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On Asymptotic Expansion of the Conormal Symbol of the Singular Bochner-Martinelli Operator on the Surfaces with Singular Points

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We study the conormal symbol of the singular Bochner-Martinelli integral on a compact closed surface with conical points $\mathcal{S}$ in $\mathbb{C}^n$ and evaluate its asymptotic expansion.

Key words: *singular Bochner-Martinelli operator, conormal symbol, conical point.*

Smooth manifolds with conical points are the simplest singular spaces in the hierarchy of stratified varieties. Differential analysis on such manifolds was perhaps initiated by Kondrat'ev [1] who invented the so-called conormal symbol of a differential operator at a singular point.

In the 1980s the analysis encompassed also pseudodifferential operators which has led to diverse algebras of pseudodifferential operators on manifolds with conical points, see [4] and the references given there. All the algebras start actually with the same typical differential operators which are of Fuchs type.

When applied to the Cauchy integral on a plane curve with corners, conormal symbols can be efficiently computed.

The work [5] was intended as an attempt to examine whether the cone theory still effectively applies to higher dimensions. To this end, we study the singular Bochner-Martinelli integral on a compact closed surface with conical points $\mathcal{S}$ in $\mathbb{C}^n$ and evaluate its conormal symbol at a conical point. Our computation demonstrates rather strikingly that the conormal symbols are no longer efficient for pseudodifferential operators in dimensions larger than 1.

The singular Bochner-Martinelli integral is of central importance in complex analysis in several variables ([2]). In [3], the C^*-algebra generated by this integral on a compact closed hypersurface without singular points is described. In contrast to the singular case, the principal homogeneous symbol is as explicit as the Bochner-Martinelli integral itself.

In this work we find the asymptotic expansion of conormal symbol.

As usual, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ under the complex structure $z_j = x_j + ix_{n+j}$, for $j = 1, \ldots, n$.

We will consider a smooth hypersurface $\mathcal{S}$ in $\mathbb{C}^n \setminus \{0\}$ with a singular point at the origin given by

$$\mathcal{S} = \{(\varphi(r)x, r) \in \mathbb{R}^{2n} : x \in X,\ r \in [0, R)\}, \tag{1}$$

where $\varphi \in C^1[0, R)$ satisfies $\varphi(0) = 0$ and $\varphi(r) > 0$ for $r \in (0, R)$, and the point $x = (x_1, \ldots, x_{2n-1})$ varies over a smooth compact hypersurface X in $\mathbb{R}^{2n-1}$ which does not meet 0.

For instance, X may be a $(2n-2)$-dimensional sphere with the centre at the origin. In any case we assume that $X = \{x \in \mathbb{R}^{2n-1} : \rho(x) = 1\}$, where ρ is a C^1 function on $\mathbb{R}^{2n-1} \setminus \{0\}$ with real values, satisfying $\nabla\rho \neq 0$ on X and $\rho(\lambda x) = \lambda^h \rho(x)$ for all $\lambda > 0$ with some $h > 0$.

The origin is a singular point of $\mathcal{S}$, for $\varphi'(0) < \infty$. If $\varphi'(0) \neq 0$ then 0 is a conical point of $\mathcal{S}$. In the case $\varphi'(0) = 0$ the point 0 is a cusp.

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Using (1) it is easy to determine a defining function of the smooth part of $\mathcal{S}$. Indeed, write $\rho(x) = \rho(z', x_n)$, where $z' = (z_1, \ldots, z_{n-1})$. Then $z \in \mathcal{S} \setminus \{0\}$ readily implies

$$\rho\Big(\frac{z'}{\varphi(\operatorname{Im} z_n)}, \frac{\operatorname{Re} z_n}{\varphi(\operatorname{Im} z_n)}\Big) = 1,$$

and so the homogeneity of ρ yields $\mathcal{S} = \{z \in \mathbb{C}^n : \operatorname{Im} z_n \in [0, R),\ \varrho(z) = 0\}$, with

$$\varrho(z) = \rho(z', \operatorname{Re} z_n) - (\varphi(\operatorname{Im} z_n))^h.$$

Given an integrable function f with compact support on $\mathcal{S}$, the singular Bochner-Martinelli integral of f is defined by

$$M_{\mathcal{S}} f\,(z) = \text{p.v.} \int\limits_{\mathcal{S}} f(\zeta) U(\zeta, z)$$

for $z \in \mathcal{S}$, where

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}}\, d\bar{\zeta}[j] \wedge d\zeta$$

and $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, while $d\bar{\zeta}[j]$ is the wedge product of all differentials $d\bar{\zeta}_1, \ldots, d\bar{\zeta}_n$ but $d\bar{\zeta}_j$. In the sequel, we drop the designation ‘p.v.’ for short.

