Journal of Siberian Federal University. Mathematics & Physics 2017, 10(4), 474–493

УДК 517.55

The Neumann Problem after Spencer

Azal Mera*
University of Babilon
Babilon, Iraq
Institute for Mathematics
University of Potsdam
Karl-Liebknecht-Str., 24/25, Potsdam (Golm), 14476
Germany

Nikolai Tarkhanov†
Institute for Mathematics
University of Potsdam
Karl-Liebknecht-Str., 24/25, Potsdam (Golm), 14476
Germany

Received 19.03.2017, received in revised form 20.05.2017, accepted 06.07.2017

When trying to extend the Hodge theory for elliptic complexes on compact closed manifolds to the case of compact manifolds with boundary one is led to a boundary value problem for the Laplacian of the complex which is usually referred to as Neumann problem. We study the Neumann problem for a larger class of sequences of differential operators on a compact manifold with boundary. These are sequences of small curvature, i.e., bearing the property that the composition of any two neighbouring operators has order less than two.

Keywords: elliptic complexes, manifolds with boundary, Hodge theory, Neumann problem.
DOI: 10.17516/1997-1397-2017-10-4-474-493.

Introduction

In the theory of elliptic linear partial differential equations the term coercive is used to describe a certain class of boundary value problems for elliptic systems $Lu = f$, in which, for functions u satisfying the boundary conditions, it is possible to estimate in relevant norm all the derivatives of u of order equal to the order m of L in terms of the norm of Lu and in terms of suitable norms for the given boundary data. That is, there is no loss in derivatives - in going from Lu to u we gain precisely m derivatives. Nowadays such boundary value problems are called simply elliptic, where the ellipticity refers to the invertibility of both interior and boundary symbols, the last condition being also known as the Shapiro-Lopatinskij condition.

In connection with the study of inhomogeneous overdetermined systems of partial differential equations, Spencer [16] proposed a method which leads in some cases to well determined elliptic boundary value problems which are however not coercive. In case the systems consists of the inhomogeneous Cauchy-Riemann equations for differential forms the resulting boundary value problem is called the $\bar{\partial}$-Neumann problem. Extending a basic inequality of [8] this problem was solved in [6] for forms on strongly pseudo-convex domains on a complex manifold. The elliptic operator L in the $\bar{\partial}$-Neumann problem is of second order, and in going from Lu to u, in a pseudo-convex domain, one gains only one derivative instead of two. This makes the problem more difficult than a coercive one, the main difficulty occuring in the proof of regularity at the

*mera@math.uni-potsdam.de; azalmera@gmail.com
†tarkhanov@math.uni-potsdam.de

boundary. The regularity proof in [6] is rather complicated. A simpler proof was found in [8]. In [7] is also presented a simpler proof which yields a raher general theorem for elliptic equations, Theorem 5 of Sec. 2. The result for the $\bar{\partial}$-Neumann problem is a very special case of this theorem.

In [7], the results are expressed in a fairly general form which may eventually prove useful in carrying out Spencer's attack on overdetermined equations. For functions u and v with values in $\mathbb{C}^{k_i}$ or in a smooth vector bundle F^i over a compact manifold with boundary $\mathcal{X}$ one considers a sesquilinear form $Q(u,v)$ which is an integral over $\mathcal{X}$ of an expression involving derivatives of u and v. For functions u, v lying in a linear space $\mathcal{D}$ determined by certain boundary conditions one is looking for a solution $u \in \mathcal{D}$ of $Q(u,v) = (f,v)$ for all $v \in \mathcal{D}$, where f is a given function with values in F^i and $(\cdot,\cdot)$ denotes the L^2 scalar product of sections in $\mathcal{X}$. The form Q is primarily assumed to be almost Hermitean and that $\Re Q(u,u) \geqslant 0$ for $u \in \mathcal{D}$. The paper [7] is aimed at obtaining solutions that are regular in $\mathcal{X}$ up to the boundary. The solutions then lie in $\mathcal{D}$ and satisfy also "free" or "natural" boundary conditions.

It was Sweeney, a PhD student of Spencer, who developed the approach of [7] within the framework of overdetermined systems, see [11], [12–15]. A differential operator A^0 is said to be overdetermined if there is a differential operator $A^1 \neq 0$ with the property that $A^1 A^0 \equiv 0$. Then, for the local solvability of the inhomogeneous equation $A^0 u = f$ it is necessary that the right-hand side satisfies $A^1 f = 0$. The above papers deal with sesquilinear forms $Q(f,g) = (A^i f, A^i g) + (A^{i-1*} f, A^{i-1*} g) + (f,g)$ called the Dirichlet forms. This work is intended as an attempt at motivating an interesting class of perturbations of the Neumann problem after Spencer. It corresponds to "small" perturbations of complexes of differential operators which are are known as quasicomplexes, see [18].

Assume that $\mathcal{X}$ is a compact n-dimensional manifold with boundary. For each nonnegative integer i let F^i be a vector bundle over $\mathcal{X}$, and let A^i be a first order differential operator which maps C^∞ sections of F^i to C^∞ sections of F^{i+1}. Suppose that the compositions $A^i A^{i-1}$ are all of order not exceeding 1 so that the operators A^i form a sequence

$$0 \longrightarrow C^\infty(\mathcal{X}, F^0) \xrightarrow{A^0} C^\infty(\mathcal{X}, F^1) \xrightarrow{A^1} \dots \xrightarrow{A^N} C^\infty(\mathcal{X}, F^N) \longrightarrow 0 \tag{0.1}$$

whose curvature $A^i A^{i-1}$ evaluated in appropriate Sobolev spaces is compact at each step.

The assumption that all of A^i have order 1 simplifies the notation essentially. This will usually not be the case in practice. However, this assumptions is fulfilled for classical complexes of differential operators which arise in differential geometry, see [17, Ch. 1].

As but one example of quasicomplexes of purely geometric origin we mention the sequence related to any connection on a smooth vector bundle over $\mathcal{X}$, see for instance [19, Ch. III].

Example 0.1. Let F be a smooth vector bundle of rank k on $\mathcal{X}$. For $i = 0, 1, \dots, n$, we denote by $\Omega^i(\mathcal{X}, F)$ the space of differential forms of degree i with C^∞ coefficients on $\mathcal{X}$ taking on their values in F. Pick a connection ∂ on F. Consider the sequence

$$0 \to C^\infty(X, F) \xrightarrow{\partial^0} \Omega^1(X, F) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{n-1}} \Omega^n(X, F) \to 0,$$

where $\partial^0 = \partial$, ∂^1 is a natural extension of ∂^0 to one-forms under preservation of the Leibniz rule, etc. Since $\partial^{i+1}\partial^i$ is a differential operator of order 0, the sequence is a quasicomplex. The principal symbols of the (formal) Laplacians Δ^i_∂ are given by

$$\sigma^2(\Delta^i_\partial)(x,\xi) = I_{F_x} \otimes \sigma^2(\Delta^i)(x,\xi),$$

where Δ^i are the Hodge-Laplace operators. We thus conclude that $\sigma^2(\Delta^i_\partial)(x,\xi)$ is invertible for all $(x,\xi) \in T^*\mathcal{X} \setminus \{0\}$. Hence, Δ^i_∂ is a second order elliptic differential operator on $\mathcal{X}$.

Note that the quasicomplex of connections is a complex if and only if the associated bundle is trivial.

Another quasicomplex of great importance in complex analytic geometry is related to certain "small" perturbations of the Dolbeault complex.

Example 0.2. Assume that $\mathcal{X}$ is a complex (analytic) manifold of dimension n. As usual, we denote by $\Omega^{0,i}(\mathcal{X})$ the space of all differential forms of bidegree $(0,i)$ with C^∞ coefficients on $\mathcal{X}$, where $0 \leqslant i \leqslant n$. Locally such a form can be written as

$$f(z) = \sum_{\substack{J=(j_1,\ldots,j_i)\\ 1\leqslant j_1<\ldots<j_i\leqslant n}} f_J(z) d\bar{z}^J,$$

where $z = (z^1, \ldots, z^n)$ are local coordinates, $d\bar{z}^J = d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_i}$ and f_I are C^∞ functions of z with complex values. Analogously to the exterior derivative d one defines the Cauchy-Riemann operator $\bar{\partial}$ which maps the differential forms of bidegree $(0,i)$ to differential forms of bidegree $(0,i+1)$ on $\mathcal{X}$, see for instance [17, 19]. Moreover, $\bar{\partial}^2 = 0$, i.e., the spaces $\Omega^{0,i}(\mathcal{X})$ are gathered together to constitute a complex of first order differential operators on $\mathcal{X}$ called the Dolbeault complex. This complex is proved to be elliptic in (the interior of) $\mathcal{X}$. Choose any differential form a of bidegree $(0,1)$ with smooth coefficients on $\mathcal{X}$ and consider the sequence

$$0 \longrightarrow \Omega^{0,0}(\mathcal{X}) \xrightarrow{\bar{\partial}+a} \Omega^{0,1}(\mathcal{X}) \xrightarrow{\bar{\partial}+a} \ldots \xrightarrow{\bar{\partial}+a} \Omega^{0,n}(\mathcal{X}) \longrightarrow 0 \tag{0.2}$$

which is equipped with differential $\bar{\partial} + a$ given by $(\bar{\partial} + a)u = \bar{\partial}u + a \wedge u$ for $u \in \Omega^{0,q}$. Since

$$\begin{array}{rcl} (\bar{\partial}+a)^2 u &=& (\bar{\partial}+a)(\bar{\partial}u + a\wedge u) \\ &=& \bar{\partial}^2 u + \bar{\partial}a \wedge u - a \wedge \bar{\partial}u + a \wedge \bar{\partial}u + a \wedge a \wedge u \\ &=& \bar{\partial}a \wedge u, \end{array}$$

the curvature of sequence (0.2) is equal to $\bar{\partial}a$. It follows that (0.2) is a quasicomplex. Moreover, it is a complex if the form a is $\bar{\partial}$-closed. The symbol sequence of (0.2) coincides with that of the Dolbeault complex, and so the quasicomplex is elliptic in $\mathcal{X}$.

The purpose of this paper is to show how one obtains existence and regularity theorems for the Neumann problem after Spencer, see (4.1), if an estimate of the form

$$\|f\|^2_{1/2} \leqslant c\left(\|A^i f\|^2 + \|A^{i-1*} f\|^2 + \|f\|^2\right)$$

holds for all smooth f satisfying certain boundary conditions. In the case of zero curvature, i.e., $A^i A^{i-1} \equiv 0$, basic results are contained in [7]. If $A^i A^{i-1} \not\equiv 0$, however, the theorems of [7] do not immediately apply. Our contribution rests on a detailed study of the boundary conditions which settles the matter of "free boundary conditions".

