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Weight q -multiplicities for Representations of $\mathfrak{sp}_4(\mathbb{C})$ **Pamela E. Harris*****Edward L. Lauber†**

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In this paper we present a closed formula for the values of the q -analog of Kostant's partition function for the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ and use this result to give a simple formula for the q -multiplicity of a weight in the representations of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$. This generalizes the 2012 work of Refaghat and Shahryari that presented a closed formula for weight multiplicities in representations of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$.

Keywords: Symplectic Lie algebra, Kostant partition functions, q -analog of Kostant partition function, weight multiplicity, weight q -multiplicity.

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1. Introduction and preliminaries

In the study of the representation theory of simple Lie algebras it is of interest to compute the multiplicity of a weight μ in a finite-dimensional complex irreducible representation of the Lie algebra \mathfrak{g} of rank r . This multiplicity is the dimension of a specific vector subspace called a weight space. The theorem of the highest weight asserts that all finite-dimensional complex irreducible representation of \mathfrak{g} are equivalent to $L(\lambda)$ a highest weight representation with dominant integral highest weight $\lambda = n_1\varpi_1 + n_2\varpi_2 + \cdots + n_r\varpi_r$ where $n_1, n_2, \dots, n_r \in \mathbb{N} := \{0, 1, 2, \dots\}$ and $\{\varpi_1, \varpi_2, \dots, \varpi_r\}$ denotes the set of fundamental weights of \mathfrak{g} . One can compute this multiplicity, denoted by $m(\lambda, \mu)$, by using Kostant's weight multiplicity formula [10]:

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho)), \quad (1)$$

where W is the Weyl group, $\ell(\sigma)$ denotes the length of σ , $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ with Φ^+ being the set of positive roots of \mathfrak{g} . If \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} , then elements of the dual of \mathfrak{h} , denoted by \mathfrak{h}^* , are called weights. In Equations (1) \wp denotes Kostant's partition function defined on weights $\xi \in \mathfrak{h}^*$ and $\wp(\xi)$ counts the number of ways the weight ξ can be expressed as a nonnegative integral sum of positive roots.

A generalization of Equations (1) was provided by Lusztig and is called the q -analog of Kostant's weight multiplicity formula [11]:

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda + \rho) - (\mu + \rho)) \quad (2)$$

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with \wp_q denoting the q -analog of Kostant's partition function defined on $\xi \in \mathfrak{h}^*$ by

$$\wp_q(\xi) = c_0 + c_1q + c_2q^2 + \cdots + c_nq^n,$$

where c_i represents the number of ways to express the weight ξ as a sum of exactly i positive roots. Note that since $\wp_q(\xi)|_{q=1} = \wp(\xi)$ for all weights $\xi \in \mathfrak{h}^*$ evaluating $m_q(\lambda, \mu)$ at $q = 1$ recovers $m(\lambda, \mu)$. The following is a celebrated result of Lusztig, which illustrates an important use of the q -analog of Kostant's weight multiplicity formula [11, Section 10, p. 226]: if \mathfrak{g} is a finite-dimensional simple Lie algebra, then $m_q(\tilde{\alpha}, 0) = q^{e_1} + q^{e_2} + \cdots + q^{e_r}$ where $\tilde{\alpha}$ is the highest root and e_1, e_2, \dots, e_r are the exponents of \mathfrak{g} . We recall that the exponents of \mathfrak{g} are related to the degrees of the basic invariants, where the degrees are equal to one more than the exponents [9].

Even though formulas, such as Equations (1) and (2), exist to compute the multiplicity and q -multiplicity of the weight μ in the irreducible representation $L(\lambda)$, respectively, the computation can be intractable. This is due to the fact that in general the number of terms appearing in the sum are factorial in the rank of the Lie algebra and there is no known closed formula for the partition function involved. There has been recent progress in addressing these complications for particular weight multiplicity computations [4, 5, 7, 8].