The properties of the Bochner-Martinelli singular integral operator on the smooth hypersurfaces are well understood. We are aimed at investigating this operator on the hypersurfaces with isolated singular points. Since $M_{\mathcal{S}} f$ is smooth away from the support of f, one can certainly assume without loss of generality that $\mathcal{S}$ is of the form (1).

We first represent $U(\zeta, z)$ in the local coordinates of $\mathcal{S}$ close to a singular point. Set

$$\nu(y) = \frac{\nabla_y \rho}{|\nabla_y \rho|},$$

for $y \in X$, and $\nu_{2n}(y, s) = -h \dfrac{\varphi'(s)}{|\nabla_y \rho|}$.

Lemma 1 ([5]). *The restriction of the Bochner-Martinelli kernel to the hypersurface $\mathcal{S}$ has the form*

$$\begin{aligned} U(\zeta, z) \quad = \quad & \frac{1}{\sigma_{2n}} \frac{\langle (\nu(y), \nu_{2n}(y,s)), (\varphi(s) y - \varphi(r) x, s - r)\rangle}{(|\varphi(s) y - \varphi(r) x|^2 + (s-r)^2)^n} (\varphi(s))^{2n-2} ds\, d\sigma(y) - \\ -i\ & \frac{1}{\sigma_{2n}} \frac{\langle i\nu_c(y,s), (\varphi(s) y - \varphi(r) x, s - r)\rangle}{(|\varphi(s) y - \varphi(r) x|^2 + (s-r)^2)^n} (\varphi(s))^{2n-2} ds\, d\sigma(y), \end{aligned}$$

where $d\sigma$ is the area form on X induced by the Lebesgue measure in $\mathbb{R}^{2n}$, σ_{2n} the area of the $(2n-1)$-dimensional sphere, and $i\nu_c = (-\nu_{n+1}, \ldots, -\nu_{2n}, \nu_1, \ldots, \nu_n)$.

The vector $i\nu_c$ is the vector lying in the tangent space to $\mathcal{S}$ and such that it is orthogonal to complex tangent space to $\mathcal{S}$. It indicates to what extent the surface $\mathcal{S}$ fits to the complex structure of $\mathbb{C}^n$, see [3].

From now on, we restrict our discussion to the hypersurfaces $\mathcal{S} \subset \mathbb{R}^n$ with conical points.

Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$, with $n > 1$. The boundary of $\mathcal{D}$ is assumed to be of the form $\mathcal{Y} \cup (\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_N)$, where $\mathcal{Y}$ is a smooth hypersurface and each $\mathcal{S}_\nu$ is diffeomorphic to a conical hypersurface $\mathcal{S}$ as above. Thus, $\partial\mathcal{D}$ is a smooth hypersurface with a finite number of conical points. Since the analysis at singular points is local, one can assume without loss of generality that $N = 1$, i.e., $\partial\mathcal{D} = \mathcal{Y} \cup \mathcal{S}$ where

$$\mathcal{S} = \{z \in \mathbb{C}^n : z = (rx, r),\ x \in X,\ r \in [0, R)\},$$

cf. (1).

For the function $f \in C_{\rm comp}(\partial\mathcal{D} \setminus \{0\})$ we define the norm

$$\|f\|_{L^{2,\gamma}(\partial\mathcal{D})} := \Big(\int\limits_{\partial\mathcal{D}} |z|^{-2\gamma}\, |f|^2\, d\Sigma \Big)^{1/2}, \tag{2}$$

where $\gamma \in \mathbb{R}$. Denote by $L^{2,\gamma}(\partial\mathcal{D})$ the completion of $C_{\rm comp}(\partial\mathcal{D} \setminus \{0\})$ with respect to this norm.

It is clear that the weight factor $|z|^{-2\gamma}$ affects the behaviour of functions in $L^{2,\gamma}(\partial\mathcal{D})$ merely at the conical point 0.