A major part of the paper is concerned with solving equations of the type $Q(u,v) = (f,v)$ for all $v \in \mathcal{D}$. The form $Q(u,v)$ is an integral of a sum of squares. In [7] also more general forms are considered, admitting a mild non-Hermitean part. Since the problem is not assumed to be coercive, one must be rather careful in handling the error terms which usually arise from derivatives of the coefficients, when deriving estimates. On assuming that $Q(u,u) \geqslant \|u\|^2$ for u in a subspace $\mathcal{D}$ (after adding (u,v) to Q), and that $Q(u,u)^{1/2}$ is compact with respect to the L^2 norm, i.e., that any sequence (u_ν) with $Q(u_\nu, u_\nu)$ bounded has a convergent subsequence in L^2, one shows that the equation can be solved, the space of solutions of the homogeneous equation is finite dimensional, and that the solution operator is compact. On assuming a gain of derivatives we present a regularity theorem for solutions.

Similarly to [7], our results are not complete in themselves, but are meant as a technical aid in obtaining more definitive results. For no indication is given when a priori estimates hold. Indeed it seems to be rather difficult to say in general when they can be established.

It is very easy to prove the existence of a Hilbert space solution of the equation $Q(u,v) = (f,v)$ for all $v \in \mathcal{D}$. But we are interested in those solutions which are smooth in $\mathcal{X}$. To do this we derive a priori estimates for the L^2 norms of derivatives of u. Near the boundary we first estimate derivatives in directions tangential to the boundary by essentially setting v equal to tangential derivatives of u. To this end, we assume that the boundary conditions are, in some sense, invariant with respect to translation along the boundary. Then, assuming the boundary to be noncharacteristic, we estimate also the normal derivatives. Then we are faced with the standard problem of going from a priori estimates of derivatives to the proof of their existence.

There is, as yet, no general theorem which states that whenever one has a priori estimates for derivatives of a function then, in fact, these derivatives exist. In each individual case one has to prove this separately, and this is often the most tedious and technical aspect of existence theorems. One way which is often used is to apply a smoothing operator to the solution. In order to apply the a priori estimates to the resulting functions it is necessary to handle the term arising from the commutator of the differential operator and the smoothing operator. This is sometimes rather complicated. This method is used extensively in the book [3], where a number of special lemmas concerned with the commutators of differential and smoothing operators are given.

In [7] another method of smoothing is used. It is more closely related to differential operators, and has proved useful in a wide class of problems. It consists in adding ε times an elliptic operator so that the resulting equation becomes elliptic and coercive under the given boundary conditions for $\varepsilon > 0$, even if the original equation is not elliptic. Thus we rely on the fact that the differentiability theorems are well known for such problems and we wish to reduce the differentiability theorems to those for coercive elliptic problems. The new equation, being coercive elliptic, has a smooth solution u_ε in $\mathcal{X}$ and, if the elliptic term has been added in a suitable way, the method of obtaining a priori estimates applies as well to the new equation as to the original one, and yields estimates for the derivatives of u_ε which are independent of ε. Letting $\varepsilon \to 0$ through a sequence ε_ν, it follows that a subsequence of the u_{ε_ν}, together with derivatives, converges to a smooth solution of the original problem.

This method, therefore, does not show that a generalised solution u is smooth, but constructs a smooth solution. If there is uniqueness among generalised solutions, then one may also infer that u is smooth.

Part 1. The Neumann problem for quasicomplexes

1. Preliminaries

Corresponding to each point $x \in \mathcal{X}$ and cotangent vector $\xi \in T^*_x\mathcal{X}$ there is associated with (0.1) a sequence of linear mappings

$$0 \longrightarrow F^0_x \stackrel{\sigma^1(A^0)(x,\xi)}{\longrightarrow} F^1_x \stackrel{\sigma^1(A^1)(x,\xi)}{\longrightarrow} \ldots \stackrel{\sigma^1(A^N)(x,\xi)}{\longrightarrow} F^N_x \longrightarrow 0, \tag{1.1}$$

where F^i_x is the fibre of the bundle F^i over x and $\sigma^1(A^i)(x,\xi)$ the principal homogeneous symbol of A^i at (x,ξ). Since $A^i A^{i-1} \equiv 0$ it follows that $\sigma^1(A^i)\sigma^1(A^{i-1}) \equiv 0$, i.e., the symbol sequence (1.1) constitutes a complex. A cotangent vector $\xi \in T^*_x\mathcal{X}$ is said to be noncharacteristic for the quasicomplex (0.1) if the symbol complex is exact.

In what follows, functional methods are used to study quasicomplex (0.1), and it will be necessary to have L^2 norms defined for sections of the vector bundles F^i. Accordingly, we shall always consider $\mathcal{X}$ to have a Riemannian structure with volume element dv, and we shall assume that each F^i has a C^∞ Hermitean inner product $(\cdot,\cdot)_x$ defined along its fibres. For arbitrary sections $f, g \in C^\infty(\mathcal{X}, F^i)$, we define

$$(f, g) = \int_{\mathcal{X}} (f(x), g(x))_x dv$$

and $\|f\| = \sqrt{(f, f)}$. Then $L^2(\mathcal{X}, F^i)$ can be defined as the completion of $C^\infty(\mathcal{X}, F^i)$ in the norm $\|\cdot\|$.

In a similar way, we use the induced area element ds on the boundary $\mathcal{S}$ of $\mathcal{X}$ to introduce the space $L^2(\mathcal{S}, F^i)$ with scalar product $(\cdot,\cdot)_{\mathcal{S}}$ and norm $\|\cdot\|_{\mathcal{S}}$.

As usual, we write A^{i-1*} for the formal adjoint of A^{i-1} as determined by the inner products in the spaces $L^2(\mathcal{X}, F^{i-1})$ and $L^2(\mathcal{X}, F^i)$. Thus A^{i-1*} is the unique differential operator from sections of F^i to sections of F^{i-1} of order 1, such that $(A^{i-1}u, g) = (u, A^{i-1*}g)$ whenever $u \in C^\infty(\mathcal{X}, F^{i-1})$ and $g \in C^\infty(\mathcal{X}, F^i)$ have support in the interior of $\mathcal{X}$.

We will also use the Sobolev norms $\|\cdot\|_s$ defined for sections of F^i, where s is a real number. Remark that if $\mathcal{X}$ the closure of an open set in $\mathbb{R}^n$, $F^i = \mathcal{X} \times \mathbb{C}^{k_i}$ and s is a nonnegative integer, then the norm $\|\cdot\|_s$ on $C^\infty(\mathcal{X}, F^i)$ is equivalent to the norm

$$f \mapsto \Big(\sum_{|\alpha| \leqslant s} \|\partial^\alpha f\|^2 \Big)^{1/2},$$

where $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$.

The construction of Sobolev spaces on the compact closed manifold $\mathcal{S}$ is more direct. We write $\|\cdot\|_{\mathcal{S},s}$ for the Sobolev norm on $C^\infty(\mathcal{S}, F^i)$ and $H^s(\mathcal{S}, F^i)$ for the corresponding function space.

2. A boundary decomposition

The operators $\Delta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ are called the Laplacians of (0.1). The unit normal vector $\nu(x)$ of the boundary $\partial\mathcal{X}$ is noncharacteristic for the quasicomplex at step i if and only if $\partial\mathcal{X}$ is noncharacteristic for the Laplacian $\Delta^i \in \mathrm{Diff}^2(\mathcal{X}; F^i)$ at x. Throughout the paper we make the standing assumption that the conormal bundle of the boundary is noncharacteristic for quasicomplex (0.1) at steps $i-1$ and i.

We can assume without loss of generality that $\mathcal{X}$ is embedded into a larger smooth manifold $\mathcal{X}'$ without boundary. Choose a smooth function ϱ in a neighbourhood U of $\partial\mathcal{X}$ in $\mathcal{X}'$ which is negative in $U \cap (\mathcal{X} \setminus \partial\mathcal{X})$, positive in $U \cap (\mathcal{X}' \setminus \mathcal{X})$ and whose differential does not vanish on $\partial\mathcal{X}$. By shrinking U if necessary, we may actually assume that $|d\varrho(x)| = 1$ holds for all $x \in \partial\mathcal{X}$, for if not, we replace ϱ by $\varrho/|d\varrho|$.

Lemma 2.1. *For $x \in \partial\mathcal{X}$, the cotangent vector $d\varrho(x) \in T_x^*\mathcal{X}$ is independent of the particular choice of ϱ.*

Proof. Let ϱ_1 and ϱ_2 be two functions with the properties described above. For each $x \in \partial\mathcal{X}$ there is a neighbourhood U_x of this point in $\mathcal{X}'$, such that $\varrho_2 = f\varrho_1$ in U_x with some smooth function f in U_x. It is clear that f is positive in $U_x \setminus \partial\mathcal{X}$. Furthermore, we get $d\varrho_2 = f d\varrho_1$ on $U_x \cap \partial\mathcal{X}$ whence $f \equiv 1$ on $U_x \cap \partial\mathcal{X}$, as desired. □

Write $\sigma^i(x)$ for the principal homogeneous symbol of A^i evaluated at the point $(x, d\varrho(x))$ of $T^*\mathcal{X}$. This is a smooth section of the bundle $\mathrm{Hom}(F^i, F^{i+1})$ whose restriction to the surface $\partial\mathcal{X}$ does not depend on the particular choice of ϱ, the latter being due to Lemma 2.1. The principal homogeneous symbol of Δ^i evaluated at $(x, d\varrho(x))$ is $\sigma^i(x)^*\sigma^i(x) + \sigma^{i-1}(x)\sigma^{i-1}(x)^*$, which we denote by $\ell^i(x)$ for short. Since the boundary is noncharacteristic for Δ^i, the map $\ell^i(x) \in \mathrm{Hom}(F_x^i)$ is invertible for all x in some neighbourhood of $\partial\mathcal{X}$ in $\mathcal{X}'$, and similarly for the symbol $\ell^{i-1}(x)$.

Theorem 2.2. *The restriction of the bundle F^i to the surface $\partial\mathcal{X}$ splits into the direct sum*

$$F^i \restriction_{\partial\mathcal{X}} = F^i_t \oplus \sigma^{i-1} F^{i-1}_t,$$

where $F^i_t = \sigma^{i*}\sigma^i(\ell^i)^{-1}F^i \restriction_{\partial\mathcal{X}}$ is a smooth subbundle of $F^i \restriction_{\partial\mathcal{X}}$.

