Journal of Siberian Federal University. Mathematics & Physics 2017, 10(3), 385–395

УДК 517.9

On Local Solvability of the System of the Equations of One Dimensional Motion of Magma

Alexander A. Papin*
Margarita A. Tokareva†
Altai State University
Lenina, 61, Barnaul, 656049
Russia

Received 01.12.2016, received in revised form 20.01.2017, accepted 20.05.2017

The local solvability of initial-boundary value problem for the system of the equations of non stationary motion of magma is proved.

Keywords: Darcy law, poroelastisity, magma, solvability, uniqueness.
DOI: 10.17516/1997-1397-2017-10-3-385-395.

1. Problem statement. Formulation of main results

A quasi-linear system of equations of composite type is considered:

$$\frac{\partial(1-\phi)\rho_s}{\partial t} + \frac{\partial}{\partial x}((1-\phi)\rho_s v_s) = 0, \quad \frac{\partial(\rho_f\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho_f \phi v_f) = 0, \tag{1}$$

$$\phi(v_f - v_s) = -k(\phi)(\frac{\partial p_f}{\partial x} - \rho_f g), \tag{2}$$

$$\frac{\partial v_s}{\partial x} = -\frac{1}{\xi(\phi)} p_e, \quad p_e = p_{tot} - p_f, \tag{3}$$

$$\frac{\partial p_{tot}}{\partial x} = -\rho_{tot} g, \quad p_{tot} = \phi p_f + (1-\phi)p_s, \quad \rho_{tot} = \phi\rho_f + (1-\phi)\rho_s. \tag{4}$$

We seek a solution of this system in the domain $(x,t) \in Q_T = \Omega \times (0,T),\ \Omega = (0,1)$, under the boundary and initial conditions

$$v_s\mid_{x=0,x=1} = v_f\mid_{x=0,x=1} = 0, \qquad \phi\mid_{t=0} = \phi^0(x), \quad \rho_f\mid_{t=0} = \rho^0(x). \tag{5}$$

This quasi-linear system of equations describes 1D non-stationary isothermal motion of magma in porous rock. The laws of conservation of mass for each phase, Darcy's law for fluid phase, taking into account the motion of a solid skeleton, the rheological law and the equation of conservation of momentum for system describe this process [1–3]. Here ρ_f, ρ_s, v_f, v_s are, respectively, real density and velocity of solid and fluid phases, ϕ is the porosity, p_f, p_s are, respectively, pressures of the fluid and solid phases; p_e is the effective pressure, p_{tot} is the total pressure, ρ_{tot} is the density of the two-phase medium, g is the density of the mass forces; $k(\phi)$ is the coefficient of filtration, $\xi(\phi)$ is the coefficient of rock shear viscosity (specified function).

*papin@math.asu.ru
†tma25@mail.ru

The problem is written in the Eulerian coordinates x, t. The real density of the solid particles ρ_s is assumed constant. The unknown quantities are ϕ, ρ_f, v_f, v_s p_f, p_s. The system of equations (1)–(4) is closed either by using the equation of state of the fluid phase $p_f = p(\rho_f)$ (in particular, the relationship may be, commonly used in applications $\dfrac{dp_f}{d\rho_f} = \dfrac{1}{\beta_f \rho_f}$, where β_f is the fluid compressibility [1–3]).

The numerical studies of various initial boundary-value problems for systems of equations (1)–(4) were carried out in [2, 3]. Some exact solutions have been constructed in [4]. In these studies the following dependencies of the functional parameters of the problem was used: $k(\phi) = \bar{k}\phi^n/\mu$, $1/\xi(\phi) = \phi^m/\nu$, where $m \in [0, 2]$, $n = 3$; ν, μ, $\bar{k}$ are positive environment settings [2].

Structurally similar systems of equations was considered in [5–7]. In these studies, based on a number of simplifying assumptions, the original system were reduced to one higher order equation. The local solvability of the Cauchy problem in Sobolev spaces was established in [5]. Travelling wave solutions have been studied in [6, 7].

In this paper the unique local solvability of problem (1)–(5) is proved in the case when $g = 0$ and ρ_f is function of pressure.

On Ω and Q_T, let us consider several function spaces, using the notation from [8]. Suppose that $||\cdot||_{q,\Omega}$ is the norm on the Lebesgue space $L_q(\Omega)$, $q \in [1,\infty]$. For brevity, let $||\cdot||_q = ||\cdot||_{q,\Omega}$, $||\cdot|| = ||\cdot||_{2,\Omega}$ .We also use the Hölder spaces $C^\alpha(\Omega)$, $C^{k+\alpha}(\Omega)$, where k is a natural number and $\alpha \in (0, 1]$ with norms:

$||f||_{C^\alpha(\Omega)} \equiv |f|_{\alpha,\Omega} = |f|_{0,\Omega} + H^\alpha_x(f)$, $|f|_{0,\Omega} = \max\limits_{x\in\overline{\Omega}} |f(x)|$,

$H^\alpha_x(f) = \sup\limits_{x_1,x_2\in\Omega} |f(x_1) - f(x_2)||x_1 - x_2|^{-\alpha}$,

$||f||_{C^{k+\alpha}(\Omega)} \equiv |f|_{k+\alpha,\Omega} = \sum\limits_{m=0}^{k} ||D^m_x f||_{0,\Omega} + H^\alpha(D^k_x f)$.