Although it is very difficult to give closed formulas for weight multiplicities in rank r Lie algebras, there has been some success in low rank cases. One such case is the work of Refaghat and Shahryari, where they provided a closed formula for Equation (1) for the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ [12]. More recently, Fernández-Núñez, García-Fuertes, and Perelomov provided a generating function for the weight multiplicities of the representations of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ [1]. Motivated by the work of Refaghat and Shahryari and of Fernández-Núñez, García-Fuertes, and Perelomov we present a new formula for Equation (2) which gives the q -multiplicities for weights of the representations of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$. This formula depends solely on a simple computation involving the values of λ and μ and not on the partition function nor on the order of the group.

Theorem 1.1. *Let ϖ_1 and ϖ_2 denote the fundamental weights of $\mathfrak{sp}_4(\mathbb{C})$ and consider $\lambda = m\varpi_1 + n\varpi_2$ and $\mu = x\varpi_1 + y\varpi_2$ with $m, n, x, y \in \mathbb{N}$. Define $a = m + n - x - y$, $b = n - y + \frac{m - x}{2}$, $c = n - x - y - 1$, and $d = -y - 1 + \frac{m - x}{2}$. Then*

$$m_q(\lambda, \mu) = \begin{cases} P - Q - R & \text{if } a, b, c, d \in \mathbb{N}, \\ P - Q & \text{if } a, b, c \in \mathbb{N} \text{ and } d \notin \mathbb{N}, \\ P - R & \text{if } a, b, d \in \mathbb{N} \text{ and } c \notin \mathbb{N}, \\ P & \text{if } a, b \in \mathbb{N} \text{ and } c, d \notin \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where

$$P = \sum_{i=0}^{\min(\lfloor \frac{a}{2} \rfloor, b)} \binom{a+b-2i}{j=\max(a-i, b)} q^j, \quad Q = \sum_{i=0}^{\lfloor \frac{c}{2} \rfloor} \binom{b+c-2i}{j=b} q^j, \quad \text{and} \quad R = \sum_{i=0}^d \binom{a+d-2i}{j=a-i} q^j.$$

The proof of Theorem 1.1 uses the following closed formula for the q -analog of Kostant's partition function.

Proposition 1.2. *Let α_1 and α_2 denote the simple roots of $\mathfrak{sp}_4(\mathbb{C})$. If $m, n \in \mathbb{N}$, then*

$$\wp_q(m\alpha_1 + n\alpha_2) = \sum_{i=0}^{\min(\lfloor \frac{m}{2} \rfloor, n)} \binom{m+n-2i}{j=\max(m-i, n)} q^j. \tag{4}$$

Equation (3) yields a generalization of the formula for Kostant’s weight multiplicity formula for $\mathfrak{sp}_4(\mathbb{C})$ presented in [12] as its evaluation at $q = 1$ recovers their result. This paper is organized as follows: Section 2 provides the necessary background to make our approach precise. Section 3 provides a proof of Proposition 1.2. Lastly, Section 4 contains the proof of Theorem 1.1 and Corollary 4.1 which considers the case of setting $q = 1$ in Theorem 1.1 thereby giving a closed formula for Kostant’s weight multiplicity for the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$.

2. Background

Following the notation of [2, 3] we now provide the necessary background to make our approach precise. Throughout this work we let α_1, α_2 denote the simple roots and ϖ_1, ϖ_2 the fundamental weights of $\mathfrak{sp}_4(\mathbb{C})$. One may change from fundamental weights to simple roots via

$$\varpi_1 = \alpha_1 + \frac{1}{2}\alpha_2, \tag{5}$$

$$\varpi_2 = \alpha_1 + \alpha_2. \tag{6}$$

We consider the case where $\lambda = m\varpi_1 + n\varpi_2$ and $\mu = x\varpi_1 + y\varpi_2$ with $m, n, x, y \in \mathbb{N}$, thereby using the fundamental weights as an initial basis for λ and μ , but we often convert to the simple roots in order to simplify partition function calculations.

