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Three Dimensional Saito Free Divisors and Singular Curves

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The purpose of the present study is to find out examples of Saito free divisors by constructing Lie algebras generated by logarithmic vector fields along them. In the course of the study, the author recognized a deep connection between Saito free divisors and deformations of curve singularities. In this paper, we will explain a method of constructing three dimensional Saito free divisors and show some examples.

Key words: *Saito free divisors, Lie algebras.*

Introduction

The notion of Saito free divisors was introduced by K. Saito (see [12]) in connection with the study of universal unfoldings of isolated hypersurface singularity. In two dimensional case, any divisor is Saito free (cf. [12], (1.7) Cor.). Since divisors in higher dimensional spaces are not Saito free in general, it is interesting to find Saito free divisors and study their properties. There are many studies on them independent of singularity theory (for examples, see [2], [7], [10], [17] and the references therein).

The primitive purpose of the present study is to find out examples of Saito free divisors by constructing Lie albegras generated by logarithmic vector fields along them. In the course of the study, the author recognized a deep connection between Saito free divisors and deformations of curve singularities. In this paper, we will explain a method of constructing three dimensional Saito free divisors and show some examples.

The contents of this paper is as follows. We will first review the results in [13], [14]. The main purpose of [14] (which is a new version of [13]) is a classification of weighted homogeneous polynomials of three variables which have the same properties as the discriminant sets of quotient spaces of irreducible Coxeter groups of rank three. In particular, they define Saito free divisors. There is a connection between such divisors and deformations of plane curves with simple singularities of exceptional types ([14]). It should be noted here that the fundamental group of the complement of each divisor classified in [14] is determined by T. Ishibe and K. Saito (cf. [11]). The second purpose of this paper is to explain a basic idea how to obtain Saito free divisors from the eight of the so-called 14 exceptional families of isolated singularities due to V. Arnol'd [5], generalizing the formulation given in [14]. The author obtained several Saito free divisors by this method. Partial results are given in [15]. In this paper, we shall show complete lists of such divisors related with singularities of Types E_{12}, Z_{11}, W_{12} and some interesting examples of divisors related with singularities of Types $E_{13}, E_{14}, Z_{12}, Z_{13}, W_{13}$.

There are many studies still to be done. The first one is to classify Saito free divisors associated with the eight singularities. The author almost succeeded to answer this question with the assistance of M. Noro (Kobe Univ.). The second one is to determine the fundamental group of the complement of each of the divisors obtained in this paper. The third one is to determine the b-functions of the

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polynomials obtained by our method. The answer to this question is quite hopefull because a software Risa/Asir is effecitive for such a purpose. In fact, the b-functions of the 17 polynomials in [14] are already determined by H. Nakayama (Kobe Univ.). The fourth is to determine the types of singularities of the divisors and give a relationship between the root systems of the singularities and those of exceptional families determined by Gabrielov [9] (see also Ebeling [8]). The work of Urabe [18] is suggestive to develop further in this direction. The fifth is to generalize the study in this paper to the case of more complicated singularities.

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1. Definition of Saito Free Divisors

Let X be the n-dimensional affine space $\mathbf{C}^n$ and let $x = (x_1, x_2, \dots, x_n)$ be its coordinate system. A polynomial $f(x)$ of x is weighted homogeneous of type $r = (r_1, \dots, r_n)$ if $f(t^{r_1}x_1, \dots, t^{r_n}x_n) = t^h f(x)$ for any $t \in \mathbf{C}$, where h is a certain number. In this note, we always take $r_1, \dots, r_n$ as positive integers. Then h is the *degree* of $f(x)$.

Let $f(x)$ be a reduced polynomial and let D be the hypersurface of X defined by $f(x) = 0$. A vector field V on X with polynomial coefficients is *logarithmic* along D if Vf/f is a polynomial. Let $Der_X(\log D)$ be the totality of logarithmic vector fields along D. It is clear that $Der_X(\log D)$ is a Lie algebra defined over $R = \mathbf{C}[x_1, \dots, x_n]$. Then D is *Saito free* if $Der_X(\log D)$ is a free R-module. Since the theory of logarithmic vector fields goes back to the paper of K. Saito [12], it is better to start with reviewing a criterion by Saito himself for a divisor to be Saito free.