For functions given on Q_T, we need the space $C^{k+\alpha,m+\beta}(Q_T)$, where k, m are natural numbers and $(\alpha, \beta) \in (0, 1]$, with norm $||f||_{C^{k+\alpha,m+\beta}(Q_T)} \equiv |f|_{k+\alpha,m+\beta,Q_T} = \sum\limits_{l=0}^{k} ||D^l_x f||_{0,Q_T} + \sum\limits_{j=1}^{m} ||D^j_t f||_{0,Q_T} + H^\alpha_x(D^k_x f) + +H^\beta_t(D^k_x f) + H^\alpha_x(D^m_t f) + H^\beta_t(D^m_t f)$, where

$H^\alpha_x(f(x,t)) = \sup\limits_{x_1,x_2\in\Omega, t\in(0,T)} |f(x_1,t) - f(x_2,t)||x_1 - x_2|^{-\alpha}$,

$H^\beta_t(f(x,t)) = \sup\limits_{t_1,t_2\in(0,T), x\in\Omega} |f(x,t_1) - f(x,t_2)||t_1 - t_2|^{-\beta}$.

In the case $k = m$ and $\alpha = \beta$, we use the notation $C^{k+\alpha}(Q_T)$.

In this paper by a solution of problem (1)–(5) we mean the set of functions $v_s \in C^{3+\alpha,1+\alpha/2}(Q_T)$ $(\phi,\, \rho_f,\, p_f,\, p_s) \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $v_f \in C^{1+\alpha,1+\alpha/2}(Q_T)$, such that $0 < \phi < 1$, $\rho_f > 0$, $p_f > 0$. These functions satisfy the equations (1)–(4) and the initial and boundary conditions (5) and regarded as continuous functions in $\overline{Q}_T$.

Let us state the main results of the paper.

Theorem 1. *Suppose that $g = 0$ and the data of problem (1)–(5) satisfies the following conditions:*
1) the functions $k(\phi), \xi(\phi), p_f(\rho_f)$ and their derivatives up to the second order are continuous for $\phi \in (0, 1), \rho_f > 0$, and satisfy the conditions

$$k_0^{-1}\phi^{q_1}(1-\phi)^{q_2} \leqslant k(\phi) \leqslant k_0\phi^{q_3}(1-\phi)^{q_4},\quad 1/\xi(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2-1},\quad 0 < R_1 \leqslant a_0(\phi) \leqslant R_2,$$

$$k_0^{-1}\rho_f^{q_5} \leqslant p_f(\rho_f) \leqslant k_0\rho_f^{q_6},\quad k_0^{-1}\rho_f^{q_7} \leqslant \frac{\partial p_f(\rho_f)}{\partial\rho_f} \leqslant k_0\rho_f^{q_8},$$

where $k_0, \alpha_i, R_i, i = 1, 2$ are positive constants, $q_1, ..., q_8$ are fixed real parameters;
2) the initial functions ϕ^0, ρ^0 satisfy the following smoothness conditions: $\phi^0 \in C^{2+\alpha}(\overline{\Omega}), \rho^0 \in C^{2+\alpha}(\overline{\Omega})$ and the matching conditions

$$\frac{dp_f(\rho^0)}{dx} \mid_{x=0,x=1} = 0,$$

as well as satisfy the inequalities

$$0 < m_0 \leqslant \phi^0(x) \leqslant M_0 < 1, \quad 0 < m_1 \leqslant \rho^0(x) \leqslant M_1 < \infty, \quad x \in \overline{\Omega},$$

where m_0, M_0, m_1, M_1 are given positive constants. Then problem (1)–(5) *has a local solution, i.e., there exists a value of $t_0 \in (0, T)$ such that*

$$v_s(x,t) \in C^{3+\alpha,1+\alpha/2}(\overline{Q}_{t_0}), \ (\phi(x,t), p_s(x,t), p_f(x,t), \rho_f(x,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}),$$

$$v_f(x,t) \in C^{1+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

Moreover, $0 < \phi(x,t) < 1, \ \rho_f(x,t) > 0$ в $\overline{Q}_{t_0}$.

2. Local solvability

Under the conditions of the theorem into force (4) we have $p_{tot} = p^0(t)$. Following [9], we rewrite the system (1)–(3). Suppose that $\bar{x} = \bar{x}(\tau, x, t)$ is a solution of the Cauchy problem

$$\frac{\partial \bar{x}}{\partial \tau} = v_s(\bar{x}, \tau), \quad \bar{x} \mid_{\tau=t} = x.$$

We set $\hat{x} = \bar{x}(0, x, t)$ and take $\hat{x}$ and t for the new variables. Then $1 - \phi(\hat{x}, t) = (1 - \phi^0(\hat{x}))\hat{J}(\hat{x}, t)$, where $\hat{J}(\hat{x}, t) = \frac{\partial \hat{x}}{\partial x}(\hat{x}, t)$ is the Jacobian of the transformation. The system of equations (1)–(3) in the new variables is of the form

$$\frac{\partial(1-\hat{\phi})}{\partial t} + \frac{(1-\hat{\phi})^2}{1-\phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = 0, \quad \frac{\partial}{\partial t}(\hat{\rho}_f\hat{\phi}) + \frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}\hat{v}_f) = v_s\frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}),$$

$$\hat{\phi}(\hat{v}_s - \hat{v}_f) = k(\hat{\phi})\frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial \hat{p}_f}{\partial \hat{x}}, \quad \frac{(1-\hat{\phi})}{1-\phi^0}\frac{\partial \hat{v}_s}{\partial \hat{x}} = -a_1(\hat{\phi})\hat{p}_e,$$

where $a_1(\phi) = 1/\xi(\phi)$.