Proof. For each $x \in \mathcal{X}$ close to the boundary, any $f \in F^i_x$ can be written in the form

$$f = t(f) + \sigma^{i-1}(x)n(f), \tag{2.1}$$

where

$$\begin{array}{rcl} t(f) & = & \sigma^i(x)^*\sigma^i(x)(\ell^i(x))^{-1}f, \\ n(f) & = & \sigma^{i-1}(x)^*(\ell^i(x))^{-1}f \end{array}$$

prove to satisfy $t \circ t = t$, $t \circ n = n$, $n \circ t = 0$ and $n \circ n = 0$. This establishes the theorem. □

Note that if $F^i = \Lambda^i T^*\mathcal{X}$ is the bundle of exterior forms of degree i over $\mathcal{X}$ then $F^i_t = \iota^* F^i$ is the pullback of F^i under the embedding $\partial\mathcal{X} \hookrightarrow \mathcal{X}$. It follows that $F^i_t = \Lambda^i T^*(\partial\mathcal{X})$.

3. Green formula

To describe natural boundary value problems for solutions of $\Delta^i u = f$ in $\mathcal{X}$, one uses a Green formula related to the Laplacian Δ^i. Such formulas are well understood in general, see for instance Lemma 3.2.10 in [17]. In this section we just compute explicitly the terms included into this formula, to get it in the form we need.

Theorem 3.1 (Green formula)**.** *For all smooth sections u and v of F^i over $\mathcal{X}$ it follows that*

$$\begin{aligned} &\int_{\partial\mathcal{X}} \Big((t(u), \imath\ell n(Av))_x - (\imath\ell n(u), t(A^*v))_x + (t(A^*u), \imath\ell n(v))_x - (\imath\ell n(Au), t(v))_x \Big) ds = \\ &= \int_{\mathcal{X}} \Big((\Delta u, v)_x - (u, \Delta v)_x \Big) dv, \end{aligned}$$

where $\imath = \sqrt{-1}$.

Proof. Let $G_A(*g, u)$ be the Green operator for a differential operator $A = A^i$, see § 2.4.2 of [17]. Here, $* : F^{i+1} \to F^{i+1\prime}$ is the fibrewise Hodge star operator determined by $\langle *g, f\rangle = (f, g)_x$ for all $f \in F^{i+1}_x$. An easy computation shows that the pullbacks of differential forms $G_A(*g, u)$ and $G_{A^*}(*u, g)$ under the inclusion $\partial\mathcal{X} \hookrightarrow \mathcal{X}$ amount to

$$\begin{array}{rcl} \iota^* G_A(*g, u) & = & (t(u), \imath\ell\, n(g))_x ds, \\ \iota^* G_{A^*}(*u, g) & = & -(\imath\ell\, n(g), t(u))_x ds \end{array}$$

on $\partial\mathcal{X}$ for all smooth sections g and u of F^{i+1} and F^i, respectively, cf. § 3.2.2 *ibid.* Applying Corollary 2.5.14 of [17] establishes the formula. □

Theorem 3.1 shows immediately that the quadrupel $t(u)$, $n(u)$, $t(A^*u)$ and $n(Au)$ gives a representation of the Cauchy data of u on the surface $\partial\mathcal{X}$ relative to the Laplacian Δ. The tangential part of the Cauchy data, $(t(u), t(A^*u))$, is usually referred to as the Dirichlet data, and the normal part of the Cauchy data, $(n(u), n(Au))$, is referred to as the Neumann data. This designation is due rather to the whimsical development of mathematics than to well-motivated choice, for, at the last step of the quasicomplex, the data $(t(u), t(A^*u))$ reduce to $t(A^*u)$, which is the classical Neumann data, and $(n(u), n(Au))$ reduce to $n(u)$, which is the classical Dirichlet data.

4. The Neumann problem

In his paper [16], Spencer proposed a method of studying the cohomology of an elliptic complex similar to (0.1) at step i. The main step involves the boundary value problem

$$\begin{array}{rclcl} \Delta^i u & = & f & \text{in} & \mathcal{X}, \\ n(u) & = & 0 & \text{on} & \partial\mathcal{X}, \\ n(Au) & = & 0 & \text{on} & \partial\mathcal{X}, \end{array} \tag{4.1}$$

where f is a given section of F^i over $\mathcal{X}$.

Example 4.1. In the special case of the de Rham complex and $i = 0$ problem (4.1) reduces to the classical Neumann problem. For $n(du)$ amounts to the normal derivative of u at $\partial\mathcal{X}$.

Even in the classical case, (4.1) is not solvable unless f satisfies additional conditions. Since (4.1) is a boundary value problem symmetric with respect to the Green formula, it is solvable only if f is orthogonal in the L^2 sense to the space $\mathcal{H}^i(\mathcal{X})$ of all $h \in C^\infty(\mathcal{X}, F^i)$ satisfying the corresponding homogeneous problem, i.e., $\Delta^i h = 0$ in $\mathcal{X}$ and $n(h) = 0$, $n(Ah) = 0$ on $\partial\mathcal{X}$. The sections of $\mathcal{H}^i(\mathcal{X})$ are called harmonic.

Lemma 4.2. *A section $h \in C^\infty(\mathcal{X}, F^i)$ is harmonic if and only if $Ah = 0$, $A^*h = 0$ in $\mathcal{X}$ and $n(h) = 0$ on $\partial\mathcal{X}$.*

Proof. The point here is that the boundary conditions of (4.1) allow us to integrate by parts without introducing integrals on the boundary. The sufficiency is obvious. To show the necessity, pick a section $h \in \mathcal{H}^i(\mathcal{X})$. On integrating by parts we readily obtain

$$0 = (\Delta^i h, h) = \|A^i h\|^2 + \|A^{i-1*} h\|^2,$$

and the lemma follows. □

The main step in the approach of [16] is to establish that $\mathcal{H}^i(\mathcal{X})$ is finite-dimensional and if $f \in C^\infty(\mathcal{X}, F^i)$ is orthogonal to $\mathcal{H}^i(\mathcal{X})$ then (4.1) can be solved for $u \in C^\infty(\mathcal{X}, F^i)$. Suppose that these solvability properties for problem (4.1) have been established. We introduce the subspace $\mathcal{N}^i(\mathcal{X})$ of $C^\infty(\mathcal{X}, F^i)$ consisting of those sections u which satisfy the boundary conditions in (4.1), i.e., $n(u) = 0$ and $n(Au) = 0$ on $\partial\mathcal{X}$. Given any $f \in C^\infty(\mathcal{X}, F^i)$, we denote by $H^i f$ the orthogonal projection of f into $\mathcal{H}^i(\mathcal{X})$. The difference $f - H^i f$ still belongs to $C^\infty(\mathcal{X}, F^i)$ and is orthogonal to $\mathcal{H}^i(\mathcal{X})$, hence there is a section $u \in \mathcal{N}^i(\mathcal{X})$ such that $\Delta^i u = f - H^i f$ in $\mathcal{X}$. Set $N^i f := u - H^i u$, thus obtaining a linear operator from $C^\infty(\mathcal{X}, F^i)$ to $\mathcal{N}^i(\mathcal{X})$. This operator is well defined, for from $u_1, u_2 \in \mathcal{N}^i(\mathcal{X})$ and $\Delta^i u_1 = \Delta^i u_2$ it follows that $u_1 - H^i u_1 = u_2 - H^i u_2$. We see that any section $f \in C^\infty(\mathcal{X}, F^i)$ can be written as

$$f = H^i f + A^{i*} A^i \, N^i f + A^{i-1} A^{i-1*} \, N^i f \tag{4.2}$$

in $\mathcal{X}$.

If the curvature of quasicomplex (0.1) vanishes at step i, i.e., $A^i A^{i-1} \equiv 0$, then the terms on the right-hand side of (4.2) are mutually orthogonal, as is easy to check. In this case formula (4.2) furnishes an isomorphism between the cohomology of (0.1) at step i and the space $\mathcal{H}^i(\mathcal{X})$ of harmonic sections, see [17, 4.1] for more details.

Part 2. Subelliptic estimates

5. The main theorem

Quasicomplex (0.1) is said to be elliptic at step i if the symbol complex (1.1) is exact at step i for each $x \in \mathcal{X}$ and for each cotangent vector $\xi \in T^*_x\mathcal{X}$ different from zero. This is equivalent to the fact that the Laplacian Δ^i is a second order elliptic operator on $\mathcal{X}$.

Theorem 5.1. *Suppose (0.1) is elliptic at steps $i-1$ and i and there is a constant c such that*

$$\|u\|^2_{1/2} \leqslant c \left(\|A^i u\|^2 + \|A^{i-1*}u\|^2 + \|u\|^2\right) \tag{5.1}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(u) = 0$. Then $\mathcal{H}^i(\mathcal{X})$ is finite-dimensional, and if $f \in C^\infty(\mathcal{X}, F^i)$ is orthogonal to $\mathcal{H}^i(\mathcal{X})$ then there exists $u \in C^\infty(\mathcal{X}, F^i)$ satisfying (4.1).

As is mentioned in the introductory remarks, this theorem is contained in [7] if the curvature of (0.1) vanishes at step i.

The first step in proving the theorem is to extend the Laplacian Δ^i to a closed operator L^i on the Hilbert space $L^2(\mathcal{X}, F^i)$. To this end we apply a classical method of (Kurt) Friedrichs, cf. [2]. In functional analysis, by the Friedrichs extension is meant a canonical self-adjoint extension of a nonnegative densely defined symmetric operator. This extension is particularly useful in situations where an operator may fail to be essentially self-adjoint or whose essential self-adjointness is difficult to show. The definition of the Friedrichs extension is based on the theory of closed positive forms on Hilbert spaces. If T is a nonnegative operator in a Hilbert space H, then $Q(u,v) = (u,Tv) + (u,v)$ is a sesquilinear form on $\operatorname{Dom} T$ and $Q(u,u) \geqslant \|u\|^2$. Thus Q defines an inner product on $\operatorname{Dom} T$. Let H_1 be the completion of $\operatorname{Dom} T$ with respect to Q. This is an abstractly defined space. For instance its elements can be represented as equivalence classes of Cauchy sequences of elements of $\operatorname{Dom} T$. It is not obvious that all elements in H_1 can be identified with elements of H. However, the canonical inclusion $\operatorname{Dom} T \hookrightarrow H$ extends to an injective continuous map $H_1 \hookrightarrow H$. We regard H_1 as a subspace of H. Define an operator T_1 in H whose domain consists of all $u \in H_1$ such that $v \mapsto Q(u,v)$ is a bounded conjugate-linear functional on H_1. Here, bounded is relative to the topology of H_1 inherited from H. Pick $u \in \operatorname{Dom} T_1$. By the Riesz representation theorem applied to the linear functional $v \mapsto \overline{Q(u,v)}$ extended to all of H, there is a unique $f \in H$ such that $Q(u,v) = (f,v)$ for all $v \in H_1$. Set $T_1 u := f$. Then T_1 is a nonnegative self-adjoint operator in H, such that $T_1 - I$ extends T. The operator $T_1 - I$ is called the Friedrichs extension of T.