Lemma 1 (cf. [12], Lemma (1.9)). *Let $V_i = \sum_{j=1}^{n} a_j^i(x)\partial_{x_j}$ $(i = 1, \dots, n)$ be vector fields on X, where $a_j^i(x)$ are polynomials. We assume that V_i $(i = 1, \dots, n)$ satisfy the conditions (i), (ii):*

(i) $[V_i, V_j] = \sum_{k=1}^{n} b_{ij}^k(x)V_k$ for some polynomials $b_{ij}^k(x)$.

(ii) $F(x) = \det(a_j^i(x))$ has no multiple factor.

Then each V_i is logarithmic along the divisor D defined by $F(x)$ and they form a system of generators of the Lie algebra consisting of vevtor fields logarithmic along D.

K. Saito showed that the discriminant set of the parameter space of a versal family of a hypersurface with an isolated singularity is a Saito free divisor.

A typical example arises from the study of rational double points. More generally, we here treat the case of real reflection groups to show examples of Saito free divisors ([12], [19], [20]). Let W be an irreducible reflection group acting on a real vetor space E of dimensions n. Let $\xi = (\xi_1, \dots, \xi_n)$ be its coordinate system and let R_ξ be the ring of polynomials of ξ. Then by the theorem of Chevalley, there are algebraically independent homogeneous polynomials $P_i(\xi)$ $(i = 1, \dots, n)$ such that $R_\xi^W = \mathbf{C}[P_1, \dots, P_n]$. We put $d_j = \deg(P_j)$ $(j = 1, 2, \dots, n)$ and assume that $d_1 \le d_2 \le \dots \le d_n$. If α_j $(j = 1, \dots, N)$ forms the set of linear functions defining reflection hyperplanes, then the product $D = \prod_{j=1}^{N} \alpha_j^2$ is W-invariant. In particular there is a polynomial $F(X_1, \dots, X_n)$ such that $D = F(P_1, \dots, P_n)$. Then $D = 0$ defines a Saito free divisor on the affine space with coordinate $(P_1, \dots, P_n)$ and $F(P_1, \dots, P_n)$ is a Saito free polynomial of type $(d_1, \dots, d_n)$.

In the subsequent sections, we restrict our attention to the case $n = 3$ and formulate a problem of finding Lie algebras generated by three vector fields containing an Euler vector with weight. In virtue of Lemma 1, it is plausible that the polynomial defined as the determinant of the 3×3 matrix consisting of coefficients of the vector fields defines a Saito free divisor. This approach leads us to very successful results in the problem of constructing Saito free divisors.

2. Singular Curves

In this section, we review some of the singular curves which are used in the subsequent considerations.

2.1. Simple Singularities

We recall the defining equations of curves with simple singularities at the origin in $\mathbf{C}^2$.

$$\begin{array}{lll} A_n & : & u^{n+1}+v^2+w^2=0\ (n\geq 1) \\ D_n & : & u(u^{n-2}+v^2)+w^2=0\ (n\geq 4) \\ E_6 & : & u^4+v^3+w^2=0 \\ E_7 & : & u(u^2+v^3)+w^2=0 \\ E_8 & : & u^5+v^3+w^2=0 \end{array} \tag{1}$$

2.2. Unimodular singularities

We use the notation in Arnol'd [5]. Our interest is related with singular curves. Among the 14 exceptional families in [5], p.93, there are 8 which are realized by curve singularities. First we give these 8 families in Table I. Putting $w=0$, we obtain curves in uv-space. In Table I, the triplet of numbers (a,b,c) and h mean the type and the degrees of the polynomials in question.