Since

$$v_s\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}) = \frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}v_s) - \hat{\rho}_f\hat{\phi}\frac{\partial v_s}{\partial \hat{x}},$$

it follows that the continuity equation for the liquid phase can be reduced to the form

$$\frac{1}{(1-\hat{\phi})}\frac{\partial}{\partial t}(\hat{\rho}_f\hat{\phi}) + \frac{1}{1-\phi^0}\frac{\partial}{\partial \hat{x}}(\hat{\rho}_f\hat{\phi}(\hat{v}_f - v_s)) + \frac{1}{1-\phi^0}\hat{\rho}_f\hat{\phi}\frac{\partial v_s}{\partial \hat{x}} = 0.$$

Using the continuity equation for the solid phase, we find that

$$\frac{\partial}{\partial t}\Big(\hat{\rho}_f\frac{\hat{\phi}}{1-\hat{\phi}}\Big) + \frac{1}{(1-\phi^0)}\frac{\partial}{\partial \hat{x}}\big(\hat{\rho}_f\hat{\phi}(\hat{v}_f - \hat{v}_s)\big) = 0.$$

Finally, passing from $(\hat{x}, t)$ to the mass Lagrangian variables (y, t) by the rule

$$\big(1-\phi^0(\hat{x})\big)d\hat{x} = dy, \quad y(\hat{x}) = \int_0^{\hat{x}} \big(1-\phi^0(\eta)\big)d\eta \in [0,1]$$

and preserving the notation y for the variable x, we obtain

$$\frac{\partial(1-\phi)}{\partial t}+(1-\phi)^2\frac{\partial v_s}{\partial x}=0,\quad \frac{\partial}{\partial t}\left(\rho_f\frac{\phi}{1-\phi}\right)+\frac{\partial}{\partial x}(\rho_f\phi(v_f-v_s))=0,$$

$$\phi(v_s-v_f)=k(\phi)(1-\phi)\frac{\partial p_f}{\partial x},$$

$$(1-\phi)\frac{\partial v_s}{\partial x}=-a_1(\phi)p_e,\quad p_e=p^0(t)-p_f.$$

Finally, we turn to the dimensionless variables

$$t'=\frac{t}{t_1},\quad x'=\frac{x}{L},\quad v_s'=\frac{v_s}{v_1},\quad v_f'=\frac{v_f}{v_1},\quad \rho_f'=\frac{\rho_f}{\rho_s},$$

$$p_f'=\frac{p_f}{p_1},\quad p_s'=\frac{p_s}{p_1},\quad p_e'=\frac{p_e}{p_1},\quad p_{tot}'=\frac{p_{tot}}{p_1},\quad a_1'(\phi)=\frac{a_1(\phi)}{a^0},\quad k'(\phi)=\frac{k(\phi)}{k_1},$$

where $L=\int_0^1(1-\phi^0(\eta))d\eta$, $t_1=\frac{L}{v_1}$, $a^0=\frac{v_1}{Lp_1}$, $k_1=\frac{v_1 L}{p_1}$, v_1,p_1 are fixed positive quantities having the dimension of velocity and pressure accordingly.

Then the domain x' is the unit interval $[0,1]$ and the system of equations will retain its structure (dashes omitted).

Using the rheological relationship, Darcy's law and the conditions $v_s|_{x=0,1}=0$, we find that

$$p^0(t)=\int_0^1\frac{a_1(\phi)}{1-\phi}p_f dx\left(\int_0^1\frac{a_1(\phi)}{1-\phi}dx\right)^{-1}\equiv P^0(\phi,\rho_f).$$

Taking into account Darcy's law, the second equation of the system assumes the form

$$\frac{\partial}{\partial t}\Big(\rho_f\frac{\phi}{1-\phi}\Big)-\frac{\partial}{\partial x}\Big(\rho_f k(\phi)(1-\phi)\frac{\partial p_f}{\partial x}\Big)=0.$$

From the first and fourth equations of the system follows that

$$\frac{1}{1-\phi}\frac{\partial\phi}{\partial t}=a_1(\phi)(p_f-p^0).$$

This equation can be written as

$$\frac{\partial G(\phi)}{\partial t}=p_f-p^0,$$

where the function $G(\phi)$ is defined by the equation

$$\frac{dG(\phi)}{d\phi}=\frac{1}{(1-\phi)a_1(\phi)}.$$

Let

$$a(\phi)=\frac{\phi}{1-\phi},\quad K(\phi)=k(\phi)(1-\phi),\quad b(\rho_f)=\rho_f\frac{\partial p_f(\rho_f)}{\partial\rho_f}.$$

Taking into account the conditions (5), we obtain the following problem for finding functions ρ_f,ϕ:

$$\frac{\partial}{\partial t}\big(a(\phi)\rho_f\big)-\frac{\partial}{\partial x}\Big(K(\phi)b(\rho_f)\frac{\partial\rho_f}{\partial x}\Big)=0, \tag{6}$$

$$\frac{\partial G(\phi)}{\partial t}=p_f(\rho_f)-p^0(t), \tag{7}$$

$$\frac{\partial\rho_f}{\partial x}\mid_{x=0,x=1}=0,\quad \rho_f\mid_{t=0}=\rho^0(x),\quad \phi\mid_{t=0}=\phi^0(x). \tag{8}$$

Lemma 1. *Let the data of problem* (6)–(8) *satisfy the conditions of the theorem. Then problem* (6)–(8) *has a unique local solution, i.e., there exists a value of* t_0 *such that*

$$(\phi(x,t), \rho_f(x,t)) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}).$$

Furthermore, $0 < \phi(x,t) < 1$, $\rho_f(x,t) > 0$ *in* $\overline{Q}_{t_0}$.