According to $\partial\mathcal{D} = \mathcal{Y} \cup \mathcal{S}$, the norm (2) can be splitted into two seminorms. The first of the two corresponds to integration over $\partial\mathcal{D} \setminus \mathcal{S}$ and controls the behaviour of functions on the smooth part of the boundary. The second seminorm corresponds to integration over $\mathcal{S}$ and specifies the behaviour of functions close to the singular points. Under the parametrisation (1), the hypersurface $\mathcal{S}$ is identified with the cylinder $X \times [0,R)$. In this manner the second seminorm actually stems from the norm

$$\|f\|_{L^{2,\gamma-n+1/2}(X\times[0,R))} := \Big(\int\limits_0^R r^{-2(\gamma-n+1/2)}\, \|f\|^2_{L^2(X)}\, \frac{dr}{r} \Big)^{1/2}$$

on the functions $f \in C_{\rm comp}(X \times (0,R))$.

Introduce a function $k(x,y;t)$ defined for $(x,y) \in X \times X$ and $t > 0$ by the equality

$$\begin{aligned} &k(x,y;t) = \\ &= \frac{1}{\sigma_{2n}} \frac{\langle (\nu(y), \nu_{2n}(y)), (y-tx, 1-t) \rangle}{(|y-tx|^2 + (1-t)^2)^n} - \frac{i}{\sigma_{2n}} \frac{\langle i\nu_c(y), (y-tx, 1-t) \rangle}{(|y-tx|^2 + (1-t)^2)^n}. \end{aligned}$$

Using this kernel, we can rewrite the singular Bochner-Martinelli integral in the form

$$M_{\mathcal{S}} f\,(x,r) = \int\limits_0^\infty \frac{ds}{s} \int\limits_X k\Big(x,y;\frac{r}{s}\Big) f(y,s)\, d\sigma(y), \tag{3}$$

where (x,r) and (y,s) are identified with $z = (rx, r)$ and $\zeta = (sy, s)$, respectively. Note that the integral over X is singular, for $k(x,y;r/s)$ has a singularity at $y = x$ provided $s = r$.

Theorem 1 ([5]). *Integral (3) induces a bounded linear operator in $L^{2,\gamma}(X \times [0,R))$ provided $1 - 2n < \gamma < 0$.*

Denote by $\mathcal{M}_{r\mapsto\lambda}$ the Mellin transform defined on functions $f(r)$ on the semi-axis. It is given by

$$\mathcal{M}_{r\mapsto\lambda} f = \int\limits_0^\infty r^{-i\lambda} f(r) \frac{dr}{r}$$

for $\lambda \in \mathbb{C}$.

Composing the singular Bochner-Martinelli operator (3) with the Mellin transform yields

$$\begin{aligned} \mathcal{M}_{r\mapsto\lambda} M_{\mathcal{S}} f(x,r) &= \int\limits_0^\infty r^{-i\lambda} \frac{dr}{r} \int\limits_0^\infty \frac{ds}{s} \int\limits_X k\Big(x,y;\frac{r}{s}\Big) f(y,s)\, d\sigma(y) = \\ &= \int\limits_0^\infty \frac{ds}{s} \int\limits_X \Big(\int\limits_0^\infty r^{-i\lambda}\, k\Big(x,y;\frac{r}{s}\Big) \frac{dr}{r} \Big) f(y,s)\, d\sigma(y) \end{aligned}$$

where (x,r) and (y,s) are identified with the points $z = (rx,r)$ and $\zeta = (sy,s)$ of $\mathcal{S}$, respectively.

In the integral over $r \in (0,\infty)$ we change the variables by $r = st$, where t runs over $(0,\infty)$. This gives

$$\begin{aligned}\mathcal{M}_{r\mapsto\lambda} M_{\mathcal{S}} f(x,r) &= \int\limits_0^\infty s^{-i\lambda}\,\frac{ds}{s}\int\limits_X \Big(\int\limits_0^\infty t^{-i\lambda}\,k(x,y;t)\frac{dt}{t}\Big) f(y,s)\,d\sigma(y) = \\ &= \int\limits_X \mathcal{M}_{t\mapsto\lambda}k(x,y;t)\,\mathcal{M}_{s\mapsto\lambda}f(y,s)\,d\sigma(y)\end{aligned}$$

for $x \in X$ and $\lambda \in \mathbb{C}$. It follows that

$$M_{\mathcal{S}} f(r) = \mathcal{M}^{-1}_{\lambda\mapsto r}\,a(\lambda)\mathcal{M}_{r'\mapsto\lambda} f(r'), \tag{4}$$

where $f(r) := f(x,r)$ is thought of as a function of $r \in (0,\infty)$ with values in functions of $x \in X$, and $a(\lambda)$ is a family of singular integral operators on X parametrised by λ varying on a horizontal line in the complex plane. The action of $a(\lambda)$ is specified by

$$a(\lambda)f(x) = \int\limits_X \mathcal{M}_{t\mapsto\lambda}k(x,y;t)\,f(y)\,d\sigma(y).$$

