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Coamoebas of Discriminants of Tetranomials

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A method of construction of a coamoeba for a discriminant of a general tetranomial equatio is presented.

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Consider a general reduced algebraic equation

$$1 + x_1 y + ...x_{n-1} y^{n-1} + y^n = 0, \tag{1}$$

where $= (x_1, \dots, x_n) \in \mathbb{C}^{n-1}$. Denote its discriminant by $\Delta = \Delta(x)$. The discriminantal set

$$\nabla = \{x \in \mathbb{C}^{n-1} : \Delta(x) = 0\}$$

admits a parameterization by $s = (s_1 : s_2 : \ldots : s_{n-1}) \in \mathbb{CP}^{n-2}$ according to [1]

$$x_k = -\frac{ns_k}{\langle \alpha, s\rangle}\left(\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle}\right)^{\frac{k}{n}}, \quad k = 1, \dots, n-1, \tag{2}$$

where α and β are integer vectors

$$\alpha = (n-1, \dots, 2, 1), \quad \beta = (1, 2, \dots, n-1).$$

We are interested in tetranomial equations, i.e. those obtained from (1) by vanishing of all coefficients x_k, except two:

$$1 + x_l y^l + x_m y^m + y^n = 0, \quad \text{where } l < m < n. \tag{3}$$

In this case the parameterization (2) of the discriminantal set in an affine coordinate $t := s_l/s_m$ of the projective line $\mathbb{CP}^1$ assumes the form:

$$\begin{aligned} x_l &= -\frac{nt}{(n-l)t + n - m}\left(\frac{(n-l)t + n - m}{lt + m}\right)^{\frac{l}{n}}, \\ x_m &= -\frac{n}{(n-l)t + n - m}\left(\frac{(n-l)t + n - m}{lt + m}\right)^{\frac{m}{n}}. \end{aligned} \tag{4}$$

Denote this parameterization by $x = \Psi(t)$ letting $x = (x_l, x_m)$.

Example 1. *In all examples we shall consider two tetranomials of degree 3 and 6.*

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1. *Consider a third degree equation* $1 + x_1y + x_2y^2 + y^3 = 0$. *Here* $n = 3$, $l = 1$, $m = 2$. *The discriminant of this equation is equal to*

$$\Delta = 27 + 4x_1^3 - 4x_2^3 + 18x_1x_2 - x_1^2x_2^2.$$

The parameterization of its zero locus ∇ *is the following*

$$x_1 = -\frac{3t}{2t+1}\left(\frac{2t+1}{t+2}\right)^{\frac{1}{3}}; \quad x_2 = -\frac{3}{2t+1}\left(\frac{2t+1}{t+2}\right)^{\frac{2}{3}}.$$

2. *For the tetranomial*

$$1 + x_1y + x_2y^2 + y^6 = 0$$

the parameterization of its discriminantal set is

$$x_1 = -\frac{6t}{5t+4}\left(\frac{5t+4}{t+2}\right)^{\frac{1}{6}}; \quad x_2 = -\frac{6}{5t+4}\left(\frac{5t+4}{t+2}\right)^{\frac{1}{3}}.$$

We shall draw the coamoeba of the discriminantal curve ∇ of equation (3).

Definition 1. *The coamoeba* $\mathcal{A}'$ *of the set* ∇ *is its image* $\mathrm{Arg}\nabla \subseteq \mathbb{R}^n$ *under the projection on the argumental subspace*

$$\mathrm{Arg}\, z = (\arg z_1, \ldots, \arg z_n).$$

In order to depict the coamoeba of the discriminantal set ∇ for equation (3) it is convenient to use the parameterization (4). This parameterization involves fractional powers of some linear in t functions. The case of integer powers has been studied in [2]. In our context this case corresponds to tetranomial equations with fixed coefficients of neighboring monomials. The coamoeba of a reduced discriminantal curve ∇ is the image of the composition of mappings $\mathrm{Arg} \circ \Psi : \mathbb{CP}_1 \to \mathbf{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. Taking into account (4), this image is parameterized by

$$\begin{aligned} \arg x_l &= \arg(-nt) - \frac{n-l}{n}\arg\big((n-l)t + n - m\big) - \frac{l}{n}\arg(lt+m), \\ \arg x_m &= \arg(-n) - \frac{n-m}{n}\arg\big((n-l)t + n - m\big) - \frac{m}{n}\arg(lt+m). \end{aligned} \tag{5}$$

The mapping $\mathrm{Arg} \circ \Psi$ is finite at a general point, i.e. it has a finite number of preimages at all points, except, possibly, their finite number. We want to know the number of preimages for all points of the torus $\mathbf{T}^2$. Following the idea of the paper [2], we consider the coamoeba A' as a 2-chain in the sense of homology theory. The support of such a chain coincides with the topological closure of the coamoeba, while the cells constituting the chain A' are composed according to the number of preimage of $\mathrm{Arg}^{-1}(p)$ of $p \in A'$. In other words, the number of preimages of any point from a relative interior of a cell is the same.