The operator Δ^i in $L^2(\mathcal{X}, F^i)$ with domain $\mathcal{N}^i(\mathcal{X})$ is nonnegative, densely defined and symmetric. The sesquilinear form $Q(u,v) = (u, \Delta^i v) + (u,v)$ on $\mathcal{N}^i(\mathcal{X})$ reduces readily to

$$D(u,v) := (A^i u, A^i v) + (A^{i-1*}u, A^{i-1*}v) + (u,v),$$

which is known as the Dirichlet scalar product on $C^\infty(\mathcal{X}, F^i)$. When completing $\mathcal{N}^i(\mathcal{X})$ in the norm $D(u) := \sqrt{D(u,u)}$, one can scarcely retain the boundary condition $n(Au) = 0$ at $\partial\mathcal{X}$. Hence, one disregards this condition from the very beginning and considers the Dirichlet inner product on the subspace of $C^\infty(\mathcal{X}, F^i)$ which consists of all u satisfying $n(u) = 0$ on $\partial\mathcal{X}$. We write $\mathcal{D}^i$ for its completion to a Hilbert space. It is not difficult to see that $\mathcal{D}^i$ can be thought of as a subspace of $L^2(\mathcal{X}, F^i)$. We now define $L^i + I$ to be the operator whose domain consists of all $u \in \mathcal{D}^i$ such that $v \mapsto D(v,u)$ extends to a bounded linear functional on $L^2(\mathcal{X}, F^i)$ and whose rule of correspondence is given by $D(u,v) = ((L^i + I)u, v)$, for all sections $v \in \mathcal{D}^i$. Then $L^i + I$ is a self-adjoint operator on $L^2(\mathcal{X}, F^i)$, and $(L^i + I)u = (\Delta^i + I)u$ if $u \in \mathcal{N}^i(\mathcal{X})$. Also, $L^i + I$ is surjective, and $(L^i + I)^{-1}$ is bounded as an operator from $L^2(\mathcal{X}, F^i)$ to $\mathcal{D}^i$. It follows by (5.1) and Rellich's theorem that $(L^i + I)^{-1}$ is a compact operator from $L^2(\mathcal{X}, F^i)$ to itself, and hence $L^i = (L^i + I) - I$ must have closed range and finite-dimensional null space. Since L^i is self-adjoint, its null space is the orthogonal complement of the range of L^i. Hence, any $f \in L^2(\mathcal{X}, F^i)$ can be written in the form $f = h + L^i u$, where h belongs to the null space of L^i and u is in the domain of L^i. The proof of Theorem 5.1 will now be complete when we establish two facts. The first of the two is that if u lies in the domain of L^i and if $L^i u$ is C^∞, then u is C^∞. The second fact is that every smooth section u in the domain of L^i must satisfy the boundary conditions $n(u) = 0$ and $n(Au) = 0$ on $\partial\mathcal{X}$. If f is C^∞, then the first statement will imply that the sections h and u in $f = h + L^i u$ are C^∞. Its proof will occupy the next three sections. The second statement will then imply that h is in $\mathcal{H}^i(\mathcal{X})$, that u is in $\mathcal{N}^i(\mathcal{X})$, and that $L^i u = \Delta^i u$. We turn to the proof of the second statement right now.

Lemma 5.2. *Every C^∞ section u in $\mathcal{D}^i$ satisfies the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$.*

Proof. Since $u \in \mathcal{D}^i$, there exists a sequence $\{u_j\}$ in $C^\infty(\mathcal{X}, F^i)$ such that $n(u_j) = 0$ on $\partial\mathcal{X}$ and $D(u - u_j) \to 0$ as $j \to \infty$. Since $n(u_j) = 0$ on $\partial\mathcal{X}$, integration by parts yields the equality $(A^* u_j, \varphi) = (u_j, A\varphi)$ for every $\varphi \in C^\infty(\mathcal{X}, F^{i-1})$. Since $D(u - u_j) \to 0$, we may pass to the limit in the equality to obtain $(A^* u, \varphi) = (u, A\varphi)$ for every φ. In view of the integration-by-parts formula (see the proof of Theorem 3.1), this means that

$$\int_{\partial\mathcal{X}} (\imath\ell n(u), t(\varphi))_x ds = 0$$

for all $\varphi \in C^\infty(\mathcal{X}, F^{i-1})$. Hence the lemma holds. □

Lemma 5.3. *Suppose the boundary is noncharacteristic for quasicomplex (0.1) at step $i - 1$. Then every $u \in C^\infty(\mathcal{X}, F^i)$ which belongs to the domain of L^i satisfies $n(Au) = 0$ on $\partial\mathcal{X}$.*

Proof. If $u \in C^\infty(\mathcal{X}, F^i)$ belongs to the domain of L^i, then for every C^∞ section v in $\mathcal{D}^i$ we get

$$\begin{array}{rcl} 0 & = & D(u, v) - ((L^i + I)u, v) \\ & = & ((A^i u, A^i v) - (A^{i*} A^i u, v)) + ((A^{i-1*} u, A^{i-1*} v) - (A^{i-1} A^{i-1*} u, v)) \\ & = & \displaystyle\int_{\partial\mathcal{X}} (\imath\ell n(Au), t(v))_x \, ds - \int_{\partial\mathcal{X}} (t(A^* u), \imath\ell n(v))_x \, ds, \end{array}$$

the last equality being due to the integration-by-parts-formula. Since $n(v) = 0$ on the surface $\partial\mathcal{X}$, the second term on the right-hand side vanishes, which gives readily

$$\int_{\partial\mathcal{X}} (\imath\ell n(Au), t(v))_x \, ds = 0$$

for all $v \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(v) = 0$ on $\partial\mathcal{X}$. On applying Theorem 2.2 we conclude that $n(Au) = 0$ on $\partial\mathcal{X}$, as desired. □

6. A priori estimates

To complete the proof of Theorem 5.1 we must prove that u is C^∞, whenever $L^i u$ is. In this section we derive certain a priori estimates which help establish this result. In what follows, c will denote a generic constant.

We shall need the norms $\|f\|_{(r,s)}$ when f is a C^∞ function with compact support in the closed half-space $\mathbb{R}^n_{\geqslant 0}$ consisting of all $x \in \mathbb{R}^n$ with $x^n \geqslant 0$. For the definition of these norms in terms of Fourier transform we refer to Section 2.5 of [3]. We only remark that if r and s are nonnegative integers, then $\|\cdot\|_{(r,s)}$ is equivalent to the norm

$$f \mapsto \Big(\sum_{\substack{|\alpha| \leqslant r+s \\ \alpha_n \leqslant r}} \int_{\mathbb{R}^n_{\geqslant 0}} |\partial^\alpha f(x)|^2 dv \Big)^{1/2}.$$

So $\|f\|_{(r,s)}$ controls the L^2 norms of those partial derivatives of f which are of total order $\leqslant r+s$ and are of order $\leqslant r$ in the normal derivative $\partial/\partial x^n$. We list the main properties of the norms $\|\cdot\|_{(r,s)}$ in

Lemma 6.1. *As defined above, the scale $\|\cdot\|_{(r,s)}$ bears the following properties:*
1) $\|f\|_{(r,0)} = \|f\|_r$, the Sobolev r-norm on $\mathbb{R}^n_{\geqslant 0}$;
2) $\|f\|_{(r,s)} \leqslant \|f\|_{(r',s')}$ if $r \leqslant r'$ and $r+s \leqslant r'+s'$;
3) $\|Pf\|_{(r,s)} \leqslant c\,\|f\|_{(r+m,s)}$ holds with some constant c independent of f, if P is a differential operator of order m;
4) $\|f\|_{(r,s)} \leqslant c\,(\|Pf\|_{(r-m,s)} + \|f\|_{(r',s')})$ holds with a constant c independent of f, if P is an elliptic differential operator of order m and $r+s = r'+s'$;
5) $\|f\|_{S,s} \leqslant \|f\|_{(1,s-1)}$, where $\|\cdot\|_{S,s}$ is the Sobolev s-norm on $\{x^n = 0\}$;
6) $2\,\Re\,(f,g) \leqslant \|f\|_{((0,s)}\|g\|_{((0,-s)}$ for any s.

Proof. Assertion 4) is Lemma 2.1.1 in [5]. The rest of the lemma is contained in Sec. 2.5 of [3]. □

Let U be a coordinate neighbourhood in $\mathcal{X}$ such that the bundles F^{i-1}, F^i, and F^{i+1} are trivial over U. Assume that the coordinate $x = (x^1, \ldots, x^n)$ on U maps U into the closed half-space $\mathbb{R}^n_{\geqslant 0}$. Then any C^∞ function with support in U can be considered as a function on $\mathbb{R}^n_{\geqslant 0}$, and hence the norms $\|f\|_{(r,s)}$ are defined for $f \in C^\infty_{\mathrm{comp}}(U)$. Now fix a frame in $F^i|_U$, that is, choose sections $e_1, \ldots, e_{k^i}$ in $C^\infty(U, F^i)$ with the property that for each $x \in U$ the elements $e_1(x), \ldots, e_{k^i}(x)$ form a basis for the fibre over x. Then each $u \in C^\infty_{\mathrm{comp}}(U, F^i)$ has component functions defined by

$$u = u^1 e_1 + \ldots + u^{k^i} e_{k^i},$$

and we may define

$$\|u\|_{(r,s)} = \Big(\sum_{j=1}^{k^i} \|u^j\|^2_{(r,s)}\Big)^{1/2}.$$

It is easy to check that the assertions in Lemma 6.1 continue to hold for these norms.