Table I

Type		Equation	(a,b,c)	h	(p,q,r)
E_{12}	:	$u^7+v^3+w^2=0$	$(6,14,21)$	42	$(2,3,7)$
E_{13}	:	$u(u^2+v^5)+w^2=0$	$(4,10,15)$	30	$(1,2,5)$
E_{14}	:	$u^8+v^3+w^2=0$	$(3,8,12)$	24	$(1,3,8)$
Z_{11}	:	$u(u^4+v^3)+w^2=0$	$(6,8,15)$	30	$(2,3,4)$
Z_{12}	:	$uv(u^3+v^2)+w^2=0$	$(4,6,11)$	22	$(1,2,3)$
Z_{13}	:	$u(u^5+v^3)+w^2=0$	$(3,5,9)$	18	$(1,3,5)$
W_{12}	:	$u^5+v^4+w^2=0$	$(4,5,10)$	20	$(2,4,5)$
W_{13}	:	$v(u^4+v^3)+w^2=0$	$(3,4,8)$	16	$(1,3,4)$

We now explain the triplet of numbers (p,q,r). For the triplet of numbers (a,b,c) and h in Table I, we define

$$\chi(t)=t^{-2}\frac{(t^{h-a}-1)(t^{h-b}-1)(t^{h-c}-1)}{(t^a-1)(t^b-1)(t^c-1)}$$

Then

$$\chi(t)=t^{-2}+t^{d_2}+t^{d_3}+\cdots+t^{d_\mu}$$

for positive integers $d_2,d_3,\ldots,d_\mu$, where μ is the Milnor number corresponding to the singularity. So we may take that $0<d_2<d_3<\cdots<d_\mu$. Then (p,q,r) are triplet of numbers proportional to (d_2,a,b).

3. Saito Free Divisors and Simple Singularities

In this section, we review the results in [14].

Let x,y,z be variables and let p,q,r be natural numbers such that $p<q<r$. In case $1<p$, we may assume that p,q,r have no common factor. We consider three vector fields on (x,y,z)-space

including the Euler vector field with weight (p, q, r):

$$\begin{cases} V_0 &= px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}, \\ V_1 &= qy\frac{\partial}{\partial x} + h_{22}(x,y,z)\frac{\partial}{\partial y} + h_{23}(x,y,z)\frac{\partial}{\partial z}, \\ V_2 &= rz\frac{\partial}{\partial x} + h_{32}(x,y,z)\frac{\partial}{\partial y} + h_{33}(x,y,z)\frac{\partial}{\partial z}, \end{cases} \tag{2}$$

where $h_{ij}(x, y, z)$ are polynomials of x, y, z. In addition, we define a 3×3 matrix M associated with the vector fields V_0, V_1, V_2 by

$$M = \begin{pmatrix} px & qy & rz \\ qy & h_{22}(x,y,z) & h_{23}(x,y,z) \\ rz & h_{32}(x,y,z) & h_{33}(x,y,z) \end{pmatrix}. \tag{3}$$

Now we consider the conditions on V_0, V_1, V_2:

Condition 1. *(i)* $[V_0, V_1] = (q-p)V_1,\ [V_0, V_2] = (r-p)V_2$.
(ii) There exist polynomials $f_j(x, y, z)$ $(j = 0, 1, 2)$ *such that*

$$[V_1, V_2] = f_0(x,y,z)V_0 + f_1(x,y,z)V_1 + f_2(x,y,z)V_2.$$

(iii) $\dfrac{\partial h_{22}}{\partial z}$ *is a non-zero constant.*
(iv) The polynomial $\det(M)$ *is not trivial. (We say that* $\det(M)$ *is trivial if it turns out to be* z^3 *by a weight preserving coordinate transformation.)*

Condition 1 (i), (ii) claim that the $\mathbf{C}[x, y, z]$-module $L(\det(M))$ spanned by V_0, V_1, V_2 becomes a Lie algebra over $\mathbf{C}[x, y, z]$ and Condition 1 (iii) does that $\deg_z \det(M) = 3$. Condition 1 (iv) is supplementary. If V_0, V_1, V_2 satisfy Condition 1, it follows that $V_j \det(M)$ is divisible by $\det(M)$ $(j = 0, 1, 2)$. Namely, V_0, V_1, V_2 and therefore all the vector fields of $L(\det(M))$ are logarithmic along the hypersurface $\{(x, y, z);\ \det(M) = 0\}$ in the sense of [12]. Conversely, it is possible to reconstruct the vector fields V_0, V_1, V_2 from the polynomial $\det(M)$.