The family $a(\lambda)$ is usually referred to as the conormal symbol of the pseudodifferential operator (3) based on the Mellin transform. To evaluate it more explicitly, we denote by Z the unique root of $\langle y-tx, y-tx\rangle + (1-t)^2 = 0$ in the upper half-plane, i.e.,

$$Z = \frac{1+\langle x,y\rangle + i\sqrt{|y-x|^2+|x|^2|y|^2-\langle x,y\rangle^2}}{1+|x|^2}. \tag{5}$$

Lemma 2 ([5]). *In the strip $0 < \operatorname{Im}\lambda < 2n-1$, the Mellin transform of $k(x,y;t)$ has the form*

$$\begin{aligned}&\mathcal{M}_{t\mapsto\lambda}k(x,y;t) = \\ &= \pi i\,\frac{(-1)^{n-1}}{(n-1)!}\,\frac{\exp\pi\lambda}{\sinh\pi\lambda}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,(i\lambda+1)(i\lambda+2)\dots(i\lambda+j-1)\times \\ &\times\ \frac{((i\lambda+j)A - i\lambda ZB)Z^{-i\lambda-j-1} + (-1)^{j-1}((i\lambda+j)A - i\lambda\bar{Z}B)\bar{Z}^{-i\lambda-j-1}}{(1+|x|^2)^n\,(Z-\bar{Z})^{2n-1-j}},\end{aligned}$$

where

$$\begin{aligned}A &= \frac{1}{\sigma_{2n}}\langle(\nu(y),\nu_{2n}(y)),(y,1)\rangle - \frac{i}{\sigma_{2n}}\langle i\nu_c(y),(y,1)\rangle,\\ B &= \frac{1}{\sigma_{2n}}\langle(\nu(y),\nu_{2n}(y)),(x,1)\rangle - \frac{i}{\sigma_{2n}}\langle i\nu_c(y),(x,1)\rangle.\end{aligned}$$

The Lemma is based on the formulas

$$\begin{aligned}&\operatorname{res}(f;Z) + \operatorname{res}(f;\bar{Z}) = \\ &= \frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{(-1)^{j+1}Z^{p-j}+\bar{Z}^{p-j}}{(Z-\bar{Z})^{2n-1-j}},\end{aligned} \tag{6}$$

where

$$f(t) = \frac{t^p}{(t-Z)^n(t-\bar{Z})^n} \tag{7}$$

and

$$\int_0^\infty t^{-\imath\lambda}k(x,y;t)\frac{dt}{t} = \pi i\,\frac{\exp\pi\lambda}{\sinh\pi\lambda}\Big(\operatorname{res}(t^{-\imath\lambda-1}k(x,y;t);Z) + \operatorname{res}(t^{-\imath\lambda-1}k(x,y;t);\bar{Z})\Big). \tag{8}$$

We denote

$$G(t) = t^{-\imath\lambda-1}k(x,y;t) = \frac{1}{a^n}\,\frac{At^{-\imath\lambda-1} - Bt^{-\imath\lambda}}{(t-Z)^n(t-\bar{Z})^n}.$$

We are now in a position to specify the inverse Mellin transform in the representation formula (4).

Theorem 2 ([5])**.** *For $|\gamma| < n - 1/2$ the singular Bochner-Martinelli integral admits the representation*

$$M_{\mathcal{S}}f\,(r) = \frac{1}{2\pi}\int\limits_{\operatorname{Im}\lambda=(n-1/2)-\gamma} r^{\imath\lambda}a(\lambda)\mathcal{M}_{r'\mapsto\lambda}f(r')\,d\lambda. \tag{9}$$

We first find asymptotics of the sum of residues of the function $f(t)$ given by the formula (7).