Let $D' = (D_1, \ldots, D_{n-1})$, where $D_j = \dfrac{1}{\sqrt{-1}}\dfrac{\partial}{\partial x^j}$. Consider the pseudodifferential operator

$$\Lambda^s = \chi(D')\,\big(1 + |D'|^2\big)^{s/2}$$

on $\mathbb{R}^{n-1}$, where $\chi \in C^\infty(\mathbb{R}^{n-1})$ is 0 on a neighbourhood of the origin and 1 outside a slightly larger set. On letting Λ^s act along the first $n-1$ coordinate directions we define $\Lambda^s f$ when f is a C^∞ function on $\mathcal{X}$ with compact support in U. And with a fixed choice of frame in F^i over U we can define $\Lambda^s u$ for $u \in C^\infty_{\mathrm{comp}}(U, F^i)$ by letting Λ^s act on the component functions of u as determined by the frame. If $\varphi \in C^\infty_{\mathrm{comp}}(U)$ and

$$T^s = \varphi \Lambda^s \varphi. \tag{6.1}$$

then T^s is an operator which acts on $C^\infty(\mathcal{X})$ and also, with a choice of local frame, an arbitrary smooth sections of F^{i-1}, F^i, or F^{i+1}.

If an appropriate frame is used to define T^s on sections of F^i, then T^s becomes a formally self-adjoint operator. In fact, let $e'_1, \ldots, e'_{k^i} \in C^\infty(\mathcal{X}, F^i)$ be such that for each $x \in U$ the elements $e'_1(x), \ldots, e'_{k^i}(x)$ form an orthonormal basis for the fibre, and let the volume element be given by $dv = v(x)dx$ in the coordinate x on U. Then define $e_j = e'_j/\sqrt{v}$, for $j = 1, \ldots, k^i$, so that if $u = u^j e_j$ and $v = v^j e_j$ have support in U, then

$$(u, v) = \int_U \sum_{j=1}^{k^i} u^i(x)\overline{v^i(x)}dx.$$

If we define $T^s u = (T^s u^j) e_j$ for $u = u^j e_j \in C^\infty(U, F^i)$, then $(T^s u, v) = (u, T^s v)$ for all C^∞ sections u and v. When letting T^s operate on sections of a bundle, we shall assume that the frame being used makes T^s self-adjoint.

Lemma 6.2. *Suppose φ, ψ, ω are C^∞ functions with compact support in U and $\varphi = 1$ on the support of ω, $\psi = 1$ on the support of φ. Let T^s be the operator defined by (6.1). Then,*

1) for each r, t there is a constant c such that $\|T^s f\|_{(r,t)} \leqslant c\, \|\psi f\|_{(r,t+s)}$;

2) if moreover P is a differential operator of order m, then for each r, t there exists a constant c such that

$$\begin{array}{rcl} \|[P, T^s] f\|_{(r,t)} & \leqslant & c\, \|\psi f\|_{(r+m,t+s-1)}, \\ \|[[P, T^s], T^s] f\|_{(r,t)} & \leqslant & c\, \|\psi f\|_{(r+m,t+2s-2)}; \end{array}$$

3) for each t there is a constant c such that

$$\|\omega f\|_{(0,t+s)} \leqslant c \left(\|T^s f\|_{(0,t)} + \|f\|_{t+s-1} \right).$$

As usual, the bracket $[P, Q]$ of two operators denotes their commutator $PQ - QP$.

Proof. Assertions 1) and 2) are well-known properties of classical pseudodifferential operators. 3) holds because T^s is tangentially elliptic on the support of ω, see Theorem 4.7 in [4]. □

Lemma 6.3. *Assume that quasicomplex (0.1) is elliptic at F^i and let $u \in C^\infty(\mathcal{X}, F^i)$ satisfy $n(u) = 0$ on $\partial \mathcal{X}$. Then there exist $v, u', u'' \in C^\infty(U, F^i)$ with support in* $\operatorname{supp} \varphi$ *such that*

1) $T^s T^s u = v + T^s u' + u''$;

2) $n(v) = 0$ on $\partial \mathcal{X}$;

3) for each t there is a constant c such that

$$\begin{array}{rcl} \|u'\|_{(1,t)} & \leqslant & c\, \|\psi u\|_{(1,t+s-1)}, \\ \|u''\|_{(1,t)} & \leqslant & c\, \|\psi u\|_{(1,t+2s-2)}. \end{array}$$

Proof. We follow the proof of Lemma 4 in [12]. Theorem 2.2 shows immediately that the homotopy formula $\sigma\, n(u) + n(\sigma u) = u$ holds for all $u \in C^\infty(\partial \mathcal{X}, F^i)$, where $n^2 = 0$. Hence, the results of [12] apply with $\mathcal{A} = n$, $\mathcal{B} = n$ and $\mathcal{R} = \sigma(x)$. Consider

$$\begin{array}{rcl} w & = & \sigma(x)\, n(T^s T^s u) \\ & = & \sigma(x)\, T^s [n, T^s] u + \sigma(x)\, [n, T^s] T^s u \\ & = & T^s w' + w'', \end{array}$$

where $w' = 2\sigma(x)\, [n, T^s] u$ and

$$\begin{array}{rcl} w'' & = & [\sigma(x)[n, T^s], T^s] u + [\sigma(x), T^s]\, [n, T^s] u \\ & = & \sigma(x)\, [[n, T^s], T^s] u + 2\, [\sigma(x), T^s]\, [n, T^s] u. \end{array}$$

Using Lemmata 6.1, 6.2 and inequality $\|\sigma(x) u\|_{\mathcal{S},s} \leqslant c\, \|u\|_{\mathcal{S},s}$ with c a constant independent of u, we infer

$$\begin{array}{rcl} \|w'\|_{\mathcal{S},t+1/2} & \leqslant & c\, \|[n, T^s] u\|_{\mathcal{S},t+1/2} \\ & \leqslant & c\, \|\psi u\|_{\mathcal{S},t+s-1/2} \\ & \leqslant & c\, \|\psi u\|_{(1,t+s-1)} \end{array}$$

and

$$\begin{array}{rcl} \|w''\|_{\mathcal{S},t+1/2} & \leqslant & c\, (\|[[n, T^s], T^s] u\|_{\mathcal{S},t+1/2} + \|[n, T^s] u\|_{\mathcal{S},t+s-1/2}) \\ & \leqslant & c\, (\|\psi u\|_{\mathcal{S},t+2s-3/2} + \|\psi u\|_{\mathcal{S},t+2s-3/2}) \\ & \leqslant & c\, \|\psi u\|_{(1,t+2s-2)}. \end{array}$$

By Theorem 2.5.7 in [3] we can choose $u', u'' \in C^\infty_{\mathrm{comp}}(U, F^i)$ such that

$$\begin{array}{rcl} u' & = & w', \\ u'' & = & w'' \end{array}$$

on the boundary of $\mathcal{X}$ and

$$\begin{array}{rcl} \|u'\|_{(1,t)} & \leqslant & c\,\|w'\|_{\mathcal{S},t+1/2}, \\ \|u''\|_{(1,t)} & \leqslant & c\,\|w''\|_{\mathcal{S},t+1/2}. \end{array}$$

In view of the estimates for w' and w'' which have already been obtained we get

$$\begin{array}{rcl} \|u'\|_{(1,t)} & \leqslant & c\,\|\psi u\|_{(1,t+s-1)}, \\ \|u''\|_{(1,t)} & \leqslant & c\,\|\psi u\|_{(1,t+2s-2)}, \end{array}$$

as required. Since

$$\begin{array}{rcl} T^s u' + u'' & = & T^s w' + w'' \\ & = & \sigma(x)\, n(T^s T^s u) \end{array}$$

on $\partial\mathcal{X}$, we can define $v = T^s T^s u - T^s u' - u''$, and the proof is complete. □

In [7] the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$ is assumed to be invariant with respect to action in the directions parallel to the boundary. This means, in particular, that if $n(u) = 0$ on $\partial\mathcal{X}$ then also $n(T^s u) = 0$, in which case Lemma 6.3 is trivial. How can the condition $\sigma(x)^* u = 0$ imply $\sigma(x)^* T^s = 0$ on the boundary? This can be achieved only in the case if $n(u) = 0$ just amounts to saying that several components of the section u of F^i vanish on $\partial\mathcal{X}$. Since quasicomplex (0.1) is elliptic at the step i, this can certainly be achieved by choosing special local frames for the bundle F^i. The decomposition of Theorem 2.2 actually gives such a vector bundle F^i_t which is a direct summand of F^i. Technically this means that all norms under consideration are independent up to equivalent norms of the particular choices of local frames, which is an ungrateful exercise in functional analysis of sections of smooth vector bundles over $\partial\mathcal{X}$.

Lemma 6.4. *For all $u \in C^\infty(\mathcal{X}, F^i)$,*

$$D(u, T^s T^s u) = D(T^s u, T^s u) + O\left(\|\psi u\|_{(1,s-1)}\right).$$

Proof. Since T^s is formally self-adjoint, the lemma reduces to Lemma 3.1 in [7]. The proof is essentially algebraic, using only self-adjointness and those properties of T^s which are mentioned in Lemma 6.2. □

Lemma 6.5. *Assume that quasicomplex (0.1) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|^2_{1/2} \leqslant c\, D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$. Then for each $s \geqslant 1/2$ there is a constant c with the property that*

$$\|T^s u\|^2_{1/2} \leqslant c\, D(T^s u, T^s u) \leqslant c\, (\|(\Delta + I)u\|^2_{s-1/2} + \|u\|^2_s) \tag{6.2}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i.

Proof. Since u is in the domain of L^i, we have $D(u,v) = ((\Delta + I)u, v)$ for all $v \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(v) = 0$ on $\partial\mathcal{X}$. Hence, this equality holds in particular for the section $v = T^s T^s u - T^s u' - u''$ described in Lemma 6.3. Thus,

$$D(u, T^s T^s u) = D(u, T^s u') + D(u, u'') + ((\Delta + I)u, T^s T^s u) + ((\Delta + I)u, T^s u' + u''). \tag{6.3}$$

We shall treat the terms on the right of (6.3) one by one.

To treat the first term we first claim that

$$D(u, T^s u') = D(T^s u, u') + O(\|\psi u\|^2_{(1,s-1)}). \tag{6.4}$$

In fact, to prove this we must majorise two terms like

$$(Au, AT^s u') - (AT^s u, Au') = (Au, [A, T^s]u') + ([T^s, A]u, Au'),$$

and by the preceding lemmata this expression is bounded by

$$\begin{array}{rcl} \|A(\psi u)\|_{(0,s-1)} \|[A, T^s]u'\|_{(0,-s+1)} + \|[T^s, A]u\| \, \|Au'\| & \leqslant & c\, \|\psi u\|_{(1,s-1)} \|u'\|_{(1,0)} \\ & \leqslant & c\, \|\psi u\|^2_{(1,s-1)}. \end{array}$$

Therefore, (6.4) holds, and since

$$\begin{array}{rcl} |D(T^s u, u')| & \leqslant & \sqrt{D(T^s u, T^s u)} \sqrt{D(u', u')} \\ & \leqslant & \dfrac{1}{4} D(T^s u, T^s u) + c\, \|u'\|^2_1 \\ & \leqslant & \dfrac{1}{4} D(T^s u, T^s u) + c\, \|\psi u\|^2_{(1,s-1)}, \end{array}$$

we get

$$|D(u, T^s u')| \leqslant \frac{1}{4} D(T^s u, T^s u) + c\, \|\psi u\|^2_{(1,s-1)}.$$

As for the second term in (6.3) we claim that $|D(u, u'')| \leqslant c\, \|\psi u\|^2_{(1,s-1)}$. In fact, for a typical term, we have

$$\begin{array}{rcl} |(Au, Au'')| & \leqslant & c\, \|A(\psi u)\|_{(0,s-1)} \|u''\|_{(1,-s+1)} \\ & \leqslant & c\, \|\psi u\|^2_{(1,s-1)}, \end{array}$$

and hence the above estimate holds.