Remark 1. *It is not trivial whether the polynomial* $\det(M)$ *has a multiple factor or not. But as will be seen Theorem 1 below, it has no multiple factor. As a result, it follows that the hypresurface defined by* $\det(M) = 0$ *is a Saito free divisor.*

Let W be a finite reflection group acting on the vector space E of Dimensions 3. Let x, y, z be basic W-invariant polynomials and let F be the discriminant of W. Then the Lie algebra of logarithmic vector fields along $F^{-1}(0)$ satisfies Condition 1. The primitive interest is to find polynomials which have properties as discriminants. Problem 1 below gives an idea to find such polynomials.

Problem 1. *Find all the triples* $\{V_0, V_1, V_2\}$ *of vector fields satisfying Condition 1. Or equivalently, find all polynomials* $F(x, y, z)$ *of the form* $F = \det(M)$.

We now state the first main theorem of [14] which answers to this problem.

Theorem 1. *Let* x, y, z *be variables and let* p, q, r *be natural numbers such that* $p < q < r$. *In case* $1 < p$, *we may assume that* p, q, r *have no common factor. Then the following assertions hold.*

(i) If $(p, q, r) \neq (2, 3, 4), (1, 2, 3), (1, 3, 5)$, *there is no triple* $\{V_0, V_1, V_2\}$ *of vector fields satisfying Condition 1.*

(ii) If (p, q, r) *is one of* $(2, 3, 4),\ (1, 2, 3),\ (1, 3, 5)$, *the polynomial* $F(x, y, z)$ *of the form* $F = \det(M)$ *is reduced to one of the following polynomials by a weight preserving coordinate transformation.*

(ii.1) The case $(p,q,r)=(2,3,4)$. (This case corresponds to the reflection group of Type A_3.)
$F_{A,1}=16x^4z-4x^3y^2-128x^2z^2+144xy^2z-27y^4+256z^3$.
$F_{A,2}=2x^6-3x^4z+18x^3y^2-18xy^2z+27y^4+z^3$.
(ii.2) The case $(p,q,r)=(1,2,3)$. (This case corresponds to the reflection group of Type B_3.)
$F_{B,1}=z(x^2y^2-4y^3-4x^3z+18xyz-27z^2)$.
$F_{B,2}=z(-2y^3+4x^3z+18xyz+27z^2)$.
$F_{B,3}=z(-2y^3+9xyz+45z^2)$.
$F_{B,4}=z(9x^2y^2-4y^3+18xyz+9z^2)$.
$F_{B,5}=xy^4+y^3z+z^3$.
$F_{B,6}=9xy^4+6x^2y^2z-4y^3z+x^3z^2-12xyz^2+4z^3$.
$F_{B,7}=\frac{1}{2}xy^4-2x^2y^2z-y^3z+2x^3z^2+2xyz^2+z^3$.
(ii.3) The case $(p,q,r)=(1,3,5)$. (This case corresponds to the reflection group of Type H_3.)
$F_{H,1}=-50z^3+(4x^5-50x^2y)z^2+(4x^7y+60x^4y^2+225xy^3)z-\frac{135}{2}y^5-115x^3y^4-10x^6y^3-4x^9y^2$.
$F_{H,2}=100x^3y^4+y^5+40x^4y^2z-10xy^3z+4x^5z^2-15x^2yz^2+z^3$.
$F_{H,3}=8x^3y^4+108y^5-36xy^3z-x^2yz^2+4z^3$.
$F_{H,4}=y^5-2xy^3z+x^2yz^2+z^3$.
$F_{H,5}=x^3y^4-y^5+3xy^3z+z^3$.
$F_{H,6}=x^3y^4+y^5-2x^4y^2z-4xy^3z+x^5z^2+3x^2yz^2+z^3$.
$F_{H,7}=xy^3z+y^5+z^3$.
$F_{H,8}=x^3y^4+y^5-8x^4y^2z-7xy^3z+16x^5z^2+12x^2yz^2+z^3$.

