УДК 517.55

Hypergeometric Systems with Polynomial Bases

Timur M.Sadykov*
Institute of Mathematics,
Siberian Federal University,
av. Svobodny 79, Krasoyarsk, 660041
Russia

Received 1.09.2007, accepted 5.12.2007

We prove that any simplicial or parallelepipedal hypergeometric configuration admits a Puiseux polynomial basis in its solution space for suitable values of its parameters.

Key words: *Puiseux polynomial basis, hypergeometric systems.*

1. Notation, Definitions and Preliminaries

The purpose of this paper is to study three important classes of hypergeometric systems of partial differential equations. The systems under study are shown to have bases in their solution spaces that consist of elementary functions. Moreover, their solution spaces split into direct sums of subspaces which are invariant under the action of monodromy. Throughout the paper we will be using the following definition.

Definition 1. *A formal Laurent series*

$$\sum_{s\in\mathbb{Z}^n} \varphi(s_1,...,s_n)x_1^{s_1}...x_n^{s_n} \tag{1}$$

is called hypergeometric if for any $i=1,...,n$ the quotient $\varphi(s+e_i)/\varphi(s)$ is a rational function in s. Throughout the paper we denote this rational function by $P_i(s)/Q_i(s+e_i)$. Here $\{e_i\}_{i=1}^n$ is the standard basis of the lattice $\mathbb{Z}^n$. By the support of this series we mean the subset of $\mathbb{Z}^n$ on which $\varphi(s)\neq 0$. We say that such a series is fully supported, if the convex hull of its support contains (a translation of) an open n-dimensional cone.

A *hypergeometric function* is a (multi-valued) analytic function obtained by means of analytic continuation of a hypergeometric series along all possible paths.

Theorem 1 (Ore, Sato [5],[2])**.** *The coefficients of a hypergeometric series are given by the formula*

$$\varphi(s) = t^s U(s)\prod_{i=1}^{m}\Gamma(\langle A_i,s\rangle + c_i), \tag{2}$$

where $t^s = t_1^{s_1}...t_n^{s_n}$, $t_i, c_i\in\mathbb{C}$, $A_i\in\mathbb{Z}^n$ and $U(s)$ is a rational function.

We will call any function of the form (2) *the Ore-Sato coefficient of a hypergeometric series.* Throughout the paper we assume (unless otherwise stated) that the parameters c_i of the Ore-Sato coefficients we are dealing with are generic.

*e-mail: sadykov@lan.krasu.ru

Definition 2. *The Horn system of an Ore-Sato coefficient. A (formal) Laurent series* $\sum_{s\in\mathbb{Z}^n}\varphi(s)x^s$ *whose coefficient satisfies the relations* $\varphi(s+e_i)/\varphi(s)=P_i(s)/Q_i(s+e_i)$ *is a (formal) solution to the following system of partial differential equations of hypergeometric type*

$$x_iP_i(\theta)y(x)=Q_i(\theta)y(x),\ \ i=1,...,n. \tag{3}$$

Here $\theta=(\theta_1,...,\theta_n)$, $\theta_i=x_i\dfrac{\partial}{\partial x_i}$. *The system (3) will be referred to as the Horn hypergeometric system defined by the Ore-Sato coefficient* $\varphi(s)$ *(see [2]) and denoted by* $\mathrm{Horn}(\varphi)$.

We will often be dealing with the important special case of an Ore-Sato coefficient (2) where $t_i=1$ for any $i=1,\dots,n$ and $U(s)\equiv 1$. The Horn system associated with such an Ore-Sato coefficient will be denoted by $\mathrm{Horn}(A,c)$, where A is the matrix with the rows $A_1,\dots,A_m$.

Definition 3. *The Ore-Sato coefficient (2), the corresponding hypergeometric series (1), and the associated hypergeometric system (3) are called nonconfluent if*

$$\sum_{i=1}^{m}A_i=0. \tag{4}$$

Definition 4. *For a pair of vectors* (a,b), $(c,d)\in\mathbb{Z}^2$ *we set*

$$\nu(a,b;c,d)=\begin{cases}\min(|ad|,|bc|), & \text{if } (a,b),\ (c,d) \text{ are in opposite open quadrants of } \mathbb{Z}^2,\\ 0 & \text{otherwise.}\end{cases}$$

The number $\nu(a,b;c,d)$ *is called the index associated to the lattice vectors* (a,b) *and* (c,d). *The index of the rows of a* 2×2 *matrix* M *will be denoted by* $\nu(M)$.