The third term in (6.3) is majorised as

$$\begin{array}{rcl} |((\varDelta + I)u, T^s T^s u)| & = & |(T^s(\varDelta + I)u, T^s u)| \\ & \leqslant & \|T^s(\varDelta + I)u\|_{(0,-1/2)} \|T^s u\|_{(0,1/2)} \\ & \leqslant & c\, \|(\varDelta + I)u\|_{s-1/2} \|T^s u\|_{1/2} \\ & \leqslant & c\, (\varepsilon^2 \|T^s u\|^2_{1/2} + \varepsilon^{-2} \|(\varDelta + I)u\|^2_{s-1/2}) \\ & \leqslant & c\, \varepsilon^2 D(T^s u, T^s u) + c\varepsilon^{-2} \|(\varDelta + I)u\|^2_{s-1/2}, \end{array}$$

where $\varepsilon > 0$ is taken so small that $c\varepsilon^2 < \dfrac{1}{4}$.

The remaining term in (6.3) can now be estimated by

$$\begin{array}{rcl} |((\varDelta + I)u, T^s u' + u'')| & \leqslant & \|\psi(\varDelta + I)u\|_{(0,s-1/2)} \|T^s u' + u''\|_{(0,-s+1/2)} \\ & \leqslant & c\, (\|(\varDelta + I)u\|^2_{s-1/2} + \|\psi u\|^2_{(1,s-1)}), \end{array}$$

and thus we have proved that

$$D(u, T^s T^s u) \leqslant \frac{1}{2}\, D(T^s u, T^s u) + c\, (\|(\varDelta + I)u\|^2_{s-1/2} + \|\psi u\|^2_{(1,s-1)}).$$

On using Lemma 6.4 and substracting the term $\dfrac{1}{2}\,D(T^s u, T^s u)$ from both sides we get

$$\frac{1}{2}\,D(T^s u, T^s u) \leqslant c\,(\|(\varDelta + I)u\|^2_{s-1/2} + \|\psi u\|^2_{(1,s-1)}).$$

To complete the proof it suffices to show that $\|\psi u\|^2_{(1,s-1)}$ is majorised by the right-hand side of (7.1). But since quasicomplex (0.1) is elliptic at step i, the operator $\varDelta^i + I$ is elliptic, and so, by part 4) of Lemma 6.1,

$$\begin{array}{rcl}\|\psi u\|^2_{(1,s-1)} & \leqslant & c\,(\|(\varDelta + I)\psi u\|^2_{(-1,s-1)} + \|u\|^2_s)\\ & \leqslant & c\,(\|(\varDelta + I)\psi u\|^2_{s-3/2} + \|u\|^2_s)\\ & \leqslant & c\,(\|(\varDelta + I)u\|^2_{s-3/2} + \|[\varDelta,\psi]u\|^2_{s-3/2} + \|u\|^2_s)\\ & \leqslant & c\,(\|(\varDelta + I)u\|^2_{s-3/2} + \|u\|^2_s),\end{array}$$

as desired. □

Recall that by ω we mean a C^∞ function with compact support in U, such that $\varphi = 1$ on the support of ω.

Lemma 6.6. *Suppose the quasicomplex (0.1) is elliptic at step i. Then for each $s \geqslant 1/2$ there is a constant c such that*

$$\|\omega u\|_{s+1/2} \leqslant c\,(\|T^s u\|_{1/2} + \|(\varDelta + I)u\|_{s-3/2} + \|u\|_{s-1/2})$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$.

Proof. Since quasicomplex (0.1) is elliptic at step i, the operator $\varDelta^i + I$ is elliptic, and part 4) of Lemma 6.1 yields

$$\begin{array}{rcl}\|\omega u\|_{s+1/2} & \leqslant & c\,(\|(\varDelta + I)\omega u\|_{s-3/2} + \|\omega u\|_{(0,s+1/2)})\\ & \leqslant & c\,(\|(\varDelta + I)u\|_{s-3/2} + \|[\varDelta,\omega]u\|_{s-3/2} + \|\omega u\|_{(0,s+1/2)})\\ & \leqslant & c\,(\|(\varDelta + I)u\|_{s-3/2} + \|u\|_{s-1/2} + \|\omega u\|_{(0,s+1/2)}).\end{array}$$

The desired estimate now follows from part 3) of Lemma 6.2. □

Theorem 6.7. *Assume that quasicomplex (0.1) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|^2_{1/2} \leqslant c\,D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$. Then for each $s \geqslant 1/2$ there is a constant c such that the estimate*

$$\|u\|_{s+1/2} \leqslant c\,\|(\varDelta + I)u\|_{s-1/2} \tag{6.5}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i.

Proof. Choose a finite covering $\{U_\nu\}$ of $\mathcal{X}$ by coordinate neighbourhoods of the form used above. For each ν, let ω_ν, φ_ν, ψ_ν and T^s_ν be as described in Lemma 6.2. We can assume that $\{\omega_\nu\}$ forms a partition of unity on $\mathcal{X}$. Then, by Lemmata 6.5 and 6.6, we get

$$\begin{array}{rcl}\|u\|_{s+1/2} & \leqslant & c\,\Big(\displaystyle\sum_\nu \|K^s_\nu u\|_{1/2} + \|(\varDelta + I)u\|_{s-3/2} + \|u\|_{s-1/2}\Big)\\ & \leqslant & c\,(\|(\varDelta + I)u\|_{s-1/2} + \|u\|_s)\end{array}$$

for all smooth u in the domain of L^i. Using the interpolation inequality

$$\|u\|_s \leqslant \varepsilon \|u\|_{s+1/2} + C(\varepsilon)\,\|u\|$$

with $\varepsilon > 0$ sufficiently small, we obtain

$$\|u\|_{s+1/2} \leqslant c\,(\|(\Delta + I)u\|_{s-1/2} + C(\varepsilon)\,\|u\|) + \frac{1}{2}\,\|u\|_{s+1/2}$$

whence

$$\|u\|_{s+1/2} \leqslant c\,(\|(\Delta + I)u\|_{s-1/2} + \|u\|). \tag{6.6}$$

Since

$$\|u\|^2 \leqslant D(u,u) = ((\Delta + I)u, u) \leqslant \|(\Delta + I)u\|\,\|u\|$$

for all u in the domain of L^i, we obtain

$$\|u\| \leqslant \|(\Delta + I)u\| \leqslant \|(\Delta + I)u\|_{s-1/2}.$$

Estimate (7.1) now follows from (6.6) and the last inequality, as desired. □

7. Elliptic regularisation

Following [7], we use the techniques of elliptic regularisation in this section to prove that u is C^∞ whenever $L^i u$ is C^∞. This will complete the proof of Theorem 5.1.

Choose a bundle F and a differential operator $\partial : C^\infty(\mathcal{X}, F^i) \to C^\infty(\mathcal{X}, F)$ of order 1 such that $\|\partial u\| \geqslant \|u\|_1$ for all $u \in C^\infty(\mathcal{X}, F^i)$. Define

$$A^i_\varepsilon = A^i \oplus \varepsilon\partial : C^\infty(\mathcal{X}, F^i) \to C^\infty(\mathcal{X}, F^{i+1}) \oplus C^\infty(\mathcal{X}, F)$$

for $\varepsilon \geqslant 0$. Except for the fact that the composition $A^i_\varepsilon A^{i-1}$ need not be of order 1 when $\varepsilon > 0$, the operators A^{i-1} and A^i_ε share most of the properties of A^{i-1} and A^i which were used in the last two sections. In particular, we can use the sesquilinear form

$$\begin{array}{rcl} D_\varepsilon(u,v) & = & (A^i_\varepsilon u, A^i_\varepsilon v) + (A^{i-1*}u, A^{i-1*}v) + (u,v) \\ & = & D(u,v) + \varepsilon^2\,(\partial u, \partial v) \end{array}$$

to define a self-adjoint operator L^i_ε on $L^2(\mathcal{X}, F^i)$ such that $D_\varepsilon(u,v) = ((L^i_\varepsilon + I)u, v)$ for all u in the domain of L^i_ε and all C^∞ sections v satisfying $n(v) = 0$ on $\partial\mathcal{X}$.

We still give $D_\varepsilon(u,v)$ the domain that consists of all $u, v \in C^\infty(\mathcal{X}, F^i)$ whose normal parts vanish on $\partial\mathcal{X}$. The only problem is on the additional boundary condition for $A^i_\varepsilon u$ for smooth sections $u \in C^\infty(\mathcal{X}, F^i)$ lying in the domain of L^i_ε. An easy verification using the Green formula shows that this free boundary condition reduces to

$$\ell^i(x) n(Au) + \varepsilon^2 (\sigma^1(\partial)(x, d\varrho(x)))^* \partial u = 0$$

on $\partial\mathcal{X}$.

Lemma 7.1. *Assume that quasicomplex (0.1) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|^2_{1/2} \leqslant c\,D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$. Then for each $s \geqslant 1/2$ there is a constant c with the property that*

$$\|u\|_{s+1/2} \leqslant c\,\|(L^i_\varepsilon + I)u\|_{s-1/2} \tag{7.1}$$

holds whenever $u \in C^\infty(\mathcal{X}, F^i)$ is in the domain of L^i_ε and $0 \leqslant \varepsilon \leqslant 1$.