Remark 2. *Let $F(x,y,z)$ be one of the polynomials in Theorem 1. Then the curve $C:\{(y,z);F(0,y,z)=0\}$ is regarded as the simple singularity of type E_6, E_7, E_8 if $F(x,y,z)$ is one of the polynomials $F_{A,j}\,(j=1,2)$, $F_{B,j}\,(j=1,\dots,7)$, $F_{H,j}\,(j=1,\dots,8)$, respectively (cf. [16]). Therefore if we regard x as a parameter, the family of curves $C_x:F(x,y,z)=0$ on yz-space is a deformation of the curve $C_0=C$.*

We state the second main theorem of [14].

Theorem 2. *(i) There is a natural bijection between the set of polynomials of (I) and that of corank one subdiagrams of the Dynkin subdiagram of Type E_6 left fixed by its non-trivial involution.*

(ii) There is a natural bijection between the set of polynomials of (II) (resp. (III)) and that of corank one subdiagrams of the Dynkin diagram of Type E_7 (resp. E_8).

Remark 3. *It is an interesting problem to determine the fundamental groups of the complements of the hypersurfaces defined by the seventeeen polynomials in Theorem 1. This is done by Ishibe and Saito (cf. [11]).*

4. A Method of Constructing Saito Free Divisors from Singular Curves

In this section, we explain an idea how to construct Saito free divisors from some of the 14 exceptional families of isolated singularities classified by Arnol'd (cf. [4]).

As before, let x,y,z be variables and let p,q,r be natural numbers such that $p<q<r$. In case $1<p$, we may assume that p,q,r have no common factor. We consider three vector fields on (x,y,z)-space including the Euler vector field of Type (p,q,r):

$$\begin{cases} V_0 &=& px\dfrac{\partial}{\partial x}+qy\dfrac{\partial}{\partial y}+rz\dfrac{\partial}{\partial z}, \\ V_1 &=& h_{21}(x,y,z)\dfrac{\partial}{\partial x}+h_{22}(x,y,z)\dfrac{\partial}{\partial y}+h_{23}(x,y,z)\dfrac{\partial}{\partial z}, \\ V_2 &=& h_{31}(x,y,z)\dfrac{\partial}{\partial x}+h_{32}(x,y,z)\dfrac{\partial}{\partial y}+h_{33}(x,y,z)\dfrac{\partial}{\partial z}, \end{cases} \tag{4}$$

where $h_{ij}(x,y,z)$ are weighted homogeneous polynomials of x,y,z. Corresponding to the vector fields V_0,V_1,V_2, we define a 3×3 matrix M by

$$M = \begin{pmatrix} px & qy & rz \\ h_{21}(x,y,z) & h_{22}(x,y,z) & h_{23}(x,y,z) \\ h_{31}(x,y,z) & h_{32}(x,y,z) & h_{33}(x,y,z) \end{pmatrix}. \tag{5}$$

Now we consider the conditions on V_0,V_1,V_2:

Condition 2. *(i)* $[V_0,V_1]=w_1V_1,\ [V_0,V_2]=w_2V_2$
for some integers w_1,w_2.
(ii) There exist polynomials $f_j(x,y,z)\,(j=0,1,2)$ *such that* $[V_1,V_2]=f_0(x,y,z)V_0+f_1(x,y,z)V_1+f_2(x,y,z)V_2$.

We now give conditions on the matrix M for the eight families of isolated singularities in Table I. First we take the triplet of integers (p,q,r) as in Table I.

(1) The cases $E_{12},\ E_{13},\ E_{14}$

$h_{21}=y^2,\ h_{31}=z,\ h_{22}(0,y,z)=az$ (a is a non-zero constant) and $\det(M)_{x=0}$ coincide with $z^3+y^7,\ z^3+y^5z,\ z^3+y^8$ in the cases $E_{12},\ E_{13},\ E_{14}$, respectively up to weight preserving coordinate transformations.

(2) The cases $Z_{11},\ Z_{12},\ Z_{13}$

$h_{21}=y^2,\ h_{31}=z,\ h_{22}(0,y,z)=ayz$ (a is a non-zero constant) and $\det(M)_{x=0}$ coincide with $y(z^3+y^4),\ yz(z^2+y^3),\ y(z^3+y^5)$ in the cases $Z_{11},\ Z_{12},\ Z_{13}$, respectively up to weight preserving coordinate transformations.