The singular points of the parameterization (5) are

$$-\frac{m}{l}, \ \frac{m-n}{n-l}, \ 0, \ \infty.$$

All these pionts lie on the axis of real values of the parameter t. They divide the axis into four intervals, while the axis itself divides the Riemann sphere $\mathbb{CP}^1 \cong \bar{\mathbb{C}}$ into two simply connected domains, where the mapping (4) admits single valued branches.

We start with the construction of zero-dimensional cells. Since the signs of linear in t functions in (5) do not change within intervals, these intervals are mapped into single-pont sets, i.e. zero-dimensional cells. The singular points are mapped to segments (1-dimensional cells), while the endpoints of the segments are images of intervals meeting at the chosen singular point.

Points from different parts of the coamoeba may have different number of preimages of Arg$\circ\Psi$. Our goal is to construct 2-cells, which are convex polygons, consisting of points with the same number of preimages.

In order to achieve this we rewrite the parameterization (4) in the form

$$x_l = (-nt)((n-l)t+n-m)^{\frac{l-n}{n}}(lt+m)^{-\frac{l}{n}},$$
$$x_m = (-n)((n-l)t+n-m)^{\frac{m-n}{n}}(lt+m)^{-\frac{m}{n}}.$$

As has been said above, the axis of real values of t divides the Riemann sphere into two simply connected domains. This allows to find the image of a branch of the mapping in the upper half-plane. The other images of branches in the upper half-plane may be obtained by 'shifting' the first one. This shift we get by changing the value of t in the parameterization (5) by $2\pi k$.

When considering branches in the upper half-plane we approximate it by removing half-circles of small radii with centers at the singular ppoints. Therefore when t moves along the real axis from left to right the value of $\arg t$ jumps by π at the singular points. Analogously, when t is moving from right to left for the branches from the lower half-plane the value of $\arg t$ jumps by $-\pi$ as the parameter goes along small half-circles in the lower half-plane.

We start with large negative values of t (i.e. from $t=-\infty$ for which $\arg t = \pi$) in the upper half-plane. Formulas (5) show that $\operatorname{Arg} x(-\infty) = (-\pi; 0)$ (this is the point p_1 on Fig. 1). The same value is preserved on the whole ray from $(-\infty)$ to $-\dfrac{m}{l}$. Passing from one vertex of a cell to another corresponds to a jump in a singular point, which happens when one of the factors in the parameterization vanishes. So, the first jump occurs at $t = -\dfrac{m}{l}$. Here x_l and x_m vanish because of the factors

$$(lt+m)^{-\frac{l}{n}} \text{ and } (lt+m)^{-\frac{m}{n}}.$$

When going around along a half-circle there occurs a jump to p_2 on Fig. 1. The jump is equal to π multiplied by $\alpha_1 = \left(\dfrac{l}{n}; \dfrac{m}{n}\right)$. Thus, we get $p_2 = p_1 + \alpha_1\pi$.

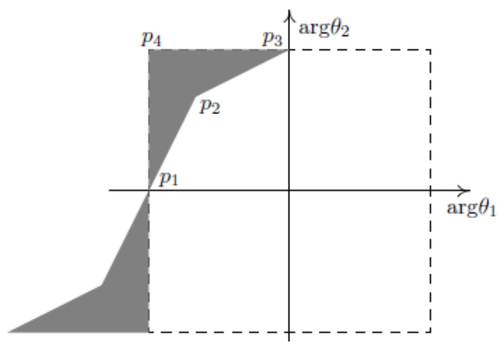

Fig. 1. Cells of the discriminant of the equation $1 + x_1y + x_2y^2 + y^3 = 0$

The jump from p_2 to p_3 on Fig. 1 occurs at $t = \dfrac{m-n}{n-l}$ when the factors

$$((n-l)t+n-m)^{\frac{l-n}{n}}; ((n-l)t+n-m)^{\frac{m-n}{n}}$$

vanish. In this case the vector $\alpha_2 = \left(-\dfrac{l-n}{n}; -\dfrac{m-n}{n}\right)$, and $p_3 = p_2 + \alpha_2\pi$.