Proof. All the arguments used to prove (7.1) continue to be valid when A^i is replaced by A^i_ε, and it is easy to see that the constant c in each of the various estimates can be chosen independently of ε. □

The reason for introducing A^i_ε is that when $\varepsilon > 0$ then the coercive estimate

$$\varepsilon^2 \|u\|_1^2 \leqslant D_\varepsilon(u,u) \tag{7.2}$$

holds for all $u \in C^\infty(\mathcal{X}, F^i)$, and it is fairly easy to obtain a regularity theorem for L^i_ε. In fact, we have

Theorem 7.2. *Suppose that quasicomplex (0.1) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|^2_{1/2} \leqslant c\, D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$ and let $0 < \varepsilon \leqslant 1$. Then for every $f \in C^\infty(\mathcal{X}, F^i)$ there is a unique section $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i_ε such that $(L^i_\varepsilon + I)u = f$.*

Proof. The operator L^i_ε was constructed in such a way that $L^i_\varepsilon + I$ automatically maps its domain onto $L^2(\mathcal{X}, F^i)$ in a one-to-one fashion. Hence, to prove the theorem, it will suffice to show that if u is in the domain of L^i_ε and if $(L^i_\varepsilon + I)u$ is C^∞, then u is also C^∞. We shall use the method of difference quotients which occurs, e.g., in [9] and [1].

If f is a function on the closed upper half-space in $\mathbb{R}^n$, if $1 \leqslant j < n$ and $h > 0$, then we write

$$\delta_{h,j} f(x) = \frac{1}{\sqrt{-1}}\, \frac{f(x^1, \ldots, x^j + h, \ldots, x^n) - f(x^1, \ldots, x^j - h, \ldots, x^n)}{2h}$$

and, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_n = 0$, we set $\delta^\alpha_h = \delta^{\alpha_1}_{h,1} \ldots \delta^{\alpha_{n-1}}_{h,n-1}$. After choosing a coordinate system $x : U \to \mathbb{R}^n$ on $\mathcal{X}$, which maps U into the closed upper half-space, and after choosing a function $\varphi \in C^\infty_{\mathrm{comp}}(U)$ we can use a local orthonormal frame to define

$$T^\alpha_h u = \varphi\, \delta^\alpha_h\, (\varphi u)\,,$$

when u is a section of one of the vector bundles F^i or of F. For details we refer to the discussion just above Lemma 6.2.

If, in Lemma 6.2, the operator T^s is replaced by the operator T^α_h with $|\alpha| = s$, then statements 1), 2), and 3) continue to hold even if the constants c are required to be independent of h for $0 < h \leqslant 1$. Consequently, Lemmata 6.3 and 6.4 also hold for the operators T^α_h, where again the constants can be chosen independent of h. Using (7.2) and the arguments in the proof of Lemma 6.5, one can show that for each $\varepsilon > 0$ and every integer $s \geqslant 1$ there is a constant c such that

$$\|T^\alpha_h u\|_1 \leqslant c \left(\|\psi (L^i_\varepsilon + I)u\|_{(0,s-1)} + \|\psi u\|_{(1,s-1)} \right), \tag{7.3}$$

provided $|\alpha| = s$, $0 < h \leqslant 1$, u belongs to the domain of L^i_ε, $\psi u \in H^{(1,s-1)}(\mathcal{X}, F^i)$ and $(L^i_\varepsilon + I)u \in C^\infty(\mathcal{X}, F^i)$. Now, if α and u satisfy these conditions, then (7.3) shows that $(T^\alpha_h u)_{0<h\leqslant 1}$ is a bounded subset of $H^1(\mathcal{X}, F^i)$. Hence, there is a sequence h_ν converging to zero such that $T^\alpha_{h_\nu} u$ converges weakly to some element f of $H^1(\mathcal{X}, F^i)$. Since $T^\alpha_{h_\nu} u$ converges in the distribution sense to $\varphi D^\alpha(\varphi u)$ as $h \to 0$, we infer that $f = \varphi D^\alpha(\varphi u)$ an hence $\varphi D^\alpha(\varphi u) \in H^1(\mathcal{X}, F^i)$. Thus, if $\varphi = 1$ on the support of $\omega \in C^\infty_{\mathrm{comp}}(U)$, we conclude that $\omega u \in H^{(1,s)}(\mathcal{X}, F^i)$.

Now let u be in the domain of L^i_ε, such that $(L^i_\varepsilon + I)u \in C^\infty(\mathcal{X}, F^i)$, and let p be a fixed point of $\partial\mathcal{X}$. Then the argument just given shows that if u is in $H^{(1,s-1)}$ on a neighbourhood U of p, then u is in $H^{(1,s)}$ on a slightly smaller neighbourhood. Thus, for each integer s there exists a function $\omega \in C^\infty_{\mathrm{comp}}(U)$ such that $\omega u \in H^{(1,s)}(U, F^i)$ and hence, by Theorem 2.5.7 in [3], the restriction of ωu to the boundary belongs to $H^s(\partial\mathcal{X}, F^i)$. It follows that $u \in H^s(\partial\mathcal{X}, F^i)$ for each s, and so $u \restriction_{\partial\mathcal{X}}$ must be C^∞ by Sobolev's lemma. Since both $(L^i_\varepsilon + I)u$ and $u \restriction_{\partial\mathcal{X}}$ are C^∞, the regularity theorem for the Dirichlet problem implies that u is C^∞ also (see for instance Theorem 9.9 in [1]). The proof of the theorem is thus complete. □

Corollary 7.3. *Suppose that quasicomplex (0.1) is elliptic at steps $i-1$ and i. Let the estimate $\|u\|^2_{1/2} \leqslant c\,D(u,u)$ hold for all $u \in C^\infty(\mathcal{X}, F^i)$ satisfying the boundary condition $n(u) = 0$ on $\partial\mathcal{X}$ and let u belong to the domain of L^i. Then,*

1) u is C^∞ if $(L^i + I)u$ is C^∞;

2) $u \in H^{s+1}(\mathcal{X}, F^i)$ if $(L^i + I)u \in H^s(\mathcal{X}, F^i)$;

3) $u \in H^{s+1}(\mathcal{X}, F^i)$ if $L^i u \in H^s(\mathcal{X}, F^i)$;

4) u is C^∞ if $L^i u$ is C^∞.

Proof. To prove 1) assume that $(L^i + I)u$ is C^∞ and for each $0 < \varepsilon \leqslant 1$ let u_ε be the unique C^∞ section satisfying $(L^i_\varepsilon + I)u_\varepsilon = (L^i + I)u$. If $s \geqslant 1/2$, then (7.1) shows that $(u_\varepsilon)_{0<\varepsilon\leqslant 1}$ is bounded in the norm $\| \dots \|_{s+1/2}$, and by Rellich's theorem there is a sequence ε_ν converging to zero, such that u_{ε_ν} converges in the norm $\| \cdot \|_s$ to an element u_0 of $H^s(\mathcal{X}, F^i)$. On passing to the limit in $D_\varepsilon(u_\varepsilon, v) = ((L^i + I)u, v)$ we obtain

$$D(u_0, v) = ((L^i + I)u, v)$$

for all $v \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(v) = 0$ on the boundary. Thus, u_0 is in the domain of L^i and $(L^i+I)u_0 = (L^i+I)u$. Since L^i+I is one-to-one, we conclude that $u_0 = u$ and so $u \in H^s(\mathcal{X}, F^i)$. Since $s \geqslant 1/2$ can be arbitrarily large, it follows that u is C^∞.

To prove 2), let $s \geqslant 0$ and assume $(L^i + I)u$ is in $H^s(\mathcal{X}, F^i)$. Choose a sequence f_ν of C^∞ sections which converges to $(L^i + I)u$ in the norm $\| \cdot \|_s$, and let u_ν be the unique C^∞ section satisfying $(L^i + I)u_\nu = f_\nu$. Then, by (7.1), the sequence u_ν converges in the norm $\| \cdot \|_{s+1}$ to some element u_0 of $H^{s+1}(\mathcal{X}, F^i)$. Since $L^i + I$ has closed graph, we get $(L^i + I)u_0 = (L^i + I)u$ and hence $u = u_0$. Thus, u belongs to $H^{s+1}(\mathcal{X}, F^i)$, as required.

If $s = 0$, then 3) follows immediately from 2). Let m be a positive integer and assume that 3) holds for all s with $0 \leqslant s \leqslant m-1$. Let $m-1 < s \leqslant m$, and assume that $L^i u$ is in $H^s(\mathcal{X}, F^i)$. Then, since $L^i u \in H^{s-1}(\mathcal{X}, F^i)$, we conclude that $u \in H^s(\mathcal{X}, F^i)$ by the inductive hypothesis, and so $(L^i + I)u$ belongs to $H^s(\mathcal{X}, F^i)$. Thus, by 2), we see that u is in $H^{s+1}(\mathcal{X}, F^i)$, as desired.

The assertion 4) follows obviously from 3) by Sobolev's lemma, and the proof is complete. □

8. A regularity theorem

In this section we assume that the curvature of quasicomplex (0.1) vanishes at step i, i.e., $A^i A^{i-1} \equiv 0$. In this case, the inhomogeneous equation $A^{i-1}u = f$ might be locally solvable only for those f which satisfy $A^i f = 0$. This is a starting point of [16].

Let T denote the operator from $L^2(\mathcal{X}, F^{i-1})$ to $L^2(\mathcal{X}, F^i)$ obtained by closing the graph of $A : C^\infty(\mathcal{X}, F^{i-1}) \to C^\infty(\mathcal{X}, F^i)$. Thus, u is in the domain of T and $Tu = f$ if and only only if there is a sequence (u_ν) in $C^\infty(\mathcal{X}, F^{i-1})$ such that $u_\nu \to u$ and $Au_\nu \to f$ in the L^2-norm. Our aim in this section is to prove

Theorem 8.1. *Assume that the quasicomplex (0.1) is elliptic at steps $i-1$, i and $i+1$, and assume that the estimate $\|f\|^2_{1/2} \leqslant c\,D(f,f)$ holds for all $f \in C^\infty(\mathcal{X}, F^i)$ satisfying $n(f) = 0$ on $\partial\mathcal{X}$. Let u be in the domain of T, let u be orthogonal to the kernel of T, and let $Tu \in H^s(\mathcal{X}, F^i)$ for some $s \geqslant 0$. Then u belongs to $H^{s+1/2}(\mathcal{X}, F^{i-1})$.*

Such a theorem has proved useful in studying counterexamples for a priori estimates like $\|f\|^2_{1/2} \leqslant c\,D(f,f)$, see, e.g., [10].

Lemma 8.2. *Under the assumptions of Theorem 8.1, for each s there is a constant c such that*

$$\|\omega Au\|_s + \|\omega A^* u\|_s \leqslant c\left(\|\omega Au\|_{(0,s)} + \|\omega A^* u\|_{(0,s)} + \|\varDelta u\|_{s-1} + \|u\|_s\right) \qquad (8.1)$$

is valid for all $u \in C^\infty(\mathcal{X}, F^i)$.