(3) The cases $W_{12},\ W_{13}$

$h_{21}=y^2,\ h_{31}=z,\ h_{22}(0,y,z)=az^2$ (a is a non-zero constant) and $\det(M)_{x=0}$ coincide with $z^4+y^5,\ z^4+y^4z$ in the cases $W_{12},\ W_{13}$, respectively up to weight preserving coordinate transformations.

5. Examples of Saito Free Divisors Related with the Eight Members of the 14 Exceptional Families

In this section, we show the totality of Saito free divisors related with the singularities of types E_{12},Z_{11},W_{12} obtained by our method explained in the previous section. The classification of these three cases is not difficult compared with the remaining five cases. The author also succeeded to classify the remaining five cases with the assistance of M.Noro (Kobe Univ.). Since we need to spend a lot of pages to show the result, we abandoned to do. Instead we only show Saito free polynomials related with these five singularities having parameters.

5.1. The Case E_{12}

The following nine polynomials $f_1,\dots,f_9$ are Saito free polynomials each of which defines deformations of a curve on (y,z)-space with a singular point of type E_{12} at the origin regarding x as a parameter. Note that f_8 and f_9 have complex coefficients and they are conjugate to each other. It is possible to prove the following statement. (For a proof, see [3]).

Let $F(x,y,z)$ be a weighted homogeneous Saito free polynomial with weight $(2,3,7)$ and degree 21. Assume that $F(0,y,z)=z^3+y^7$. Then $F(x,y,z)$ coincides with one of the 9 polynomials $f_1,\dots,f_9$ up to weight preserving coordinate transformation.

```
f1 = x^6*y^3/32 + 3*x^3*y^5/28 + 3*y^7/49 - 3/16*x^4*y^2*z - 3/7*x*y^4*z + z^3;
f2 = -1/864*x^6*y^3 + 5*x^3*y^5/84 + 3*y^7/49 - 1/48*x^4*y^2*z - 3/7*x*y^4*z + z^3;
f3 = x^3*y^5/21 + 3*y^7/49 - 3/7*x*y^4*z + z^3;
f4 = 3*y^7/49 - 3/7*x*y^4*z + z^3;
f5 = 78125*x^9*y/200120949 + 44375*x^6*y^3/4840416 + 107*x^3*y^5/1372 + 3*y^7/49 -
     6250*x^7*z/3176523 - 1375*x^4*y^2*z/16464 - 3/7*x*y^4*z + z^3;
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f6 = 64*x^9*y/823543 + 208*x^6*y^3/453789 + 68*x^3*y^5/1029 + 3*y^7/49 +
     48*x^7*z/117649 - 40*x^4*y^2*z/1029 - 3/7*x*y^4*z + z^3;
f7 = -448*x^9*y/243 + 16*x^6*y^3/9 - 4*x^3*y^5/7 + 3*y^7/49 - 112*x^7*z/27 +
     8/3*x^4*y^2*z - 3/7*x*y^4*z + z^3;
f8 = -752*x^9*y/823543 - 2017*I*x^9*y/(823543*Sqrt[3]) - 397*x^6*y^3/33614 +
     323*I*Sqrt[3]*x^6*y^3/33614 + 39*x^3*y^5/686 + 9/686*I*Sqrt[3]*x^3*y^5 +
     3*y^7/49 + 1763*x^7*z/235298 - 249*I*Sqrt[3]*x^7*z/235298 +
     3/686*x^4*y^2*z - 37/686*I*Sqrt[3]*x^4*y^2*z - 3/7*x*y^4*z + z^3;
f9 = -752*x^9*y/823543 + 2017*I*x^9*y/(823543*Sqrt[3]) - 397*x^6*y^3/33614 -
     323*I*Sqrt[3]*x^6*y^3/33614 + 39*x^3*y^5/686 - 9/686*I*Sqrt[3]*x^3*y^5 +
     3*y^7/49 + 1763*x^7*z/235298 + 249*I*Sqrt[3]*x^7*z/235298 +
     3/686*x^4*y^2*z + 37/686*I*Sqrt[3]*x^4*y^2*z - 3/7*x*y^4*z + z^3;
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5.2. The Case Z_{11}

The following polynomials f_1, f_2, f_3, f_4 are Saito free polynomials each of which defines deformations of a curve on (y, z)-space with a singular point of Type Z_{11} at the origin regarding x as a parameter.

Let $F(x, y, z)$ be a weighted homogeneous Saito free polynomial with the weight $(2, 3, 4)$ and degree 30. Assume that $F(0, y, z) = y(z^3 + y^4)$. Then $F(x, y, z)$ coincides with one of the 4 polynomials f_1, f_2, f_3, f_4 below up to weight preserving coordinate transformation.