The solvability of problem (6)–(8) is established by using the Tikhonov- Schauder fixed-point theorem: if V is a compact convex closed set of Banach space B and the operator Λ maps V into itself continuously in the norm of B, then there is a fixed point on V [10, pp. 227].

Since the function $\psi = G(\phi)$ is strictly monotone, at $\phi \in (0,1)$, that the inverse function is exist: $\phi = G^{-1}(\psi)$. Assuming that $\rho(x,t) = \rho_f(x,t) - \rho^0(x)$, $\omega(x,t) = G(\phi) - G(\phi^0)$. We represent the equations (6),(7) in the form

$$\frac{\partial}{\partial t}\left(a(\omega)(\rho+\rho^0)\right) = \frac{\partial}{\partial x}\left(K(\omega)b(\rho+\rho^0)\frac{\partial(\rho+\rho^0)}{\partial x}\right), \tag{9}$$

$$\frac{\partial \omega}{\partial t} = p_f(\rho+\rho^0) - p^0(t). \tag{10}$$

Here $a(\omega) = \dfrac{\phi(\omega)}{1-\phi(\omega)}$, $K(\omega) = k(\phi(\omega))(1-\phi(\omega))$, $\phi(\omega) = G^{-1}(\omega + G(\phi^0))$. Moreover,

$$\rho\mid_{t=0} = \omega\mid_{t=0} = \frac{\partial(\rho+\rho^0)}{\partial x}\Big|_{x=0,x=1} = 0. \tag{11}$$

For the Banach space, we choose the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$, where β is any number from the interval $(0,\alpha)$, $\alpha \in [0,1)$. Let

$$V = \Big\{\bar\rho(x,t), \bar\omega(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_{t_0}) | \quad \bar\rho\mid_{t=0} = \bar\omega\mid_{t=0} = \frac{\partial\bar\rho}{\partial x}\Big|_{x=0,x=1} = 0,$$

$$\frac{m_1}{2} - \rho^0(x) \leqslant \bar\rho(x,t) \leqslant 2M_1 - \rho^0(x) < \infty,$$

$$G(m_0/2) - G(\phi^0) \leqslant \bar\omega(x,t) \leqslant G(\frac{M_0+1}{2}) - G(\phi^0) < \infty, \quad (x,t) \in Q_{t_0},$$

$$(|\bar\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}}, |\bar\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}}) \leqslant K_1,\ (|\bar\omega|_{2+\alpha,(2+\alpha)/2,Q_{t_0}}, |\bar\rho|_{2+\alpha,(2+\alpha)/2,Q_{t_0}}) \leqslant K_1 + K_2\Big\},$$

where K_1 is an arbitrary positive constant, while the positive constant K_2 will be given later. We note that on the set V following inequalities hold: $0 < \dfrac{m_0}{2} \leqslant \phi(\bar\omega) \leqslant \dfrac{M_0+1}{2} < 1$, $a(\bar\omega) > 0$, $K(\bar\omega) > 0$.

Let us construct an operator Λ mapping V in V. Suppose that $\bar\omega, \bar\rho \in V$. Using (10), we define the function ω by the equality

$$\omega = \int_0^t \left(p_f(\bar\rho(x,\tau)+\rho^0(x)) - \int_0^1 \frac{a_1(\phi(\bar\omega))}{1-\phi(\bar\omega)} p_f(\bar\rho(x,\tau)+\rho^0(x))dx \left(\int_0^1 \frac{a_1(\phi(\bar\omega))}{1-\phi(\bar\omega)}dx\right)^{-1}\right) d\tau. \tag{12}$$

From the representation (12) it follows that smoothness ω is determined by the smoothness of functions $\bar\rho, \rho^0$ and p^0. In particular, we have an estimate

$$|\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}} = C_1(m_0, M_0, m_1, M_1, K_1, T, |\rho^0|_{2+\alpha,\Omega})(1 + t_0|\bar\rho_{xx}|_{\alpha,\alpha/2,\Omega}).$$

Lemma 2. *Let function $a_1(\phi)$, $\phi \in (0,1)$ satisfies the following condition*

$$(1-\phi)a_1(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2}, \quad 0 < R_1 \leqslant a_0(\phi) \leqslant R_2,$$

where $R_i > 0$, $\alpha_i > 0$, $i = 1, 2$. Then we have the estimate of the form

$$R_2|G(\phi_1) - G(\phi_2)| \geqslant |\phi_1 - \phi_2|.$$

Proof. Assume without loss of generality that $0 < \phi_1 \leqslant \phi_2 < 1$. From the definition of functions $G(\phi)$ and $a_1(\phi)$, we have

$$0 < \Delta G \equiv G(\phi_2) - G(\phi_1) = \int_{\phi_1}^{\phi_2} \frac{ds}{(1-s)a_1(s)} \geqslant \frac{1}{R_2}(\phi_2 - \phi_1).$$

Lemma 2 is proved. □

In this way, we have estimate

$$|\phi(x,t) - \phi^0(x)| \leqslant \delta(t), \quad \delta(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0,$$

which implies, that there exists a value $t_1 = t_1(m_0, M_0, m_1, M_1)$, such that for all $t_0 \leqslant t_1$ the following inequality holds

$$0 < \frac{m_0}{2} \leqslant \phi(x,t) \leqslant \frac{M_0+1}{2}, \quad (x,t) \in Q_{t_0}. \tag{13}$$

Taking into account (13) we also have the estimate for function $\omega(x,t)$: $G\left(\frac{m_0}{2}\right) \leqslant \omega(x,t) + G(\phi^0) \leqslant G\left(\frac{M_0+1}{2}\right)$.