The set of positive roots of $\mathfrak{sp}_4(\mathbb{C})$ is given by $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ and hence

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = 2\alpha_1 + \frac{3}{2}\alpha_2 = \varpi_1 + \varpi_2. \tag{7}$$

The Weyl group of $\mathfrak{sp}_4(\mathbb{C})$ is denoted by W and its elements are generated by the root reflections s_1 and s_2 , which are perpendicular to the simple roots α_1 and α_2 , respectively. The eight elements of W and their lengths are presented in Tab. 1. The action of s_1 and s_2 on the simple roots and

Table 1. The elements of the Weyl group of $\mathfrak{sp}_4(\mathbb{C})$ and their length

$\sigma \in W$	1	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$	$s_2s_1s_2$	$s_1s_2s_1s_2$
$\ell(\sigma)$	0	1	1	2	2	3	3	4

the fundamental weights is given by

$$s_1(\alpha_1) = -\alpha_1, \quad s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad s_1(\alpha_2) = 2\alpha_1 + \alpha_2, \quad s_2(\alpha_2) = -\alpha_2 \tag{8}$$

and for $1 \leq i, j \leq 2$

$$s_i(\varpi_j) = \begin{cases} \varpi_j & \text{if } i \neq j, \\ \varpi_j - \alpha_j & \text{if } i = j. \end{cases} \tag{9}$$

The action of any other element of the Weyl group is acquired by noting that the action of s_1 and s_2 is linear. For example, $s_1s_2(3\varpi_2) = 3s_1(s_2(\varpi_2)) = 3s_1(\varpi_2 - \alpha_2) = 3(s_1(\varpi_2) - s_1(\alpha_2)) = 3(\varpi_2 - (2\alpha_1 + \alpha_2)) = 3((\alpha_1 + \alpha_2) - (2\alpha_1 + \alpha_2)) = -3\alpha_1$.

3. The q -analog of Kostant’s partition function

The following sections consider a weight ξ and analyze the value of $\wp_q(\xi)$ when using the positive roots of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$. In this analysis we note that combinatorially the

positive roots of the Lie algebra of $\mathfrak{sp}_4(\mathbb{C})$, $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ all but $2\alpha_1 + \alpha_2$ are positive roots of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, whose positive roots are α_1, α_2 , and $\alpha_1 + \alpha_2$. Hence, we first present a closed formula for the q -analog of the partition function for the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ and use this result in our proof for the closed formula for the q -analog of the partition function for the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$.

3.1. Formula for \wp_q on $\mathfrak{sl}_3(\mathbb{C})$

By examining the partition function value on the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, we can begin to better understand the partition function value on $\mathfrak{sp}_4(\mathbb{C})$. We consider the q -analog of Kostant's partition function on $m\alpha_1 + n\alpha_2$ with $m, n \in \mathbb{N}$ in the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

Proposition 3.1. *If $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $m, n \in \mathbb{N}$, then $\wp_q(m\alpha_1 + n\alpha_2) = \sum_{j=\max(m,n)}^{m+n} q^j$.*

Proof. The number of possible ways to write $m\alpha_1 + n\alpha_2$ as a nonnegative integral sum of the positive roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ is determined entirely by the number of times $\alpha_1 + \alpha_2$ is used. If a partition of $m\alpha_1 + n\alpha_2$ includes $c(\alpha_1 + \alpha_2)$, where $0 \leq c \leq \min(m, n)$, then there must be $m - c$ uses of α_1 and $n - c$ uses of α_2 . The total number of roots used in this partition will be $m + n - c$. Since we know that c ranges in value from 0 to $\min(m, n)$, and that there is one and only one possible partition for each value of c , it follows that the number of roots used in a partition of $m\alpha_1 + n\alpha_2$ ranges between $m + n - \min(m, n) = \max(m, n)$ and $m + n$. Thus,

$$\wp_q(m\alpha_1 + n\alpha_2) = \sum_{j=\max(m,n)}^{m+n} q^j. \quad \square$$

The next corollary follows directly from Proposition 3.1.