Definition 5. *By the initial exponent of a multiple hypergeometric series*

$$x_1^{\alpha_1}\dots x_n^{\alpha_n}\sum_{s\in\mathbb{Z}^n}\varphi(s_1,\dots,s_n)x_1^{s_1}\dots x_n^{s_n}$$

we will mean the vector $(\alpha_1,\dots,\alpha_n)\in\mathbb{C}^n$. *Observe that the initial exponent of such a series is only defined up to shifts by integer vectors.*

Definition 6. *A series solution* $f(x)=\sum_{\alpha\in A}c_\alpha x^\alpha$ *to a Horn system is called pure if for any* $\alpha,\beta\in A$ *we have* $\alpha=\beta \mod \mathbb{Z}^n$. *In other words, a series solution is called pure if it is given by the product of a monomial and a Laurent series. A set of series* $\{f_k(x)\}_{k=1}^r$ *is called a pure basis of the solution space of a Horn system in a neighbourhood of a nonsingular point* $x\in\mathbb{C}^n$ *if every* f_k *converges at* x, *is a pure solution and together they span a linear space whose dimension equals the holonomic rank of the Horn system.*

Since a Horn system has polynomial coefficients, it follows that any of its Puiseux series solutions can be written as a finite linear combination of pure solutions to the same system of equations. Moreover, in a neighbourhood of a nonsingular point, a pure basis in the local solution space of a Horn system is defined uniquely up to permutation and multiplication of its elements with nonzero constants. In this paper we will neglect this unessential difference between pure bases of solutions. The pure basis of a hypergeometric system is especially convenient for computing monodromy since, within the domain of convergence of the basis series, the monodromy matrices are diagonal.

Definition 7. *The support* S *of a series solution to (3) is called irreducible if there exists no series solution to (3) supported in a proper nonempty subset of* S.

Definition 8. *Let* $\varphi(s)$ *be an Ore-Sato coefficient and let* $f(x)$ *be a Puiseux polynomial solution to Horn(*φ*). If there exists a multi-index* $I=\{i_1,\dots,i_n\}\subset\{1,\dots,m\}$ *with different components such that for any* $s\in\mathrm{supp}f$ *there exists* $j\in I$ *and* $k\in\{0,\dots,j-1\}$ *such that* $\langle A_j,s\rangle+c_j+k=0$ *then* f *is called stable Puiseux polynomial solution to Horn(*φ*).*

Any Puiseux series solution (centered at the origin) of a Horn system with generic parameters is either a fully supported series or a stable Puiseux polynomial. Indeed, for a polynomial to be a solution to a hypergeometric system, its exponents must satisfy a system of linear algebraic equations. The generic parameters assumption implies that the right-hand sides of these equations are also generic and hence the system of linear algebraic equations is defined by a square nondegenerate matrix. The corresponding solutions to the hypergeometric system are precisely stable polynomials.

We will be using the following notation throughout the paper. For a vector $x = (x_1, \ldots, x_n)$, we will denote by e^x the vector $(e^{x_1}, \ldots, e^{x_n})$. By $\mathrm{diag}(x)$ we will denote the diagonal matrix with the elements $x_1, \ldots, x_n$ on the main diagonal.

2. Atomic Horn Systems

Let $M \in GL(n, \mathbb{Z})$ be an integer nondegenerate square matrix and $\alpha \in \mathbb{C}^n$ a parameter vector. The (confluent) Horn system $\mathrm{Horn}(M, \alpha)$ associated with this data will be called *atomic.* An atomic system can be transformed into a system of differential equations with constant coefficients by means of the isomorphism in Corollary 5.2 in [1]. In accordance with the Malgrange-Ehrenpreis-Palamodov fundamental principle (see Chapter 6 in [3]), an atomic system only has elementary solutions which can be expressed in terms of Puiseux polynomials and exponential functions. In the present section, we carry out a detailed analysis of the properties of a general atomic hypergeometric system. The reason for studying atomic systems is that the supports and the initial exponents of a Horn system with generic parameters can be recovered from an atomic systems associated with it.