Using (9), (11) and $\bar{\omega}(x,t)$ we find the function $\rho(x,t)$ as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):

$$\begin{aligned} &\frac{\partial}{\partial t}\big(a(\bar{\omega})(\rho+\rho^0)\big) = \frac{\partial}{\partial x}\Big(K(\bar{\omega})b(\bar{\rho})\frac{\partial(\rho+\rho^0)}{\partial x}\Big), \\ &\rho\,|_{t=0} = \frac{\partial \rho_f}{\partial x}\Big|_{x=0,x=1} = 0, \quad \frac{\partial \rho^0}{\partial x}\Big|_{x=0,x=1} = 0. \end{aligned} \tag{14}$$

The equation for $\rho(x,t)$ is uniformly parabolic. In view of the properties of $\bar{\omega}(x,t)$ and $\rho^0(x)$ problem (14) has a classical solution [8]. In addition, we have the following estimate:

$$\Big|\frac{1}{a(\bar{\omega})}\frac{\partial a(\bar{\omega})}{\partial t}\Big| \leqslant C_0(m_0, M_0, m_1, M_1, \max_{0\leqslant t\leqslant T}|p^0(t)|).$$

Under the additional condition smallness for the value of the time interval the following statement holds [9].

Lemma 3. *For $t_0 \leqslant \min(t_1, t_2)$, $\quad t_2 = \ln 2/C_0(m_0, M_0, m_1, M_1)$, the classical solution of problem (14) satisfies the following inequality in Q_{t_0} :*

$$0 < \frac{m_1}{2} \leqslant \rho(x,t) + \rho^0(x) \leqslant 2M_1 < \infty.$$

Proof. Further, setting $U(x,t) = \rho(x,t) + \rho^0(x)$, we can express problem (14) in the form

$$\frac{\partial}{\partial t}\Big(a(\bar{\omega})U\Big) = \frac{\partial}{\partial x}\Big(K(\bar{\omega})b(\bar{\rho})\frac{\partial U}{\partial x}\Big), \quad \frac{\partial U}{\partial x}\Big|_{x=0,x=1} = 0, \quad U\big|_{t=0} = \rho^0. \tag{15}$$

First, we show that $U(x,t) \geqslant 0$, $(x,t) \in Q_{t_0}$. In equation (15), let us make the change $U(x,t) = -z(x,t)$. Then

$$z\frac{\partial a}{\partial t} + a\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}(Kb\frac{\partial z}{\partial x}).$$

Let

$$z^{(0)}(x,t) = \max\{z,0\}, \quad z^{(0)}(x,t)\mid_{t=0} = \max\{-\rho^0, 0\} = 0,$$
$$\sigma_\varepsilon(x,t) = z^{(0)}(x,t)(|z^{(0)}(x,t)|^2 + \varepsilon)^{-1/2}, \ \varepsilon > 0.$$

Let us multiply the equation for the function z by σ_ε and then integrate over Ω. We obtain the equality

$$\begin{gathered}\frac{d}{dt}\int_0^1 a(|z^{(0)}|^2+\varepsilon)^{1/2}dx + \int_0^1 \frac{\partial a}{\partial t}(z\sigma_\varepsilon - (|z^{(0)}|^2+\varepsilon)^{1/2})dx + \\ +\varepsilon\int_0^1 Kb\frac{\partial z}{\partial x}\frac{\partial z^{(0)}}{\partial x}(|z^{(0)}|^2+\varepsilon)^{-3/2}dx = 0.\end{gathered} \tag{16}$$

Let $A^+(t) = \{x \in \Omega |\, z(x,t) > 0\}$, $A^-(t) = \{x \in \Omega |\, z(x,t) \leqslant 0\}$. Then

$$\int_0^1 \frac{\partial a}{\partial t}(z\sigma_\varepsilon - (|z^{(0)}|^2+\varepsilon)^{1/2})dx = -\varepsilon\int\limits_{A^+(t)}\frac{\partial a}{\partial t}(|z|^2+\varepsilon)^{-1/2}dx - \varepsilon^{1/2}\int\limits_{A^-(t)}\frac{\partial a}{\partial t}dx,$$

$$\int_0^1 a(|z^{(0)}|^2+\varepsilon)^{1/2}dx = \int\limits_{A^+(t)} a(|z|^2+\varepsilon)^{1/2}dx + \varepsilon^{1/2}\int\limits_{A^-(t)} adx,$$

$$\int_0^1 a(|z^{(0)}|^2+\varepsilon)^{1/2}\big|_{t=0}dx = \varepsilon^{1/2}\int_0^1 a\big|_{t=0}dx, \quad \int\limits_{A^+(t)} a(|z|^2+\varepsilon)^{1/2}dx \geqslant \int\limits_{A^+(t)} a|z|dx = \int_0^1 az^{(0)}dx.$$