The last vertex corresponds to vanishing of linear factors, but in fact in this case only x_l becomes zero. Since the coefficent at t is negative, $\alpha_3 = (-\pi, 0)$, and we get the point $p_4 = p_3 + \alpha_3\pi$. The jump at ∞ corresponds to the segment $[p_4; p_1]$. Thus, we get a closed polygonal line inside which the chosen branch from the upper half-plane is projected.

The image of the lower half-plane is constructed in the same way, taking into account jumps by $-\pi$. The images of upper and lower branches are symmetric with respect to the point p_1, since passing from the upper half-plane to the lower one around ∞ the image turns by π. In this way we get the first cell of the coamoeba. The remaining cells are constructed in the same way, but it is necessary to pick the general starting points of cells.

The parameterization involves multi-valued functions, therefore the coamoeba has n images of upper and lower half-planes. In order to determine the points p_1 of these cells we return to the original parameterization (4). To determine the coordinates of the remaining starting points we need to sum the coordinates of the starting point $p_1 = (-\pi; 0)$ and the vector of exponents of fractional expressions multiplied by $2\pi k$, $k = 0, \ldots, n-1$, i.e the starting points have the coordinates $(-\pi; 0) + \Big(\dfrac{2\pi kl}{n}, \dfrac{2\pi km}{n}\Big)$. The rest of the construction of cells is performed analogously.

Example 2. *Let us determine starting points for our examples.*

1. $1 + x_1 y + x_2 y^2 + y^3 = 0$. *On the interval* $(-\infty; -2)$ *the arguments are* $(-\pi; 0)$. *The first jump at* $t = -2$ *takes us to* $\Big(-\dfrac{2\pi}{3}; \dfrac{2\pi}{3}\Big)$. *The point* $(0; \pi)$ *corresponds to the jump at* $t = \dfrac{1}{2}$, *while the last point* $(\pi; -\pi)$ *to the jump at* $t = 0$.

   Correspondingly, when going in the lower half-plane we get the values

$$(\pi; 0), \Big(-\frac{4\pi}{3}; -\frac{2\pi}{3}\Big), (-2\pi; -\pi), (\pi; -\pi).$$

2. $1 + x_1 y + x_2 y^2 + y^6 = 0$. *The axis of* t *is divided into intervals*

$$(-\infty, -2), \Big(-2, -\frac{4}{5}\Big), \Big(-\frac{4}{5}, 0\Big), (0, +\infty).$$

   At the points of jumps we get

$$(-\pi, 0), \Big(\frac{5\pi}{6}, \frac{\pi}{3}\Big), (0, \pi), (\pi, \pi).$$

   The starting cells look as follows (Fig.2).

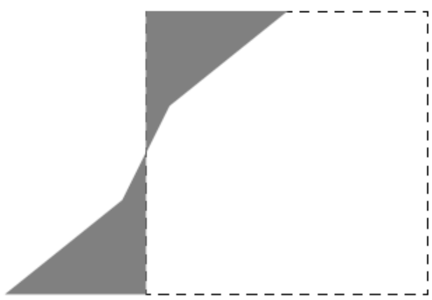

Fig. 2. Cells of the discriminant of $1 + x_1 y + x_2 y^2 + y^6 = 0$

Determine now coordinates of general points of the remaining cells.

Example 3. *We determine the number of groups of symmetric cells with the help of the degree of polynomial.*

1. $1 + x_1y + x_2y^2 + y^3 = 0.$

This equation is of degree 3, hence we have three groups of symmetric cells. The coordinates of the first one we have already found.

The second group begins at $(-\pi; 0) + 2\pi\Big(\frac{1}{3}; \frac{2}{3}\Big) = \Big(-\frac{\pi}{3}; \frac{4\pi}{3}\Big) = \Big(-\frac{2\pi}{3}; \frac{-2\pi}{3}\Big)$. *The third one at the point* $\Big(\frac{\pi}{3}, \frac{2\pi}{3}\Big)$

Taking the union of all cells on the fundamental square and making necessary shifts we get the coamoeba of the discriminant of an equation of third degree (Fig.3).