Theorem 2. *Let $M \in GL(n, \mathbb{Z})$ be an integer nondegenerate square matrix, $\alpha \in \mathbb{C}^n$, and let* $\mathrm{Horn}(M, \alpha)$ *denote an atomic system of hypergeometric differential equations defined by this data.*

1. *The dimension of the space $\mathcal{F}$ of fully supported solutions to* $\mathrm{Horn}(M, \alpha)$ *at a point $x \in (\mathbb{C}^*)^n$ equals $|\det M|$. A basis in this space is given by any maximal set of linearly independent germs of the* generating solution
$$x^{-M^{-1}\alpha} \exp\left(-\sum_{j=1}^{n} x^{-M^{-1}e_j}\right), \tag{5}$$
where $e_j = (0, \ldots, 1, \ldots, 0)$ (1 in the j-th position).

2. *Apart from (5) and its analytic continuations, an atomic Horn system only has stable Puiseux polynomial solutions. The subspace of Puiseux polynomial solutions to* $\mathrm{Horn}(M, \alpha)$ *and the subspace of its fully supported solutions are both invariant under the action of monodromy.*

3. *A solution to an atomic system can only be singular on the union of coordinate hyperplanes $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$. The initial exponents of the elements of the pure basis in the subspace of fully supported solutions to* $\mathrm{Horn}(M, \alpha)$ *are given by $\{w_k\}_{k=1}^{|\det M|} = (-M^{-1}(\mathbb{Z}^n + \alpha))/\mathbb{Z}^n$. Here $\mathbb{Z}^n + \alpha$ denotes the shift of the integer lattice by the vector α. Denote by $w^{(1)}, \ldots, w^{(n)}$ the columns of the matrix whose rows are the vectors w_k, $k = 1, \ldots, |\det M|$. The action of monodromy on the pure basis of the subspace $\mathcal{F}$ is given by the matrices $\{\mathrm{diag}(e^{2\pi i w^{(j)}})\}_{j=1}^{n}$.*

Proof. (1) The number of fully supported solutions to an atomic system can be computed by means of Corollary 5.2 in [1] and the formula for the holonomic rank of the Gelfand-Kapranov-Zelevinsky system of equations. Let us show that the generating solution (5) is indeed a solution to the hypergeometric system $\mathrm{Horn}(M, \alpha)$. We denote by $M_1, \ldots, M_n$ the rows of the matrix M and choose $\varepsilon > 0$ to be sufficiently small. For $k \in \mathbb{N}_0^n$ let $\tau(k) = \{s \in \mathbb{C}^n : |\langle M_j, s\rangle + \alpha_j + k_j| = \varepsilon,$ for any $j = 1, \ldots, n\}$ be the n-dimensional cycle around one of the singularities of the Ore-Sato coefficient defining an atomic system. Furthermore let $\mathcal{C} = \sum_{k\in\mathbb{N}_0^n} \tau(k)$. By Proposition 4 in [4] a solution to Horn(M,α) can be represented by the following multiple Mellin-Barnes integral which by computing residues can be transformed into a hypergeometric Puiseux series:

$$\begin{aligned} &\frac{|M|}{(2\pi i)^n}\int_{\mathcal{C}} \prod_{j=1}^{n} \Gamma(\langle M_j, s\rangle + \alpha_j)\, x_1^{s_1}\dots x_n^{s_n}\, ds_1\dots ds_n = \\ &= \sum_{k\in\mathbb{N}_0^n} \frac{(-1)^{|k|}}{k!} x^{-M^{-1}(k+\alpha)} = x^{-M^{-1}\alpha} \sum_{k\in\mathbb{N}_0^n} \frac{1}{k!}\prod_{j=1}^{n}\left(-x^{-M^{-1}e_j}\right)^{k_j} = \\ &= x^{-M^{-1}\alpha}\exp\left(-\sum_{j=1}^{n} x^{-M^{-1}e_j}\right). \end{aligned} \tag{6}$$

Thus the function (5) satisfies the hypergeometric system in question. The dimension of the linear span of all analytic continuations of (5) equals $|\det M|$ and hence any maximal linearly independent set of germs of the generating solution at a nonsingular point is a basis in the subspace of fully supported solutions to an atomic system.