Corollary 3.2. *Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. If $m, n \in \mathbb{N}$, then $\wp(m\alpha_1 + n\alpha_2) = \min(m, n) + 1$.*

Proof. We note that $\wp_q(m\alpha_1 + n\alpha_2)|_{q=1} = \wp(m\alpha_1 + n\alpha_2)$. Thus, we set $q = 1$ and find that

$$\left(\sum_{j=\max(m,n)}^{m+n} q^j \right) \Big|_{q=1} = \sum_{j=\max(m,n)}^{m+n} 1 = m + n - (\max(m, n) - 1) = \min(m, n) + 1. \quad \square$$

3.2. Formula for \wp_q on $\mathfrak{sp}_4(\mathbb{C})$

We now consider the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ with positive roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2$.

Proof of Proposition 1.2. We note that every partition of a weight that is possible in $\mathfrak{sl}_3(\mathbb{C})$ is also possible in $\mathfrak{sp}_4(\mathbb{C})$. However, $\mathfrak{sp}_4(\mathbb{C})$ also has $2\alpha_1 + \alpha_2$ as a positive root, so we must consider all partitions of a weight using this root. Let $m, n \in \mathbb{N}$. It is clear that any partition of $m\alpha_1 + n\alpha_2$ can contain i copies of the positive root $2\alpha_1 + \alpha_2$ so long as $0 \leq i \leq \min(\lfloor \frac{m}{2} \rfloor, n)$. It follows that when using $0 \leq i \leq \min(\lfloor \frac{m}{2} \rfloor, n)$ copies of the root $2\alpha_1 + \alpha_2$ to partition $m\alpha_1 + n\alpha_2$, the remainder $(m - 2i)\alpha_1 + (n - i)\alpha_2$ must be partitioned using only the roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$. Thus, by Proposition 3.1 the number of ways to partition $(m - 2i)\alpha_1 + (n - i)\alpha_2$ using only the positive roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ is given by

$$\sum_{j=\max(m-2i, n-i)}^{m+n-3i} q^j. \quad (10)$$

In our count we must add i to the exponents of q in every term of expression (10) to account for the i copies of the root $2\alpha_1 + \alpha_2$ used in the partition of $m\alpha_1 + n\alpha_2$. Doing this yields the polynomial

$$q^{\max(m-i,n)} + q^{\max(m-i,n)+1} + \dots + q^{m+n-2i}.$$

By accounting for the fact that $0 \leq i \leq \min(\lfloor \frac{m}{2} \rfloor, n)$ we arrive at the desired result

$$\varphi_q(m\alpha_1 + n\alpha_2) = \sum_{i=0}^{\min(\lfloor \frac{m}{2} \rfloor, n)} \binom{m+n-2i}{j=\max(m-i,n)} q^j. \quad \square$$

We now obtain a closed formula for Kostant's partition function on $\mathfrak{sp}_4(\mathbb{C})$.

Corollary 3.3. *If $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$ and $m, n \in \mathbb{N}$, then*

$$\varphi(m\alpha_1 + n\alpha_2) = \begin{cases} \left(\lfloor \frac{m}{2} \rfloor + 1\right) \left(m - \lfloor \frac{m}{2} \rfloor + 1\right) & \text{if } n \geq m, \\ \frac{(n+1)(n+2)}{2} & \text{if } m \geq 2n, \\ \frac{2mn - m^2 - n^2 + m + n}{2} + \lfloor \frac{m}{2} \rfloor \left(m - \lfloor \frac{m}{2} \rfloor\right) + 1 & \text{if } 2n > m > n. \end{cases}$$

Proof. Setting $q = 1$ into Equation (4) we find that

$$\begin{aligned} \varphi(m\alpha_1 + n\alpha_2) &= \left(\sum_{i=0}^{\min(\lfloor \frac{m}{2} \rfloor, n)} \min(m-i, n) \right) - \\ &\quad - \frac{1}{2} \min\left(\lfloor \frac{m}{2} \rfloor, n\right) \left(\min\left(\lfloor \frac{m}{2} \rfloor, n\right) + 1\right) + \min\left(\lfloor \frac{m}{2} \rfloor, n\right) + 1. \end{aligned} \quad (11)$$