Lemma 3. *The sum of residues of the function $f(t)$ at Z and $\bar{Z}$ has no singularity as $\operatorname{Im}Z \to 0$, and*

$$\lim_{\operatorname{Im}Z\to 0}\Big(\operatorname{res}(f;Z) + \operatorname{res}(f;\bar{Z})\Big) = \frac{p(p-1)\dots(p-2n+2)}{(2n-1)!}\,Z^{p-2n+1}.$$

Moreover,

$$\operatorname{res}(f;Z) + \operatorname{res}(f;\bar{Z}) =$$

$$= \frac{1}{2(n-1)!}\sum_{s=0}^{\infty}\frac{p\cdots(p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\Big(\bar{Z}^{p-2n-s+1} + (-1)^s Z^{p-2n-s+1}\Big).$$

Proof. Set $\Sigma = \operatorname{res}(f;Z) + \operatorname{res}(f;\bar{Z})$. By the formula (6),

$$\Sigma =$$

$$= \frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,Z^{p-2n+1}\frac{(-1)^{j+1} + \left(\dfrac{\bar{Z}}{Z}\right)^{p-j}}{\left(1-\dfrac{\bar{Z}}{Z}\right)^{2n-1-j}}.$$

Setting $Q := 1 - \bar{Z}/Z$ we rewrite Σ in the form

$$\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,Z^{p-2n+1}\frac{(-1)^{j+1} + (1-Q)^{p-j}}{Q^{2n-1-j}},$$

which splits into two sums

$$Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{(-1)^{j+1}}{Q^{2n-1-j}},$$

$$Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{\sum\limits_{k=0}^{\infty}\binom{p-j}{k}(-Q)^k}{Q^{2n-1-j}}.$$

The binomial series in the latter sum converges only for $|Q| < 1$. If $|Q| = 1$ it should be replaced by a Taylor polynomial of sufficiently large degree N along with a remainder $O((\operatorname{Im}Z)^{N+1})$.

Set $l = j + k$ in the second sum and transform it. We obtain

$$Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=0}^{n-1}\,\sum_{l=j}^{\infty}\,\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,\frac{p(p-1)\dots(p-l+1)}{(l-j)!}\,\frac{(-1)^{l-j}Q^l}{Q^{2n-1}}.$$

Interchanging the order of summation and substituting j for l and k for j immediately yields

$$Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=0}^{n-1}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\,\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!} +$$

$$+\;\; Z^{p-2n+1}\,\frac{(-1)^n}{(n-1)!}\,\sum_{j=n}^{\infty}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\,\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}.$$

Summarizing we get

$$\begin{aligned}
\Sigma &= Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\frac{(2n-2-j)!}{j!\,(n-1-j)!}\,p(p-1)\dots(p-j+1)\,\frac{(-1)^{j+1}}{Q^{2n-1-j}}+\\
&+ Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=0}^{n-1}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}+\\
&+ Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\sum_{j=n}^{\infty}\binom{p}{j}\frac{(-1)^jQ^j}{Q^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}.
\end{aligned}$$

Lemma 3 will be proved once we prove the lemma below. The latter is of an independent interest.

Lemma 4. *We have*

$$\begin{aligned}
\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!} &= \frac{(2n-2-j)!}{(n-1-j)!}, && \text{if } j=0,1,\dots,n-1;\\
\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!} &= 0, && \text{if } j=n,n+1,\dots,2n-2;\\
\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!} &= (-1)^{n-1}(n-1)!, && \text{if } j=2n-1,\\
\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!} &=\\
=\sum_{l=2n-1}^{j}\frac{(-1)^{l+n}(l-2n+2)\cdots(l-n)}{l!(j-l)!}&, && \text{if } j>2n-1.
\end{aligned}$$

Proof. Consider the function

$$F(z)=\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}z^{n-1-k}.$$

A trivial verification shows that

$$\begin{aligned}
F(z) &= \left(\frac{\partial}{\partial z}\right)^{n-1}\sum_{k=0}^{j}(-1)^k\binom{j}{k}z^{2n-2-k}=\\
&= \left(\frac{\partial}{\partial z}\right)^{n-1}z^{2n-2}\left(1-\frac{1}{z}\right)^j=\\
&= \left(\frac{\partial}{\partial z}\right)^{n-1}\left(z^{2n-2-j}(z-1)^j\right).
\end{aligned}$$

For $z=1$ we then get

$$\begin{aligned}
\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!} &= \binom{n-1}{j}j!\,(2n-2-j)(2n-3-j)\dots n=\\
&= \frac{(2n-2-j)!}{(n-1-j)!}
\end{aligned}$$

whenever $j=0,1,\dots,n-1$.