(2) The singular locus of the integrand of the Mellin-Barnes integral representing a solution to an atomic system contains the smallest possible number of families of singular hyperplanes (equal to the dimension n of the ambient space). Thus by definition any series solution to an atomic system is either a fully supported (Puiseux) series or a stable (Puiseux) polynomial. The subspace of Puiseux polynomial solutions of any system of equations (not necessarily atomic or even hypergeometric) is unvariant under the action of monodromy. The invariance of the subspace of fully supported solutions follows from the form of the generating solution (5).

(3) By (1) and (2) any element of the pure basis of an atomic system is either a Puiseux polynomial or the product of a monomial and an entire function. Thus the singular locus of an atomic system is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$. The branching of such a family of functions is the same as that of a family of Puiseux monomials. Thus the action of monodromy on the pure basis of the subspace $\mathcal{F}$ is given by diagonal matrices determined by the initial exponents of the fully supported solutions. These exponents are derived from the Puiseux series expansion (6) for the generating solution to the atomic system. $\square$

Example 1. *Let the hypergeometric configuration be defined by the matrix*

$$M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

the atomic Horn system Horn$(M,(\alpha,\beta,\gamma)^T)$ *has rank two. A basis in its solution space is given by the functions*

$$x^{-\alpha+\gamma}y^{-\frac{1}{2}(-\alpha+\beta+\gamma)}z^{-\frac{1}{2}(-\alpha-\beta+3\gamma)}\exp\left(\frac{x}{\sqrt{y}z^{3/2}} + \frac{\sqrt{z}}{\sqrt{y}} + \frac{\sqrt{y}\sqrt{z}}{x}\right),$$

$$x^{-\alpha+\gamma}y^{-\frac{1}{2}(-\alpha+\beta+\gamma)}z^{-\frac{1}{2}(-\alpha-\beta+3\gamma)}\exp\left(-\frac{x}{\sqrt{y}z^{3/2}} - \frac{\sqrt{z}}{\sqrt{y}} - \frac{\sqrt{y}\sqrt{z}}{x}\right).$$

In the case of two variables it is possible to tell exactly how many Puiseux polynomial solutions an atomic system might have and what their initial exponents are. The following theorem is a consequence of Theorem 5.3 and Lemma 4.5 in [1] and the rank formula for the Gelfand-Kapranov-Zelevinsky hypergeometric system.

Theorem 3. *1. For any 2×2 nondegenerate integer matrix*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and any $c \in \mathbb{C}^2$

$$\operatorname{rank}(\operatorname{Horn}(M,c)) = |\det(M)| + \nu(M).$$

Moreover, there exist $|\det(M)|$ *fully supported series solutions of* $\operatorname{Horn}(M,c)$ *while the remaining* $\nu(M)$ *solutions are stable Puiseux polynomials.*

2. *In the case when* $\nu(M) > 0$ *we may without loss of generality assume that* $a, b > 0$ *and* $c, d < 0$ *(see proof of Lemma 6.3 in [1]). Under this assumption, the initial exponents of the Puiseux polynomial solutions to* $\operatorname{Horn}(M,c)$ *are given by* $M^{-1}\mathcal{R}_M$, *where*

$$\mathcal{R}_M = \begin{cases} \{(u,v) \in \mathbb{N}^2 : u < b,\ v < -c\}, & \text{if } |ad| > |bc|, \\ \{(u,v) \in \mathbb{N}^2 : u < a,\ v < -d\}, & \text{if } |ad| < |bc|. \end{cases}$$

Definition 9. *For* $m \geq n$ *let* A *be a* $m \times n$ *integer matrix of rank* n *with the rows* $A_1, \ldots, A_m$ *and let* $c \in \mathbb{C}^m$ *be a vector of parameters. Let* $I = (i_1, \ldots, i_n)$ *be a multi-index such that the square matrix* A_I *with the rows* $A_{i_1}, \ldots, A_{i_n}$ *is nondegenerate. Let* c_I *denote the vector* $(c_{i_1}, \ldots, c_{i_n})$. *The hypergeometric system* $\operatorname{Horn}(A_I, c_I)$ *will be referred to as an atomic system associated with the system* $\operatorname{Horn}(A,c)$. *The number of atomic systems associated with a hypergeometric system* $\operatorname{Horn}(A,c)$ *equals the number of maximal nondegenerate square submatrices of the matrix* A.