Integrating relation (16) with respect to time, we obtain

$$\int\limits_{A^+(t)} a(|z|^2+\varepsilon)^{1/2}dx + \varepsilon^{1/2}\int\limits_{A^-(t)} adx + \varepsilon\int\limits_0^t\int\limits_{A^+(\tau)} Kb|\frac{\partial z}{\partial x}|^2(z^2+\varepsilon)^{-3/2}dxd\tau =$$

$$= \varepsilon\int_0^t\int\limits_{A^+(\tau)}\frac{\partial a}{\partial \tau}(|z|^2+\varepsilon)^{-1/2}dxd\tau + \varepsilon^{1/2}\int_0^t\int\limits_{A^-(\tau)}\frac{\partial a}{\partial \tau}dxd\tau + \varepsilon^{1/2}\int_0^1 a\mid_{t=0} dx.$$

Therefore,

$$\int_0^1 az^{(0)}dx \leqslant \varepsilon^{1/2}\int_0^t\int_0^1\Big|\frac{\partial a}{\partial \tau}\Big|dxd\tau + \varepsilon^{1/2}\int_0^1 a\big|_{t=0}dx.$$

Passing to the limit as $\varepsilon \to 0$, we find that $z^{(0)} = 0$, i.e. $U \geqslant 0$.

After multiplication by $U^{l-1}(x,t), l > 2$, equation (15) can be expressed as

$$\frac{1}{l}\frac{\partial(aU^l)}{\partial t} + (l-1)KbU^{l-2}\Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{l-1}{l}U^l\frac{\partial a}{\partial t} = \frac{\partial}{\partial x}\Big(KbU^{l-1}\frac{\partial U}{\partial x}\Big).$$

Then

$$\frac{1}{l}\frac{d}{dt}\int_0^1 aU^l dx \leqslant \frac{l-1}{l}\max_{0\leqslant x\leqslant 1}\Big|\frac{1}{a}\frac{\partial a}{\partial t}\Big|\int_0^1 aU^l dx.$$

Therefore,

$$y'(t) \leqslant \frac{l-1}{l}\max_{0\leqslant x\leqslant 1}\Big|\frac{1}{a}\frac{\partial a}{\partial t}\Big| y(t), \quad y^l(t) = \int_0^1 (a^{1/l}U)^l dx,$$

$$y(t) \leqslant y(0)\exp\Big\{\frac{l-1}{l}\int_0^t \max_{0\leqslant x\leqslant 1}\Big|\frac{1}{a}\frac{\partial a}{\partial t}\Big| d\tau\Big\}.$$

After passing to the limit as $l \to \infty$, we obtain

$$\max_{0\leqslant x\leqslant 1} U(x,t) \leqslant \max_{0\leqslant x\leqslant 1}\rho^0(x)\exp\Big\{\int_0^t \max_{0\leqslant x\leqslant 1}|\frac{1}{a}\frac{\partial a}{\partial t}|d\tau\Big\}.$$

Taking into account the inequality $\max\limits_{0\leqslant x\leqslant 1}\rho^0(x) \leqslant M_1$ and choosing t from the condition

$$t \leqslant t_2, \quad \exp\Big\{\int_0^{t_2}\max_{0\leqslant x\leqslant 1}|\frac{1}{a}\frac{\partial a}{\partial t}|d\tau\Big\} \leqslant 2,$$

we obtain upper bound for ρ. To obtain a lower estimate we represent equation (15) in the form $(z(x,t) = 1/U(x,t))$

$$\frac{1}{l}\frac{\partial(az^l)}{\partial t} + (l+1)Kbz^{l-2}\Big(\frac{\partial z}{\partial x}\Big)^2 - \frac{l+1}{l}z^l\frac{\partial a}{\partial t} = \frac{\partial}{\partial x}\Big(Kbz^{l-1}\frac{\partial z}{\partial x}\Big).$$

Then we obtain inequality

$$\frac{1}{l}\frac{d}{dt}\int_0^1 az^l dx \leqslant \frac{l+1}{l}\max_{0\leqslant x\leqslant 1}\Big|\frac{1}{a}\frac{\partial a}{\partial t}\Big|\int_0^1 az^l dx,$$

then we obtain the estimate

$$\max_{0\leqslant x\leqslant 1}\frac{1}{U(x,t)} \leqslant \max_{0\leqslant x\leqslant 1}\frac{1}{\rho^0(x)}\exp\Big\{\int_0^t \max_{0\leqslant x\leqslant 1}\Big|\frac{1}{a}\frac{\partial a}{\partial t}\Big|d\tau\Big\} \leqslant \frac{2}{m_1}.$$

Lemma 3 is proved. □

In view of Lemma 3 and the properties of $\bar\omega$, we have the following estimates [8, Sec. 3]:

$$|\rho|_{\alpha,\alpha/2,Q_{t_0}} \leqslant C_2,$$

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leqslant C_3\left(1 + |\rho^0|_{2+\alpha,\Omega} + |\bar\rho_x|_{\alpha,\alpha/2,Q_{t_0}} + |\bar\omega_t|_{\alpha,\alpha/2,Q_{t_0}} + |\bar\omega_x|_{\alpha,\alpha/2,Q_{t_0}}\right),$$

in which the constant C_2, C_3 depends on K_1, m_0, m_1, M_0, M_1. Therefore

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leqslant C_4(K_1, m_0, m_1, M_0, M_1).$$

Let $C_5 = \max\{C_1, C_4\}$. Choose K_2 so that $C_5 \leqslant \dfrac{K_1+K_2}{2}$. Then, for $t_0 < \min(t_1, t_2, (K_1 + K_2)^{-1})$ we obtain

$$|\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leqslant K_1 + K_2, \quad |\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leqslant K_1 + K_2.$$