We now consider each case individually. If $n \geq m$, then Equation (11) simplifies to $\left(\lfloor \frac{m}{2} \rfloor + 1\right) \left(m - \lfloor \frac{m}{2} \rfloor + 1\right)$. If $m \geq 2n$, then Equation (11) simplifies to $\frac{(n+1)(n+2)}{2}$. Finally, if $2n > m > n$, then Equation (11) yields

$$\left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \min(m-i, n) \right) - \frac{\lfloor \frac{m}{2} \rfloor \left(\lfloor \frac{m}{2} \rfloor + 1\right)}{2} + \lfloor \frac{m}{2} \rfloor + 1. \quad (12)$$

Let us consider the first term of expression (12). If $i \leq m - n$, then $n \leq m - i$ and hence $\min(m - i, n) = n$. Similarly, if $i > m - n$, then $n > m - i$ and hence $\min(m - i, n) = m - i$. Thus

$$\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \min(m-i, n) = \frac{2mn - m^2 - n^2 + m + n}{2} + m \lfloor \frac{m}{2} \rfloor - \frac{\lfloor \frac{m}{2} \rfloor \left(\lfloor \frac{m}{2} \rfloor + 1\right)}{2}. \quad (13)$$

Substituting Equation (13) into Equation (12) yields the desired result. \square

4. The q -analog of Kostant's weight multiplicity formula for $\mathfrak{sp}_4(\mathbb{C})$

Proposition 1.2 provided a closed formula for the q -analog of the partition function for the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$. We now use this formula to provide a proof of Theorem 1.1, but first we

must examine the partition function value input $\sigma(\lambda + \rho) - (\mu + \rho)$ for all $\sigma \in W$ as appearing in Equation (1). Throughout the rest of this section we let $\lambda = m\varpi_1 + n\varpi_2$ and $\mu = x\varpi_1 + y\varpi_2$ with $m, n, x, y \in \mathbb{N}$. To illustrate the computations $\sigma(\lambda + \rho) - (\mu + \rho)$ we consider the case when $\sigma = s_1$. Using Equations (5), (6), and (8) we find that

$$\begin{aligned} s_1(\lambda + \rho) - (\mu + \rho) &= \\ &= s_1 \left((m + n + 2)\alpha_1 + \left(\frac{m}{2} + n + \frac{3}{2}\right)\alpha_2 \right) - \left((x + y + 2)\alpha_1 + \left(\frac{x}{2} + y + \frac{3}{2}\right)\alpha_2 \right) \\ &= (n + 1)\alpha_1 + \left(\frac{m}{2} + n + \frac{3}{2}\right)\alpha_2 - \left((x + y + 2)\alpha_1 + \left(\frac{x}{2} + y + \frac{3}{2}\right)\alpha_2 \right) \\ &= (n - x - y - 1)\alpha_1 + \left(n - y + \frac{m - x}{2}\right)\alpha_2 \end{aligned}$$

Repeating this procedure with every element of the Weyl group generates the content of Tab. 2.

Table 2. Computing $\sigma(\lambda + \rho) - (\mu + \rho)$ for all $\sigma \in W$

σ	$\ell(\sigma)$	$\sigma(\lambda + \rho) - (\mu + \rho)$
1	0	$(m + n - x - y)\alpha_1 + (n - y + \frac{m-x}{2})\alpha_2$
s_1	1	$(n - x - y - 1)\alpha_1 + (n - y + \frac{m-x}{2})\alpha_2$
s_2	1	$(m + n - x - y)\alpha_1 + (-y - 1 + \frac{m-x}{2})\alpha_2$
s_1s_2	2	$(-n - x - y - 3)\alpha_1 + (-y - 1 + \frac{m-x}{2})\alpha_2$
s_2s_1	2	$(n - x - y - 1)\alpha_1 + (-y - 2 - \frac{m+x}{2})\alpha_2$
$s_1s_2s_1$	3	$(-m - n - x - y - 4)\alpha_1 + (-y - 2 - \frac{m+x}{2})\alpha_2$
$s_2s_1s_2$	3	$(-n - x - y - 3)\alpha_1 + (-n - y - 3 - \frac{m+2}{2})\alpha_2$
$s_1s_2s_1s_2$	4	$(-m - n - x - y - 4)\alpha_1 + (-n - y - 3 - \frac{m+2}{2})\alpha_2$