The importance of atomic hypergeometric systems lies in the fact that, as long as the supports of series solutions are concerned, a generic hypergeometric system is built of associated atomic systems. Indeed, the set of supports of solutions to a hypergeometric system with generic parameters consists of supports of solutions to associated atomic systems. In particular, the initial exponents of Puiseux polynomial solutions to a hypergeometric system are precisely the initial exponents of Puiseux polynomials which satisfy the associated atomic systems.

Example 2. *The atomic hypergeometric system defined by the matrix*

$$\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$$

and the zero parameter vector has the form

$$\begin{array}{l} x(3\theta_x - 5\theta_y)(3\theta_x - 5\theta_y + 1)(3\theta_x - 5\theta_y + 2) - (-\theta_x + 2\theta_y), \\ y(-\theta_x + 2\theta_y)(-\theta_x + 2\theta_y + 1) - \prod_{j=0}^{4}(3\theta_x - 5\theta_y + j). \end{array} \tag{7}$$

The pure basis in the solution space of this system is given by the functions 1, $x^{-2}y^{-1}$, $x^{-4}y^{-2}$, $6x^{-5}y^{-3} + x^{-6}y^{-3}$, $24x^{-7}y^{-4} + x^{-8}y^{-4}$, $e^{-xy - x^3y^2}$.

Example 3. *The atomic hypergeometric system defined by the matrix*

$$\begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$$

and the zero parameter vector has the form

$$\begin{array}{l} x\prod_{j=0}^{3}(4\theta_x - 2\theta_y + j) - (-3\theta_x + \theta_y)(-3\theta_x + \theta_y + 1)(-3\theta_x + \theta_y + 2), \\ y(-3\theta_x + \theta_y) - (4\theta_x - 2\theta_y)(4\theta_x - 2\theta_y + 1). \end{array} \tag{8}$$

A basis in the solution space of this system is given by the functions 1, $x^{1/2}y^{3/2}$, $2xy^2 - xy^3$, $6x^{3/2}y^{7/2} - x^{3/2}y^{9/2}$, $e^{-x^{1/2}y^{3/2} - xy^2}$, $e^{x^{1/2}y^{3/2} - xy^2}$.

3. Configurations with Polynomial Bases

In this section, we consider a class of hypergeometric systems whose solution space consists of (Puiseux) polynomials. Apart from systems with rational bases of solutions, such systems have the simplest possible monodromy representation since the corresponding monodromy groups are generated by diagonal matrices.

3.1. Simplicial Configurations

An important special instance of a general nonconfluent Horn system is the system defined by a matrix whose rows are the vertices of an n-dimensional integer simplex. More precisely, let $M \in GL(n,\mathbb{Z})$ be an integer nondegenerate square matrix and $\alpha \in \mathbb{C}^n$ a parameter vector. Let $\tilde{\alpha} = (\alpha, \alpha_{n+1}) \in \mathbb{C}^{n+1}$. Denote by $M_1, \ldots, M_n$ the rows of the matrix M and let $M_{n+1} = -M_1 - \ldots - M_n$. Let $\tilde{M}$ be the $(n+1)\times n$ matrix with the rows $M_1, \ldots, M_{n+1}$. The (nonconfluent) Horn system $\mathrm{Horn}(\tilde{M}, \tilde{\alpha})$ associated with this data will be called *simplicial.*

Proposition 1. *A simplicial Horn system* $\mathrm{Horn}(\tilde{M}, \tilde{\alpha})$ *admits the following generating solution:*

$$x^{-M^{-1}\alpha}\left(1 + \sum_{j=1}^{n} x^{-M^{-1}e_j}\right)^{-|\tilde{\alpha}|}, \tag{9}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ *(1 in the j-th position). Any solution to the Horn system* $\mathrm{Horn}(\tilde{M}, \tilde{\alpha})$ *is either in the linear span of analytic continuations of the generating solution or is a Puiseux polynomial.*