It remains to verify conditions

$$|\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leqslant K_1, \quad |\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leqslant K_1.$$

Integrating equation (14) with respect to time, we obtain $|\rho|_{0,Q_{t_0}} \leqslant C_6 t_0$. From the equation (12) we obtain $|\omega|_{0,Q_{t_0}} \leqslant C_7 t_0$. Further, using for ρ, ω an inequality of the form [11, pp. 35]

$$|u|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leqslant C|u|^{c}_{2+\alpha,1+\alpha/2,Q_{t_0}} |u|^{1-c}_{0,Q_{t_0}}, \quad c = (1+\alpha)(2+\alpha)^{-1},$$

we find that there exists a sufficiently small value of t_0, depending on K_1 and K_2, such that the required estimates hold: $|\rho|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leqslant K_1$, $|\omega|_{1+\alpha,(1+\alpha)/2,Q_{t_0}} \leqslant K_1$.

Thus, the operator Λ maps the set V into itself for sufficiently small values of t_0. Using the estimates obtained above, we can easily show the continuity of the operator Λ in the norm of the space $C^{2+\beta,1+\beta/2}(\overline{Q}_{t_0})$. By the Tikhonov-Schauder theorem, there exists a fixed point $(\rho,\omega) \in V$ of the operator Λ.

Let us establish uniqueness of the solution of problem (6)–(8).

Suppose that $(\rho_f^{(1)}, \phi^{(1)})$ and $(\rho_f^{(2)}, \phi^{(2)})$ are two different solutions of problem. Their difference $\rho = \rho_f^{(1)} - \rho_f^{(2)}, \phi = \phi^{(1)} - \phi^{(2)}$ is the solution of the linear homogeneous system

$$\frac{\partial}{\partial t}(d_0\rho + d_1\phi) - \frac{\partial}{\partial x}\left(d_2\frac{\partial \rho}{\partial x} + d_3\rho + d_4\phi\right) = 0, \tag{17}$$

$$\frac{\partial}{\partial t}(h_0\phi) - h_1\rho + V(t) = 0, \tag{18}$$

$$V(t) = P^0\big(\phi^{(1)}, \rho_f^{(1)}\big) - P^0\big(\phi^{(2)}, \rho_f^{(2)}\big) = \int_0^1 \big(h_2(x,t)\rho(x,t) + h_3(x,t)\phi(x,t)\big)dx,$$

with zero initial and boundary conditions $\phi\,|_{t=0} = \rho\,|_{t=0} = \left|\dfrac{\partial \rho}{\partial x}\right|_{x=0,x=1} = 0$.

The coefficients

$$d_0 = a\big(\phi^{(1)}\big) > 0, \quad d_1 = \frac{\big(a(\phi^{(1)}) - a(\phi^{(2)})\big)\rho_f^{(2)}}{\phi^{(1)} - \phi^{(2)}} > 0, \quad d_2 = K\big(\phi^{(2)}\big) b\big(\rho_f^{(2)}\big) > 0,$$

$$d_3 = K\big(\phi^{(1)}\big)\frac{b\big(\rho_f^{(1)}\big) - b\big(\rho_f^{(2)}\big)}{\rho_f^{(1)} - \rho_f^{(2)}}\frac{\partial \rho_f^{(1)}}{\partial x}, \quad d_4 = b\big(\rho_f^{(2)}\big)\frac{K\big(\phi^{(1)}\big) - K\big(\phi^{(2)}\big)}{\phi^{(1)} - \phi^{(2)}}\frac{\partial \rho_f^{(1)}}{\partial x},$$

$$h_0 = \frac{G(\phi^{(1)}) - G(\phi^{(2)})}{\phi^{(1)} - \phi^{(2)}} > 0, \quad h_1 = \frac{p(\rho_f^{(1)}) - p(\rho_f^{(2)})}{\rho_f^{(1)} - \rho_f^{(2)}},$$

$$h_2 = \frac{a_1(\phi^{(1)})}{1 - \phi^{(1)}}\frac{p_f(\rho_f^{(1)}) - p_f(\rho_f^{(2)})}{\rho_f^{(1)} - \rho_f^{(2)}}\left(\int_0^1 \frac{a_1(\phi^{(1)})}{1 - \phi^{(1)}}dx\right)^{-1},$$

$$h_3 = \left(\frac{a_1(\phi^{(1)})}{1 - \phi^{(1)}} - \frac{a_1(\phi^{(2)})}{1 - \phi^{(2)}}\right)\big(\phi^{(1)} - \phi^{(2)}\big)^{-1}\times$$

$$\times\left(p_f(\rho_f^{(2)})\left(\int_0^1 \frac{a_1(\phi^{(1)})}{1 - \phi^{(1)}}dx\right)^{-1} - \int_0^1 \frac{a_1\,(\phi^{(2)})}{1 - \phi^{(2)}}p_f(\rho_f^{(2)})dx\left(\int_0^1 \frac{a_1(\phi^{(1)})}{1 - \phi^{(1)}}dx\int_0^1 \frac{a_1(\phi^{(2)})}{1 - \phi^{(2)}}dx\right)^{-1}\right)$$

are bounded for all $x \in [0,1]$, $t \in [0,T]$.