Proof. Using the ellipticity of the quasicomplex at F^{i-1} and F^{i+1}, one checks readily that

$$(g,h) \mapsto (A^*g, Ag + A^*h, Ah)$$

is an elliptic operator from sections of $F^{i-1} \oplus F^{i+1}$ to sections of $F^{i-2} \oplus F^i \oplus F^{i+2}$. Hence, by Lemma 6.1, part 4),

$$\begin{array}{rl} \|\omega A^* u\|_s + \|\omega A u\|_s \leqslant c \, (\|\omega A^* u\|_{(0,s)} + \|\omega A u\|_{(0,s)} + \\ + \quad \|A^*(\omega A^* u)\|_{s-1} + \|A(\omega A^* u) + A^*(\omega A u)\|_{s-1} + \|A(\omega A u)\|_{s-1}), \end{array}$$

and since the commutators $[A^*, \omega]$, $[A, \omega]$, etc., have order zero, and the operators A^*A^*, AA have order one, we get

$$\begin{array}{rcl} \|A^*(\omega A^* u)\|_{s-1} & \leqslant & c \, \|u\|_s, \\ \|A(\omega A^* u) + A^*(\omega A u)\|_{s-1} & \leqslant & c \, (\|\Delta u\|_{s-1} + \|u\|_s) \, , \\ \|A(\omega A u)\|_{s-1} & \leqslant & c \, \|u\|_s. \end{array}$$

Estimate (8.2) now follows. □

Lemma 8.3. *Under the assumptions of Theorem 8.1, for each $s \geqslant 1/2$ there is a constant c such that*

$$\|A^* u\|_s \leqslant c \, \|(\Delta + I)u\|_{s-1/2} \tag{8.2}$$

holds for each $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i.

Proof. By Lemma 6.2 and Lemma 8.2 we have

$$\begin{array}{rcl} \|\omega A^* u\|_s^2 & \leqslant & c \left(\|\omega A^* u\|_{(0,s)}^2 + \|\omega A u\|_{(0,s)}^2 + \|\Delta u\|_{s-1}^2 + \|u\|_s^2 \right) \\ & \leqslant & c \left(\|T^s A^* u\|^2 + \|T^s A u\|^2 + \|\Delta u\|_{s-1}^2 + \|u\|_s^2 \right) \\ & \leqslant & c \left(D(T^s u, T^s u) + \|\Delta u\|_{s-1}^2 + \|u\|_s^2 \right) \end{array}$$

for all $u \in C^\infty(\mathcal{X}, F^i)$. If u belongs to the domain of L^i, then by Lemma 6.5

$$\begin{array}{rcl} \|\omega A^* u\|_s^2 & \leqslant & c \left(\|(\Delta + I)u\|_{s-1/2}^2 + \|u\|_s^2 \right) \\ & \leqslant & c \, \|(\Delta + I)u\|_{s-1/2}^2. \end{array}$$

Now cover $\mathcal{X}$ with a finite number of neighbourhoods U_ν of the kind used in Lemma 6.2 and choose the corresponding functions ω_ν to form a partition of unity on $\mathcal{X}$. Then

$$\|A^* u\|_s \leqslant \sum_\nu \|\omega_\nu A^* u\|_s \leqslant c \, \|(\Delta + I)u\|_{s-1/2}$$

for all $u \in C^\infty(\mathcal{X}, F^i)$ in the domain of L^i, as desired. □

Lemma 8.4. *Under the assumptions of Theorem 8.1, let $u \in L^2(\mathcal{X}, F^i)$ belong to the domain of L^i, and assume that $L^i u \in H^{s-1/2}$ for some $s \geqslant 1/2$. Then u is in the domain of T^* and $T^* u$ belongs to $H^s(\mathcal{X}, F^{i-1})$.*

Proof. In view of part 2) of Corollary 7.3 we get $u \in H^{s+1/2}(\mathcal{X}, F^i)$ and hence $(L^i + I)u \in H^{s-1/2}(\mathcal{X}, F^i)$. Choose a sequence (f_ν) in $C^\infty(\mathcal{X}, F^i)$ which converges to $(L^i + I)u$ in the norm $\| \cdot \|_{s-1/2}$ and let $u_\nu \in C^\infty(\mathcal{X}, F^i)$ be the unique solution to

$$(L^i + I)u_\nu = f_\nu.$$

Then by (7.1) the sequence (u_ν) converges in the norm $\| \cdot \|_{s+1/2}$, and since $L^i + I$ gas closed graph, the limit must be u. Now Lemma 5.2 and the Green formula show that each u_ν is in the domain of T^*, and $T^* u_\nu = A^* u_\nu$. The estimate (8.2) now implies that $(T^* u_\nu)$ converges in the norm $\| \cdot \|_s$. Since T^* has closed graph, we conclude that $u = \lim u_\nu$ is in the domain of T^* and $T^* u = \lim T^* u_\nu$ is in $H^s(\mathcal{X}, F^{i-1})$. The proof is complete. □

As is remarked in Section 5., any $f \in L^2(\mathcal{X}, F^i)$ can be written as $f = h + L^i u$, where h lies in the null space of L^i and u is in the domain of L^i. If we require that u be orthogonal to the null space of L^i, then f determines u uniquely and the correspondence $f \mapsto u$ defines an operator $N^i : L^2(\mathcal{X}, F^i) \to L^2(\mathcal{X}, F^i)$ which, as one easily sees, is self-adjoint and bounded.

Proof of Theorem 8.1. Let u be in the domain of T^{i-1}, let u be orthogonal to the kernel of T^{i-1}, and assume that Tu is in $H^s(\mathcal{X}, F^i)$ for some $s \geqslant 0$. Then, since $Tu = h + L^i(NTu)$, where $h \in \mathcal{H}^i(\mathcal{X})$ is C^∞ on $\mathcal{X}$, Lemma 8.4 shows that NTu is in the domain of T^* and $T^* NTu$ belongs to $H^{s+1/2}(\mathcal{X}, F^{i-1})$. To complete the proof we show that

$$u = T^* NTu.$$

In fact, if $v \in C^\infty(\mathcal{X}, F^{i-1})$ is an arbitrary section with support in the interior of $\mathcal{X}$, then $Av = h + \Delta NAv$, where $h \in \mathcal{H}^i(\mathcal{X})$. Hence,

$$Av - AA^* NAv = h + A^* ANAv.$$

Since $A^i A^{i-1} \equiv 0$, the terms on the right-hand side of this equality are orthogonal to the terms on the left-hand side. It follows that $A(I - A^* NA)v = 0$ and so $(I - A^* NA)v$ is in the null space of T. Since u is orthogonal to the null space of T, we obtain

$$\begin{aligned} 0 &= (u, (I - A^* NA)v) \\ &= ((I - T^* NT)u, v), \end{aligned}$$

and $u = T^* NTu$ now follows. □

The first author gratefully acknowledges the financial support of the Ministry of High Education of Iraq.

References

[1] S.Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand, Princeton, N.J., 1965.

[2] A.Alfonso, B.Simon, The Birman-Krein-Vishik theory of self-adjoint extensions of semi-bounded operators, *J. Operator Theory*, **4**(1980), 251–270.

[3] L.Hörmander, Linear Partial Differential Operators, Springer-Verlag, Berlin et al., 1963.

[4] L.Hörmander, Pseudodifferential operators, *Comm. Pure Appl. Math.*, **18**(1965), 501–517.

[5] L.Hörmander, Pseudo-differential operators and nonelliptic boundary problems, *Ann. of Math.*, **83**(1966), 129–209.

[6] J.J.Kohn, Harmonic integrals on strongly pseudo-convex manifolds, *P. I: Ann. of Math.*, **78**(1963), 112–148; *P. II: Ann. of Math.*, **79**(1964), 450–472.

[7] J.J.Kohn, L.Nirenberg, Estimates for elliptic complexes of higher order, *Comm. Pure Appl. Math.*, **18**(1965), 443–492.

[8] C.B.Morrey, Jr., The $\bar{\partial}$-Neumann problem on strongly pseudo-convex manifolds, Outlines of Joint Soviet-American Symposium on Partial Differential Equations, Novosibirsk, 1963, 171–178.

[9] L.Nirenberg, Remarks on strongly elliptic partial differential equations, *Comm. Pure Appl. Math.*, **8**(1955), 648–674.

[10] W.J.Sweeney, A non-compact Dirichlet norm, *Proc. Nat. Acad. Sci. USA*, **58**(1967), 2193–2195.

[11] W.J.Sweeney, A uniqueness theorem for the Neumann problem, *Ann. of Math.*, **90**(1969), no. 2, 353–360.

[12] W.J.Sweeney, Estimates for elliptic complexes of higher order, *J. Diff. Eq.*, **10**(1971), 112–140.

[13] W.J.Sweeney, Coerciveness in the Neumann problem, *J. Diff. Geom.*, **6**(1971/72), 375–393.

[14] W.J.Sweeney, A condition for subellipticity in Spencer's Neumann problem, *J. Diff. Eq.*, **21**(1976), no. 2, 316–362.

[15] W.J.Sweeney, Subelliptic estimates for certain complexes of pseudodifferential operators, *J. Diff. Eq.*, **61**(1986), no. 2, 250–267.

[16] D.C.Spencer, Harmonic integrals and Neumann problems associated with linear partial differential equations, Outlines of Joint Soviet-American Symposium on Partial Differential Equations, Novosibirsk, 1963, 253–260.

[17] N.Tarkhanov, Complexes of Differential Operators, Kluwer Academic Publishers, Dordrecht, NL, 1995.

[18] N.Tarkhanov, *Euler characteristic of Fredholm quasicomplexes*, Funct. Anal. and its Appl. **41**(2007), no. 4, 87-93.

[19] R.O.Wells, Differential Analysis on Complex Manifolds, Springer-Verlag, New York, 1980.

Задача Неймана по Спенсеру

Азал Мера
Николай Тарханов
Потсдамский университет
Карл-Либкнехт-Штр., 24/25, Потсдам, 14476
Германия

Попытка распространить теорию Ходжа для эллиптических комплексов на компактных замкнутых многообразиях на случай компактных многообразий с краем приводит к краевой задаче для лапласиана комплекса, которая обычно называется задачей Неймана. Мы изучаем задачу Неймана для более широкого класса последовательностей дифференциальных операторов на компактном многообразии с краем. Это последовательности малой кривизны, т.е. обладающие свойством, что композиция любых двух соседних операторов имеет порядок меньший, чем два.

Ключевые слова: эллиптические комплексы, многообразия с границей, теория Ходжа, задача Неймана.