Proof. By Proposition 4 in [4] the multiple Mellin-Barnes integral

$$\frac{1}{(2\pi i)^n} \int\limits_{\mathcal{C}} \prod_{i=1}^{n+1} \Gamma(\langle M_i, s\rangle + \alpha_i)\, x_1^{s_1} \ldots x_n^{s_n}\, ds_1 \ldots ds_n \tag{10}$$

is a solution of the simplicial Horn system $\mathrm{Horn}(M, \alpha)$. Here the contour of integration $\mathcal{C}$ is defined to be the sum of all elements of a suitable basis for n-dimensional homologies of the complement of the singularities of the integrand in (10) as in Proposition 4 in [4]. Since the matrix M is nondegenerate, the function $\prod_{i=1}^{n} \Gamma(\langle M_i, s\rangle + \alpha_i)$ has isolated singularities at the solutions of the system of linear equations

$$M_i s + \alpha_i = -m_i, \qquad i = 1, \ldots, n, \tag{11}$$

where $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$. The residues at these singularities are $(-1)^{|m|}/(m!\,|M|)$. The value of the weight function $\Gamma(\langle M_{n+1}, s\rangle + \alpha_{n+1})$ at the solution of (11) is given by

$$\Gamma(\langle M_{n+1}, M^{-1}(-m-\alpha)\rangle + \alpha_{n+1}) = \Gamma\left(\langle \sum_{i=1}^{n} M_i, M^{-1}(m+\alpha)\rangle + \alpha_{n+1}\right) = \Gamma(|m| + |\tilde{\alpha}|). \tag{12}$$

Using the summation formula

$$\sum_{m \in \mathbb{N}_0^n} (-1)^{|m|} \Gamma(|m| + \lambda)/m!\, x^m = \Gamma(\lambda)(1 + |x|)^{-\lambda}$$

and Theorem 5 in [4] we arrive at (9). The second claim of the proposition is proved as in Theorem 2. □

Example 4. *The Horn system*

$$\begin{aligned} &x(2\theta_x + \theta_y - 7)(2\theta_x + \theta_y - 6)(\theta_x + 2\theta_y - 7) - (\theta_x + \theta_y - 3)(\theta_x + \theta_y - 10/3)(\theta_x + \theta_y - 11/3),\\ &y(2\theta_x + \theta_y - 7)(\theta_x + 2\theta_y - 7)(\theta_x + 2\theta_y - 6) - (\theta_x + \theta_y - 3)(\theta_x + \theta_y - 10/3)(\theta_x + \theta_y - 11/3) \end{aligned}$$

is holonomic and its holonomic rank equals 9. The pure basis in its solution space is given by the Puiseux polynomials $y^4/x,\ y^{11/3}/x^{1/3},\ x^{1/3}y^{10/3},\ x^{10/3}y^{1/3},\ x^{11/3}/y^{1/3},\ x^4/y,$

$$x^3 + 2x^2y - 54x^3y + 2xy^2 - 108x^2y^2 + y^3 - 54xy^3,\ 10x^{7/3}y^{4/3} + 10x^{4/3}y^{7/3} - 9x^{7/3}y^{7/3},$$

$$4x^{8/3}y^{2/3} + 6x^{5/3}y^{5/3} - 27x^{8/3}y^{5/3} + 4x^{2/3}y^{8/3} - 27x^{5/3}y^{8/3}.$$

The six Puiseux monomials are stable while the last three solutions are not.

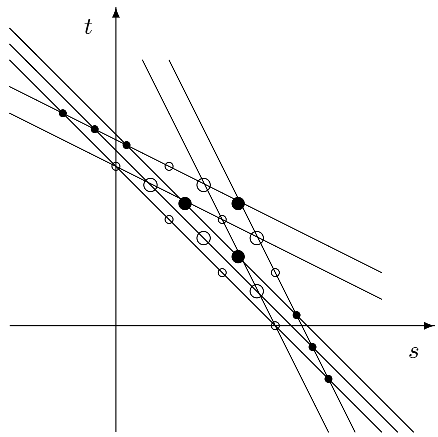

Fig. 1. The supports of solutions

In this picture, the small filled circles correspond to the monomial solutions, the big filled circles indicate the Puiseux trinomial while the remaining two polynomials correspond to the small and big empty circles respectively.

3.2. Parallelepipeds

Let $M \in GL(n, \mathbb{Z})$ be an integer nondegenerate square matrix and let $\alpha, \beta \in \mathbb{C}^n$ be two parameter vectors. Denote by $\tilde{M}$ the $2n \times n$ matrix obtained by joining together the rows of the matrices M and $-M$. The rows of such a matrix define the vertices of a parallelpiped of nonzero n-dimensional volume. Let $\tilde{\alpha}$ be the vector with the components $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$. It turns out that the corresponding Horn system $\mathrm{Horn}(\tilde{M}, \tilde{\alpha})$ admits a simple basis of solutions.