In just the same way we evaluate the second sum. Suppose $j=n,n+1,\dots,2n-2$. Consider the function

$$F(z)=\sum_{k=0}^{n-1}(-1)^k\binom{j}{k}\frac{(2n-2-k)!}{(n-1-k)!}z^{n-k-1}$$

which is actually equal to

$$\Big(\frac{\partial}{\partial z}\Big)^{n-1}\left(z^{2n-2-j}(z-1)^j\right),$$

as it is easy to check. Hence we readily deduce that $F(1)=0$, as desired.

Let us prove the third equality corresponding to $j=2n-1$. For this purpose, consider the function

$$F(z)=\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{k}\frac{(2n-2-k)!}{(n-1-k)!}z^{n-k-1}.$$

An easy computation shows that

$$\begin{aligned}
F(z) &= \Big(\frac{\partial}{\partial z}\Big)^{n-1}\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{k}z^{2n-2-k}= \\
&= \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}z^{2n-2-k}-(-1)^{2n-1}\frac{1}{z}\Big)= \\
&= \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\frac{(z-1)^{2n-1}}{z}\Big)+\Big(\frac{\partial}{\partial z}\Big)^{n-1}\frac{1}{z}= \\
&= \Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\frac{(z-1)^{2n-1}}{z}\Big)+(-1)^{n-1}(n-1)!\,z^{-n}.
\end{aligned}$$

For $z=1$ the first term vanishes, and so $F(1)=(-1)^{n-1}(n-1)!$

Consider the last equality corresponging to $j>2n-1$. We have

$$F(z)=\Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{l=0}^{n-1}\frac{(-1)^l z^{2n-l-2}}{l!(j-l)!}\Big)=$$

$$=\Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{l=0}^{j}\frac{(-1)^l z^{2n-l-2}}{l!(j-l)!}-\sum_{l=2n-1}^{j}\frac{(-1)^l z^{2n-l-2}}{l!(j-l)!}\Big)=$$

$$=\Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\frac{z^{2n-j-2}(z-1)j}{j!}\Big)+\Big(\frac{\partial}{\partial z}\Big)^{n-1}\Big(\sum_{l=2n-1}^{j}\frac{(-1)^{l+1}z^{2n-l-2}}{l!(j-l)!}\Big).$$

Therefore

$$F(1)=\sum_{2n-1}^{j}\frac{(-1)^{l+n}(l-2n+2)\cdots(l-n)}{l!(j-l)!},$$

which proves the lemma. □

We are now in a position to complete the proof of Lemma 3. To this end, we observe that the first and the second sums in the expression for $\varSigma$ cancel. In the third sum only the terms corresponding to $j\geqslant 2n-1$ do not vanish. Hence it follows that

$$\begin{aligned}
\lim_{\operatorname{Im} Z\to 0}\varSigma &= Z^{p-2n+1}\frac{(-1)^n}{(n-1)!}\binom{p}{2n-1}\frac{(-1)^{2n-1}Q^{2n-1}}{Q^{2n-1}}=(-1)^{n-1}(n-1)! \\
&= \binom{p}{2n-1}Z^{p-2n+1}.
\end{aligned}$$

Then by Lemma 4

$$\varSigma=Z^{p-2n+1}\sum_{j=2n-1}^{\infty}(-1)^j p(p-1)\cdots(p-j+1)Q^{j-2n-1}\sum_{l=2n-1}^{j}\frac{(-1)^{l+n}(l-2n+2)\cdots(l-n)}{l!(j-l)!}.$$

Substituting $j=k+2n-1$ and $l=s+2n-1$, we get

$$\varSigma=Z^{p-2n+1}\sum_{k=0}^{\infty}(-1)^{k+1}p\cdots(p-k-2n+2)Q^k\sum_{s=0}^{k}\frac{(-1)^{s+n-1}(s+1)\cdots(s+n-1)}{(s+2n-1)!(k-s)!}=$$

$$= Z^{p-2n-1}\sum_{s=0}^{\infty}\sum_{k=s}^{\infty}\frac{(-1)^{k+s+n}p\cdots(p-k-2n+2)(s+1)\cdots(s+n-1)Q^k}{(s+2n-1)!(k-s)!} =$$

$$= Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^{s+n}(j+1)\cdots(j+n-1)}{(s+2n-1)!}\sum_{k=s}^{\infty}\frac{(-1)^k p\cdots(p-k-2n+2)Q^k}{(k-s)!} =$$

$$= Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^{s+n}(j+1)\cdots(j+n-1)}{(s+2n-1)!}\sum_{m=0}^{\infty}\frac{(-1)^{s+m}p\cdots(p-s-m-2n+2)Q^{s+m}}{m!} =$$

$$= Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^{n}(j+1)\cdots(j+n-1)Q^s}{(s+2n-1)!}\sum_{m=0}^{\infty}\frac{(-1)^{m}p\cdots(p-s-m-2n+2)Q^{m}}{m!}.$$