Taking into account (18), equation (17) can be represented as

$$\frac{\partial}{\partial t}(d_0\rho) + \frac{d_1}{h_0}\big(h_1\rho - V(t)\big) + h_0\phi\frac{\partial}{\partial t}\left(\frac{d_1}{h_0}\right) - \frac{\partial}{\partial x}\left(d_2\frac{\partial \rho}{\partial x} + d_3\rho + d_4\phi\right) = 0. \tag{19}$$

Multiplying the equation (19) by $\rho(x,t)$ and integrating by x from 0 to 1, we obtain

$$\frac{d}{dt}\int_0^1 \rho_1^2(x,t)dx \leqslant C\Big(\int_0^1 \rho_1^2(x,t)dx + \int_0^1 u^2(x,t)dx + V^2(t)\Big), \tag{20}$$

where $\rho_1(x,t) = d_0^{1/2}|\rho(x,t)|$, $u(x,t) = h_0\phi(x,t)$. Here the constant C depends on T and quantities

$$\max_{(x,t)\in Q_T}\frac{1}{\phi^{(i)}(x,t)},\quad \max_{(x,t)\in Q_T}\frac{1}{1-\phi^{(i)}(x,t)},\quad \max_{(x,t)\in Q_T}\rho_f^{(i)}(x,t),\quad \max_{(x,t)\in Q_T}\frac{1}{\rho_f^{(i)}(x,t)},$$

$$\max_{(x,t)\in Q_T}\left\|\frac{\partial\phi^{(i)}(x,t)}{\partial t}\right\|,\quad \max_{(x,t)\in Q_T}\left\|\frac{\partial\rho_f^{(i)}(x,t)}{\partial t}\right\|,\quad \max_{(x,t)\in Q_T}\left\|\frac{\partial\rho_f^{(i)}(x,t)}{\partial x}\right\|,\quad i=1,2.$$

For $V(t)$, we also have $V(t) \leqslant C\displaystyle\int_0^1 (\rho_1(x,t) + |u(x,t)|)dx$.

Integrating equation (18) by time and taking into account the estimate for $V(t)$, we obtain

$$|u(x,t)| \leqslant C\int_0^t \big(\rho_1(x,\tau)+|V(\tau)|\big)d\tau \leqslant C\Big(\int_0^t \rho_1(x,\tau)d\tau + \int_0^t\int_0^1 \rho_1(x,\tau)dxd\tau + \int_0^t\int_0^1 |u(x,\tau)|dxd\tau\Big).$$

Integrating last inequality by x from 0 to 1, we obtain Gronwall inequality for function $\displaystyle\int_0^1 |u(x,t)|dx$:

$$\int_0^1 |u(x,t)|dx \leqslant C\Big(\int_0^t\int_0^1 \rho_1(x,\tau)dxd\tau + \int_0^t\int_0^1 |u(x,\tau)|dxd\tau\Big).$$

Therefore

$$\int_0^1 |u(x,t)|dx \leqslant C\int_0^t\int_0^1 \rho_1(x,\tau)dxd\tau, \quad |V(t)| \leqslant C\Big(\int_0^1 \rho_1(x,t)dx + \int_0^t\int_0^1 \rho_1(x,\tau)dxd\tau\Big),$$

and consequently $|u(x,t)| \leqslant C\Big(\displaystyle\int_0^t \rho_1(x,\tau)d\tau + \int_0^t\int_0^1 \rho_1(x,\tau)dxd\tau\Big)$. Hence we obtain from (20):

$$\frac{d}{dt}\|\rho_1(t)\|^2 \leqslant C\Big(\|\rho_1(t)\|^2 + \int_0^t \|\rho_1(\tau)\|^2 d\tau\Big). \tag{21}$$

We set $w(t) = \displaystyle\int_0^t \|\rho_1(\tau)\|d\tau$, then from (21) we obtain $\dfrac{d^2w}{dt^2} \leqslant C\Big(\dfrac{dw}{dt} + w(t)\Big)$. This yields $\dfrac{d}{dt}\Big(e^t\Big(\dfrac{dw}{dt} - (C+1)w\Big)\Big) + we^t \leqslant 0$, so we have inequality $\dfrac{dw}{dt} \leqslant (C+1)w$. Therefore $w(t) = 0$ и $\rho = 0$, $\phi = 0$. Lemma 1 is proved.

After finding ϕ and ρ_f, we find $p_{tot} = P^0(\rho_f, \phi)$. Then we find $p_s = (p^0 - \phi p_f)(1-\phi)^{-1}$. We find v_s from the equation $\dfrac{\partial v_s}{\partial x} = -a_1(\phi)(1-\phi)^{-1}(p_{tot} - p_f)$, and from the Darcy's law we obtain $v_f = v_s - k(\phi)(1-\phi)\phi^{-1}\dfrac{\partial p_f}{\partial x}$.

Since $(\phi, \rho_f) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$, then we have: $v_s \in C^{3+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$, $v_f \in C^{1+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$, $(p_f, p_s) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$. □

This work was partially supported by the grants RFBR 16-08-00291 "Hydroelastic and thermodynamic effects with inter-action of poroelastic ice and structures" and state assignment of the Ministry of Education and Science no. 2014/2.

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Локальная разрешимость системы уравнений одномерного движения магмы

Александр А. Папин
Маргарита А. Токарева
Алтайский государственный университет
Ленина, 61, Барнаул, 656049
Россия

Для системы уравнений одномерного нестационарного движения магмы доказана однозначная локальная разрешимость начально-краевой задачи.

Ключевые слова: закон Дарси, пороупругость, магма, разрешимость, единственность.