Proposition 2. *The Horn system* $\mathrm{Horn}(\tilde{M}, \tilde{\alpha})$ *admits the following generating solution:*

$$x^{-M^{-1}\alpha} \prod_{j=1}^{n} \left(1 + x^{-M^{-1}e_j}\right)^{-\alpha_j - \beta_j}, \tag{13}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ *(1 in the j-th position). Any solution to the Horn system* $\mathrm{Horn}(\tilde{M}, \tilde{\alpha})$ *is either in the linear span of analytic continuations of the generating solution or is a Puiseux polynomial.*

The proof of this proposition is analogous to the proof of Proposition 1 above.

Corollary 1. *Any Horn system defined by a matrix whose rows are the vertices of a simplex or a parallelepiped admits a basis of Puiseux polynomials for suitable values of its parameters. In particular, the solution space of such a system splits into the direct sum of one-dimensional invariant subspaces. The monodromy representation of such a Horn system is reducible.*

Example 5. *The Horn system*

$$\begin{gathered} x(\theta_x - 2\theta_y - 2)(2\theta_x - \theta_y - 2)(2\theta_x - \theta_y - 1) - (2\theta_x - \theta_y + 1)(2\theta_x - \theta_y + 2)(\theta_x + 2\theta_y - 2), \\ y(\theta_x - 2\theta_y + 1)(\theta_x - 2\theta_y + 2)(-2\theta_x + \theta_y - 2) - (\theta_x - 2\theta_y - 1)(2\theta_x - \theta_y - 2)(\theta_x - 2\theta_y - 2) \end{gathered}$$

is holonomic and its holonomic rank equals 5. The pure basis in its solution space is given by the Puiseux polynomials x^2y^2, $x^{-2}y^{-2}$, $9 + 1/x + x + 1/y + 4/(xy) + y + 4xy$,

$$4x^{-5/3}y^{-4/3} + 6x^{-2/3}y^{-4/3} + 24x^{-2/3}y^{-1/3} + 16x^{1/3}y^{-1/3} + x^{-2/3}y^{2/3} +$$

$$+24x^{1/3}y^{2/3} + 6x^{4/3}y^{2/3} + 4x^{4/3}y^{5/3},$$
$$4x^{-4/3}y^{-5/3} + 6x^{-4/3}y^{-2/3} + 24x^{-1/3}y^{-2/3} + x^{2/3}y^{-2/3}+$$
$$+16x^{-1/3}y^{1/3} + 24x^{2/3}y^{1/3} + 6x^{2/3}y^{4/3} + 4x^{5/3}y^{4/3}.$$

The only stable solutions in this example are the Laurent monomials x^2y^2 *and* $x^{-2}y^{-2}$.

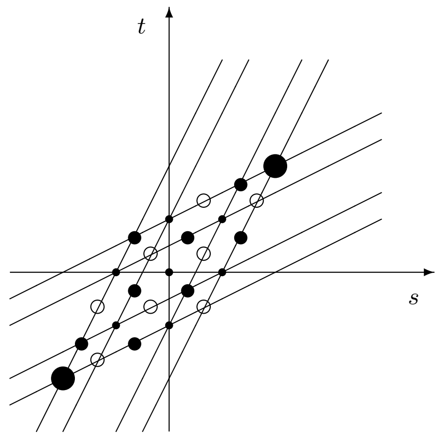

Fig. 2. The supports of solutions

Gratefully acknowledge the financial support of the SFU by SM project, grant 45.2007.

References

[1] A.Dickenstein, L.Matusevich, T.Sadykov, Bivariate hypergeometric D-modules, *Adv. in Math.*, **196**(2005), no. 1, 78-123.

[2] I.M.Gelfand, M.I.Graev, V.S.Retach, General hypergeometric systems of equations and series of hypergeometric type, *Russian Math. Surveys*, **47**(1992), no. 4, 1-88.

[3] V.P.Palamodov, Linear differential operators with constant coefficients, Moscow, Nauka, 1967 (Russian).

[4] T.M.Sadykov, On a multidimensional system of hypergeometric differential equations, *Siberian Math. J.*, **39**(1998), 986-997.

[5] M.Sato, Theory of prehomogeneous vector spaces (algebraic part), *Nagoya Math. J.*, **120** (1990), 1-34.