The sum

$$\sum_{m=0}^{\infty}\frac{(-1)^{m}p\cdots(p-s-m-2n+2)Q^{m}}{m!} =$$

$$= p\cdots(p-s-2n+2)\sum_{m=0}^{\infty}\frac{(-1)^{m}(p-2n-s+1)\cdots(p-s-m-2n+2)Q^{m}}{m!} =$$

$$= p\cdots(p-s-2n+2)(1-Q)^{p-2n-s-1}.$$

Therefore,

$$\Sigma = Z^{p-2n+1}\sum_{s=0}^{\infty}\frac{(-1)^n(s+1)\cdots(s+n-1)p\cdots(p-2n-s+2)Q^s(1-Q)^{p-2n-s+1}}{(s+2n-1)!} =$$

$$= (-1)^n\cdot Z^{p-2n+1}\cdot(1-Q)^{p-2n+1}\sum_{s=0}^{\infty}\frac{(s+1)\cdots(s+n-1)p\cdots(p-2n-s+2)Q^s(1-Q)^{-s}}{(s+2n-1)!}.$$

Since $Q = 1-\bar{Z}/Z$, then $1-Q=\bar{Z}/Z$ and

$$\frac{Q}{1-Q} = \frac{Z-\bar{Z}}{\bar{Z}}.$$

From here

$$\Sigma = (-1)^n\bar{Z}^{p-2n-1}\sum_{s=0}^{\infty}\frac{p\cdots(p-2n-s+2)}{s!(s+n)\cdots(s+2n-1)}\cdot\frac{(Z-\bar{Z})^s}{\bar{Z}^s}.$$

Hence

$$\operatorname{res}(f;Z)+\operatorname{res}(f;\bar{Z}) = \frac{\bar{Z}^{p-2n+1}}{(n-1)!}\sum_{s=0}^{\infty}\frac{p\cdots(p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)\bar{Z}^s} =$$

$$= \frac{1}{2(n-1)!}\sum_{s=0}^{\infty}\frac{p\cdots(p-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\cdot\left(\bar{Z}^{p-2n-s+1}+(-1)^sZ^{p-2n-s+1}\right),$$

as desired. □

Theorem 3. *The function $\mathcal{M}_{t\mapsto\lambda}k(x,y;t)$ admits an asymptotic expansion*

$$\begin{aligned}\mathcal{M}_{t\mapsto\lambda}k(x,y;t) &=\\ &= \pi\frac{(i\lambda+1)\ldots(i\lambda+2n-2)}{(2n-1)!}\frac{\exp\pi\lambda}{\sinh\pi\lambda}\frac{((i\lambda+2n-1)\operatorname{Im}A - i\lambda\operatorname{Re}Z\operatorname{Im}B)Z^{-i\lambda-2n}}{(1+|x|^2)^n} +\\ &+ O(\operatorname{Im}Z)\end{aligned}$$

as $\operatorname{Im}Z\to 0$.

Moreover,

$$\mathcal{M}_{t\mapsto\lambda}k(x,y;t) = \frac{i\pi}{2(2n-1)!}\frac{\exp\pi\lambda}{(1+|x|^2)^n\sinh\pi\lambda}\sum_{s=0}^{\infty}\frac{(i\lambda+1)\cdots(i\lambda+2n+s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)}\times$$

$$\times\left((-1)^{s+1}\bar{Z}^{-i\lambda-2n-s}(i\lambda B\bar{Z}+A(i\lambda+2n+s-1)) - \bar{Z}^{-i\lambda-2n-s}(i\lambda BZ+A(i\lambda+2n+s-1))\right).$$

PROOF. Using Lemma 2 and Lemma 3 we obtain

$$\lim_{\operatorname{Im} Z\to 0} \mathcal{M}_{t\mapsto\lambda} k(x,y;t) =$$
$$= \quad -\pi i\, \frac{(i\lambda+1)\dots(i\lambda+2n-2)}{(2n-1)!}\, \frac{\exp\pi\lambda}{\sinh\pi\lambda}\, \frac{((i\lambda+2n-1)A - i\lambda ZB)Z^{-i\lambda-2n}}{(1+|x|^2)^n}.$$