We now consider the q -analog of Kostant's partition function on the expressions $\sigma(\lambda + \rho) - (\mu + \rho)$ as listed in Tab. 2. We note that $m, n, x, y \in \mathbb{N}$ and that the q -analog of Kostant's partition function returns 0 if the coefficient of either α_1 or α_2 is negative or not an integer. Thus, the Weyl group elements s_1s_2 , s_2s_1 , $s_1s_2s_1$, $s_2s_1s_2$, and $s_1s_2s_1s_2$ never contribute to the q -analog of Kostant's weight multiplicity formula since at least one of the coefficients of α_1 or α_2 in the expression $\sigma(\lambda + \rho) - (\mu + \rho)$ will always be negative whenever $\lambda = m\varpi_1 + n\varpi_2$ and $\mu = x\varpi_1 + y\varpi_2$ with $m, n, x, y \in \mathbb{N}$. With this observation at hand we now present the proof of our main result Theorem 1.1.

Proof of Theorem 1.1. We need only consider the contribution of $(-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda + \rho) - (\mu + \rho))$ for the Weyl group elements $\sigma = 1, s_1$, and s_2 . Let $a = m + n - x - y$, $b = n - y + \frac{m-x}{2}$, $c = n - x - y - 1$, and $d = -y - 1 + \frac{m-x}{2}$. Then from Tab. 2 we note that

$$\begin{aligned} 1(\lambda + \rho) - (\mu + \rho) &= a\alpha_1 + b\alpha_2, \\ s_1(\lambda + \rho) - (\mu + \rho) &= c\alpha_1 + b\alpha_2, \\ s_2(\lambda + \rho) - (\mu + \rho) &= a\alpha_1 + d\alpha_2. \end{aligned}$$

By Proposition 1.2 if $\sigma = 1$ and $a, b \in \mathbb{N}$, then

$$P = (-1)^{\ell(1)} \wp_q(a\alpha_1 + b\alpha_2) = \sum_{i=0}^{\min(\lfloor \frac{a}{2} \rfloor, b)} \left(\sum_{j=\max(a-i, b)}^{a+b-2i} q^j \right). \quad (14)$$

Observe that if $a \notin \mathbb{N}$ or $b \notin \mathbb{N}$, then $\wp_q(a\alpha_1 + b\alpha_2) = 0$ and hence $P = 0$. Note that if $m, n, x, y \in \mathbb{N}$, then

$$-x - y - 1 < -y - \frac{x}{2} \iff n - x - y - 1 < n - y + \frac{m - x}{2} \iff c < b.$$

Thus, Proposition 1.2 allows us to compute the following: If $\sigma = s_1$ and $b, c \in \mathbb{N}$ with $c < b$, then

$$Q = (-1)^{\ell(s_1)} \wp_q(c\alpha_1 + b\alpha_2) = - \sum_{i=0}^{\lfloor \frac{c}{2} \rfloor} \left(\sum_{j=b}^{b+c} q^j \right). \quad (15)$$

However, if $b \notin \mathbb{N}$ or $c \notin \mathbb{N}$, then $\wp_q(c\alpha_1 + b\alpha_2) = 0$ and hence $Q = 0$. Finally, since $m, n, x, y \in \mathbb{N}$, we have

$$\frac{n - y}{2} > -y - 1 \iff \frac{n - y + m - x}{2} > -y - 1 + \frac{m - x}{2} \iff \frac{a}{2} > d.$$

Thus, Proposition 1.2 allows us to compute the following: If $\sigma = s_2$ and if $a, d \in \mathbb{N}$ with $a > 2d$, then

$$R = (-1)^{\ell(s_2)} \wp_q(a\alpha_1 + d\alpha_2) = - \sum_{i=0}^d \left(\sum_{j=a-i}^{a+d-2i} q^j \right). \quad (16)$$

Again, if $a \notin \mathbb{N}$ or $d \notin \mathbb{N}$, then $\wp_q(a\alpha_1 + d\alpha_2) = 0$ and hence $R = 0$. Equation (3) now follows from taking the sum of Equations (14)–(16). \square

Our last result follows from setting $q = 1$ in Equation (3) of Theorem 1.1 and using Corollary 3.3.