```
f1 = y*(3*y^4 - 3*x*y^2*z + z^3)
f2 = y*(x^3*y^2 + 9*y^4 - 9*x*y^2*z + 3*z^3)
f3 = y*(-4*x^3*y^2 - 27*y^4 + 16*x^4*z + 144*x*y^2*z - 128*x^2*z^2 + 256*z^3)
f4 = y*(2*x^6 + 18*x^3*y^2 + 27*y^4 - 3*x^4*z - 18*x*y^2*z + z^3)
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Note that $f_3 = y \cdot F_{A,1}(x^2, y, z)$, $f_4 = y \cdot F_{A,2}(x^2, y, z)$, where $F_{A,1}$, $F_{A,2}$ are the polynomials introduced in Theorem 1.

5.3. The Case W_{12}

The following polynomials f_1, f_2, f_3, f_4 are Saito free polynomials each of which defines deformations of a curve on (y, z)-space with a singular point of Type W_{12} at the origin regarding x as a parameter.

Let $F(x, y, z)$ be a weighted homogeneous Saito free polynomial with weight $(2, 4, 5)$ and degree 20. Assume that $F(0, y, z) = y^5 + z^4$. Then $F(x, y, z)$ coincides with one of the 4 polynomials f_1, f_2, f_3, f_4 below up to weight preserving coordinate transformation.

```
f1=2560*x^4*y^3 - 95*x^2*y^4 + y^5 - 65536*x^5*z^2 + 2560*x^3*y*z^2 - 30*x*y^2*z^2 + z^4
f2=16*x^6*y^2 + 24*x^4*y^3 + 9*x^2*y^4 + y^5 - 8*x^3*y*z^2 - 6*x*y^2*z^2 + z^4
f3=25*x^8*y + 100*x^6*y^2 + 110*x^4*y^3 + 20*x^2*y^4 + y^5 - 2*x^5*z^2 - 20*x^3*y*z^2 -
   10*x*y^2*z^2 + z^4
f4=x^2*y^4 + y^5 - 2*x*y^2*z^2 + z^4
```

5.4. The Cases $E_{13}, E_{14}, Z_{12}, Z_{13}, W_{13}$

The following six polynomials are Saito free as polynomials of x, y, z. Each of them contains a parameter t.

```
g1 = z*(t*x^2*y^4 + 2*y^5 - 10*x*y^2*z + 5*z^2)
g2 = -1728*x^6*y^6 + t^2*x^6*y^6 - 2*t*x^3*y^7 + y^8 + 432*x^4*y^4*z +
     12*t*x^4*y^4*z - 12*x*y^5*z + z^3
g31= z*(y^4 + 2*x*y^2*z + t*x^2*z^2 - y*z^2)
g32= y*z*(t*x^2*y^2 + y^3 + 2*x*y*z - z^2)
g4 = y*(243*t^2*x^6*y^3 - 54*t*x^3*y^4 + 3*y^5 + 324*t*x^4*y^2*z - 36*x*y^3*z +
     108*x^2*y*z^2 + z^3)
g5 = z*(-12*y^4 + 288*x^2*y^2*z + t^2*x^2*y^2*z - 1728*x^4*z^2 + 2*t*x*y*z^2 + z^3)
```

Remark 4. *With the assistance of M. Noro, the author succeeded to construct 33, 33, 34, 35, 23 numbers of Saito free polynomials with real coefficients and without parameters related with the singularities* $E_{13}, E_{14}, Z_{12}, Z_{13}, W_{13}$*, respectively. There are many Saito free polynomials with complex coefficients related with the singularities* $E_{13}, E_{14}, Z_{12}, Z_{13}, W_{13}$ *as in the case of the polynomials* f_8, f_9 *in subsection 6.1.*

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