Let us estimate the sum $B+\bar{B}$. Since $\langle\nabla_y\rho, y\rangle = h$ it follows that the real part of B is

$$\frac{B+\bar{B}}{2} = \frac{2}{\sigma_{2n}}\, \frac{\langle\nabla_y\rho, x\rangle - h}{|\nabla_y\rho|} = \frac{2}{\sigma_{2n}}\, \frac{\langle\nabla_y\rho, x-y\rangle}{|\nabla_y\rho|},$$

which is $O(\operatorname{Im} Z)$ as $\operatorname{Im} Z\to 0$. On the other hand, A is purely imaginary, for

$$\langle(\nu(y), \nu_{2n}(y)), (y,1)\rangle = 0.$$

This establishes the first formula.

Consider the last formula. Since

$$\operatorname{res}(G;Z) + \operatorname{res}(G;\bar{Z}) =$$
$$= \frac{B}{2(n-1)!a^n}\sum_{s=0}^{\infty} \frac{(-i\lambda)\cdots(-i\lambda-2n-s+2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \left(\bar{Z}^{-i\lambda-2n-s+1} + (-1)^s Z^{-i\lambda-2n-s+1}\right) +$$
$$+ \frac{A}{2(n-1)!a^n}\sum_{s=0}^{\infty} \frac{(-i\lambda-1)\cdots(-i\lambda-2n-s+1)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \left(\bar{Z}^{-i\lambda-2n-s} + (-1)^s Z^{-i\lambda-2n-s}\right) =$$
$$= \frac{1}{2(n-1)!a^n}\sum_{s=0}^{\infty} \frac{(-i\lambda+1)\cdots(-i\lambda-2n-s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \times$$
$$\times \left((-1)^{s+1}(i\lambda B\bar{Z}^{-i\lambda-2n-s+1} + A(i\lambda+2n+s-1)\bar{Z}^{-i\lambda-2n-s}) -\right.$$
$$\left. -\ (i\lambda B Z^{-i\lambda-2n-s+1} + A(i\lambda+2n+s-1)Z^{-i\lambda-2n-s})\right) =$$
$$= \frac{1}{2(n-1)!a^n}\sum_{s=0}^{\infty} \frac{(-i\lambda+1)\cdots(-i\lambda-2n-s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \times$$
$$\times \left((-1)^{s+1}\bar{Z}^{-i\lambda-2n-s}(\lambda B\bar{Z} + A(i\lambda+2n+3-1)) -\right.$$
$$\left. -Z^{-i\lambda-2n-s}(\lambda BZ + A(i\lambda+2n+3-1))\right).$$

Then, using the equality (8), we get

$$\int_0^{\infty} t^{-\imath\lambda} k(x,y;t)\frac{dt}{t} = \pi i\, \frac{\exp\pi\lambda}{\sinh\pi\lambda}\Big(\operatorname{res}(G(t);Z) + \operatorname{res}(G(t);\bar{Z})\Big) =$$
$$= \frac{i\pi\exp\pi\lambda}{2(n-1)!a^n\sinh\pi\lambda}\sum_{s=0}^{\infty} \frac{(-i\lambda+1)\cdots(-i\lambda-2n-s-2)(Z-\bar{Z})^s}{s!(s+n)\cdots(s+2n-1)} \times$$
$$\times \left((-1)^{s+1}\bar{Z}^{-i\lambda-2n-s}(\lambda B\bar{Z} + A(i\lambda+2n+3-1)) -\right.$$
$$\left. -Z^{-i\lambda-2n-s}(\lambda BZ + A(i\lambda+2n+3-1))\right).$$

$\square$

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References

[1] V.A.Kondrat'ev, Boundary value problems for elliptic equations in domains with conical points, *Trudy Mosk. Mat. Obshch.,* **16**(1967), 209-292 (Russian).

[2] A.Kytmanov, The Bochner-Martinelli Integral, and Its Applications, Birkhäuser Verlag, Basel et al., 1995.

[3] N.Tarkhanov, N.Vasilevski, Microlocal Analysis of the Bochner-Martinelli Integral, Preprint 2005/25, Universität Potsdam, 2005, 9 pp.

[4] B.-W.Schulze, Boundary Value Problems and Singular Pseudo-Differential Operators, J. Wiley, Chichester, 1998.

[5] A.M.Kytmanov, S.G.Myslivets, Singular Bochner-Martinalli operator on the hypersurfaces with singularities, *Vestnik of Novosibirsk State Univ.*, (2007), no. 2, 17-32 (Russian).