Corollary 4.1. *Let $\lambda = m\varpi_1 + n\varpi_2$ and $\mu = x\varpi_1 + y\varpi_2$ with $m, n, x, y \in \mathbb{N} := \{0, 1, 2, \dots\}$ be weights of $\mathfrak{sp}_4(\mathbb{C})$ and define $a = m + n - x - y$, $b = n - y + \frac{m - x}{2}$, $c = n - x - y - 1$, and $d = -y - 1 + \frac{m - x}{2}$. Then*

$$m(\lambda, \mu) = \begin{cases} P - Q - R & \text{if } a, b, c, d \in \mathbb{N}, \\ P - Q & \text{if } a, b, c \in \mathbb{N} \text{ and } d \notin \mathbb{N}, \\ P - R & \text{if } a, b, d \in \mathbb{N} \text{ and } c \notin \mathbb{N}, \\ P & \text{if } a, b \in \mathbb{N} \text{ and } c, d \notin \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

where

$$P = \begin{cases} \left(\left\lfloor \frac{a}{2} \right\rfloor + 1 \right) \left(a - \left\lfloor \frac{a}{2} \right\rfloor + 1 \right) & \text{if } b \geq a, \\ \frac{(b+1)(b+2)}{2} & \text{if } a \geq 2b, \\ \frac{2ab - a^2 - b^2 + a + b}{2} + \left\lfloor \frac{a}{2} \right\rfloor \left(a - \left\lfloor \frac{a}{2} \right\rfloor \right) + 1 & \text{if } 2b > a > b, \end{cases}$$

$$Q = \left\lfloor \frac{c+2}{2} \right\rfloor^2,$$

$$R = \frac{(d+1)(d+2)}{2}.$$

We end by providing a computational proof that $m_q(\tilde{\alpha}, 0) = q^1 + q^3$ where $\tilde{\alpha}$ is the highest root and 1 and 3 are the exponents of $\mathfrak{sp}_4(\mathbb{C})$.

Example 1. Let $\lambda = \tilde{\alpha} = 2\varpi_1$ and $\mu = 0$. Then $m = 2$, $n = x = y = 0$, and $a = m+n-x-y = 2$, $b = n - y + \frac{m-x}{2} = 1$, $c = n - x - y - 1 = -1$, and $d = -y - 1 + \frac{m-x}{2} = 0$. By Theorem 1.1 as $a, b, d \in \mathbb{N}$ and $c \notin \mathbb{N}$ we must compute P and R . Observe that

$$P = \sum_{i=0}^1 \left(\sum_{j=\max(2-i,1)}^{3-2i} q^j \right) = q + q^2 + q^3 \quad \text{and} \quad R = \sum_{i=0}^0 \left(\sum_{j=2}^{2-2i} q^j \right) = q^2.$$

Thus $m_q(\tilde{\alpha}, 0) = P - R = q + q^3$. Lastly, note that $m_q(\tilde{\alpha}, 0)|_{q=1} = 2$ recovers the rank of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$.

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Вес q -кратностей для представлений $\mathfrak{sp}_4(\mathbb{C})$

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США

В настоящей работе мы приводим замкнутую формулу для значений q -аналога функции обобщения Константа для алгебры Ли $\mathfrak{sp}_4(\mathbb{C})$ и используем этот результат, чтобы дать простую формулу для q -кратности веса в представлениях алгебры Ли $\mathfrak{sp}_4(\mathbb{C})$. Это обобщает работу Рефагата и Шахрияри в 2012 г., которые дали замкнутую формулу для кратности веса в представлениях алгебры Ли $\mathfrak{sp}_4(\mathbb{C})$.

Ключевые слова: симплектическая алгебра Ли, статистическая сумма Константа, q -аналог статистической суммы Константа, кратность веса, вес q -кратности.