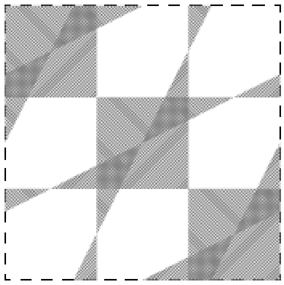

Fig. 3. The coamoeba of the discriminant of the cubic equation $1 + x_1y + x_2y^2 + y^3 = 0$

In this example no more than two two-dimensional cells overlap, this means that points of the coamoeba have at most two preimages.

2. $1 + x_1y + x_2y^2 + y^6 = 0$ *The starting points have the coordinates*

$$(0; 0), \Big(-\frac{2\pi}{3}; \frac{2\pi}{3}\Big), \Big(-\frac{\pi}{3}; -\frac{2\pi}{3}\Big), \Big(\frac{\pi}{3}; \frac{2\pi}{3}\Big), \Big(\frac{2\pi}{3}; -2\frac{\pi}{3}\Big).$$

The coamoeba looks as follows Fig.4

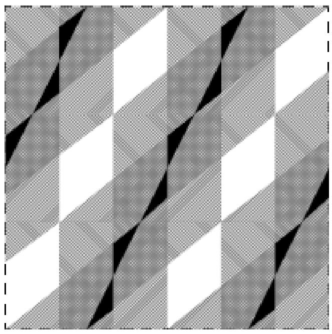

Fig. 4. The coamoeba of the discriminant of the equation $1 + x_1y + x_2y^2 + y^6 = 0$

In contrast to the previous example, the maximal number of overlapping cells is three, i.e. some points of the coamoeba have three preimages. Black triangles on the figure depicting the coamoeba correspond to such regions.

Coamoebas in these examples have non-empty complements. Let us give an example of a tetranomial whose discriminant's amoeba covers whole torus (Fig. 5).

Such an example is delivered by the equation $1 + x_2 y^2 + x_3 y^3 + y^6 = 0$.

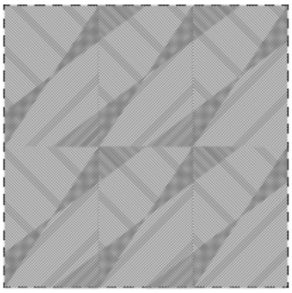

Fig. 5. The coamoeba of the discriminant of the equation $1 + x_2 y^2 + x_3 y^3 + y^6 = 0$

Example 4. *Consider now a system of equations*

$$\begin{cases} y_1^3 + a y_1 y_2 - 1 = 0, \\ y_2^3 + b y_1 y_2 - 1 = 0. \end{cases}$$

The parameterization of the discriminantal set of this system is the following:

$$\begin{aligned} a &= -\frac{3}{s-2}\left(\frac{s-2}{1+s}\right)^{\frac{1}{3}}\left(\frac{1-2s}{1+s}\right)^{\frac{1}{3}} = -3(s-2)^{-\frac{2}{3}}(1+s)^{-\frac{2}{3}}(1-2s)^{\frac{1}{3}}, \\ b &= -\frac{3s}{1-2s}\left(\frac{s-2}{1+s}\right)^{\frac{1}{3}}\left(\frac{1-2s}{1+s}\right)^{\frac{1}{3}} = -3s(s-2)^{\frac{1}{3}}(1+s)^{-\frac{2}{3}}(1-2s)^{-\frac{2}{3}}. \end{aligned} \tag{6}$$

The set of singular points of this parameterization is

$$s : -\infty, -1, 0, \frac{1}{2}, 2.$$

The starting points of cells of the coamoeba:

$$\left(-\frac{\pi}{3}; -\frac{\pi}{3}\right), \ \left(\frac{\pi}{3}; \frac{\pi}{3}\right), \ (\pi; \pi).$$

The coamoeba of the discriminantal set is on Fig. 6.

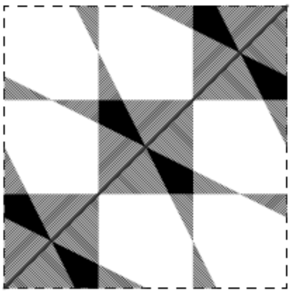

Fig. 6. The coamoeba of the discriminant for a system of equations.

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Коамебы дискриминантов для тетраномов

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Приводится способ построения коамебы дискриминанта общего тетрономиального уравнения.

Ключевые слова: коамеба, дискриминант, тетраном.