform the origin to a point of some open set $V \subset B^{(3)}_{m,n}$. Then f holomorphically extends in the domain $B^{(3)}_{m,n}$ to a function $F \in \sigma(\bar{B}^{(3)}_{m,n})$.*

5^0. Let Δ_ψ be an analytic disc

$$\Delta_\psi = \{Z : Z = \psi(t\Lambda^0), |t| < 1\},$$

where Λ^0_r is a fixed point of the skeleton $X^{(3)}_{m,n}$, and ψ is an automorphism of the domain $B^{(3)}_{m,n}$. Then the boundary T_ψ of the analytic disc lies on $X^{(3)}_{m,n}$, since the automorphism maps points of the skeleton to the points of the skeleton.

From the Morera theorems we obviously get a corollary on functions with one-dimensional holomorphic extension property along analytic discs.

Corollary 3. *If the function $f \in C(X^{(3)}_{m,n})$ exdends holomorphically (in t) in analytic discs Δ_ψ for all automorphisms ψ (or for all automorphisms ψ that transform the origin to a point of some fixed open set $V \subset B^{(3)}_{m,n}$), then the function f extends holomorphically in $B^{(3)}_{m,n}$.*

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Граничная теорема Морера для матричного шара третьего типа

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В этой статье рассматривается граничный вариант теоремы Мореры для матричного шара третьего типа.

Ключевые слова: матричный шар первого типа, матричный шар третьего типа, ядро Пуассона, теорема Морера.