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On Local Solvability of the System of the Equations of One Dimensional Motion of Magma

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The local solvability of initial-boundary value problem for the system of the equations of non stationary motion of magma is proved.

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1. Problem statement. Formulation of main results

A quasi-linear system of equations of composite type is considered:

\[ \frac{\partial (1 - \phi)\rho_s}{\partial t} + \frac{\partial}{\partial x}((1 - \phi)\rho_s v_s) = 0, \quad \frac{\partial (\rho_f \phi)}{\partial t} + \frac{\partial}{\partial x}(\rho_f \phi v_f) = 0, \]

\[ \phi (v_f - v_s) = -k(\phi)\left(\frac{\partial p_f}{\partial x} - \rho_f g\right), \]

\[ \frac{\partial v_s}{\partial x} = -\frac{1}{\xi(\phi)}\rho_e, \quad p_e = p_{tot} - p_f, \]

\[ \frac{\partial p_{tot}}{\partial x} = -\rho_{tot}g, \quad p_{tot} = \phi p_f + (1 - \phi)\rho_s, \quad \rho_{tot} = \phi \rho_f + (1 - \phi)\rho_s. \]

We seek a solution of this system in the domain \((x, t) \in Q_T = \Omega \times (0, T), \quad \Omega = (0, 1)\), under the boundary and initial conditions

\[ v_s \mid_{x=0, x=1} = v_f \mid_{x=0, x=1} = 0, \quad \phi \mid_{t=0} = \phi^0(x), \quad \rho_f \mid_{t=0} = \rho^0(x). \]

This quasi-linear system of equations describes 1D non-stationary isothermal motion of magma in porous rock. The laws of conservation of mass for each phase, Darcy’s law for fluid phase, taking into account the motion of a solid skeleton, the rheological law and the equation of conservation of momentum for system describe this process [1–3]. Here \(\rho_f, \rho_s, v_f, v_s\) are, respectively, real density and velocity of solid and fluid phases, \(\phi\) is the porosity, \(\rho_f, \rho_s\) are, respectively, pressures of the fluid and solid phases; \(p_e\) is the effective pressure, \(p_{tot}\) is the total pressure, \(\rho_{tot}\) is the density of the two-phase medium, \(g\) is the density of the mass forces; \(k(\phi)\) is the coefficient of filtration, \(\xi(\phi)\) is the coefficient of rock shear viscosity (specified function).
The problem is written in the Eulerian coordinates $x, t$. The real density of the solid particles $\rho_s$ is assumed constant. The unknown quantities are $\phi, \rho_f, v_f, p_s, p_f$. The system of equations (1)–(4) is closed either by using the equation of state of the fluid phase $p_f = p(\rho_f)$ (in particular, the relationship may be, commonly used in applications $\frac{dp_f}{d\rho_f} = \frac{1}{\beta_f \rho_f}$, where $\beta_f$ is the fluid compressibility [1–3]).

The numerical studies of various initial boundary-value problems for systems of equations (1)–(4) were carried out in [2, 3]. Some exact solutions have been constructed in [4]. In these studies the following dependencies of the functional parameters of the problem was used: $k(\phi) = \frac{\phi^3}{\rho_s \mu}$, $l(\phi) = \phi^{m/\nu}$, where $m \in [0, 2], n = 3; \nu, \mu, \bar{k}$ are positive environment settings [2].

Structurally similar systems of equations was considered in [5–7]. In these studies, based on a number of simplifying assumptions, the original system were reduced to one higher order equation. The local solvability of the Cauchy problem in Sobolev spaces was established in [5]. Travelling wave solutions have been studied in [6, 7].

In this paper the unique local solvability of problem (1)–(5) is proved in the case when $g = 0$ and $\rho_f$ is function of pressure.

On $\Omega$ and $Q_T$, let us consider several function spaces, using the notation from [8]. Suppose that $||f||_{q, \Omega}$ is the norm on the Lebesgue space $L_q(\Omega)$, $q \in [1, \infty]$. For brevity, let $||f|| = ||f||_{q, \Omega}$. We also use the Hölder spaces $C^{\alpha}(\Omega)$, $C^{k+\alpha}(\Omega)$, where $k$ is a natural number and $\alpha \in (0, 1]$ with norms:

$$
||f||_{C^{\alpha}(\Omega)} = \sup_{x \in \Omega} |f(x)|, \\
H_2^\alpha(f) = \sup_{x_1, x_2 \in \Omega} |f(x_1) - f(x_2)||x_1 - x_2|^{-\alpha}, \\
H_{m+\alpha}(f) = \sup_{x_1, x_2 \in \Omega} |f(x_1) - f(x_2)||x_1 - x_2|^{-m-\alpha}.
$$

For functions given on $Q_T$, we need the space $C^{k+\alpha, m+\beta}(Q_T)$, where $k, m$ are natural numbers and $(\alpha, \beta) \in (0, 1]$, with norm

$$
||f||_{C^{k+\alpha, m+\beta}(Q_T)} = \sup_{t \in Q_T} ||f||_{L_q(Q_T)} + \sum_{j=1}^m ||D_j^f||_{0, Q_T} + H^\alpha(D^1 f) + H^\alpha(D^2 f) + H^\alpha(D^3 f) + H^\alpha(D^4 f),
$$

where

$$
H_2^\alpha(f(x, t)) = \sup_{x_1, x_2 \in \Omega, t \in (0, T)} |f(x_1, t) - f(x_2, t)||x_1 - x_2|^{-\alpha}, \\
H_2^\alpha(f(x, t)) = \sup_{t_1, t_2 \in (0, T), x \in \Omega} |f(x, t_1) - f(x, t_2)||t_1 - t_2|^{-\beta}.
$$

In the case $k = m$ and $\alpha = \beta$, we use the notation $C^{k+\alpha}(Q_T)$.

In this paper by a solution of problem (1)–(5) we mean the set of functions $v_s \in C^{3+\alpha, 1+\alpha/2}(Q_T)$ and $v_f \in C^{2+\alpha, 1+\alpha/2}(Q_T)$, such that $0 < \phi < 1$, $\rho_f > 0$, $p_f > 0$. These functions satisfy the equations (1)–(4) and the initial and boundary conditions (5) and regarded as continuous functions in $Q_T$.

Let us state the main results of the paper.

**Theorem 1.** Suppose that $g = 0$ and the data of problem (1)–(5) satisfies the following conditions:

1) the functions $k(\phi), \xi(\phi), p_f(\rho_f)$ and their derivatives up to the second order are continuous for $\phi \in (0, 1), \rho_f > 0$, and satisfy the conditions

$$
k_0^{-1} \phi^q (1 - \phi)^{q_2} \leq k(\phi) \leq k_0 \phi^q (1 - \phi)^{q_3}, \\
1/\xi(\phi) = a_0(\phi)\phi^{\alpha_1}(1 - \phi)^{\alpha_2 - 1}, \quad 0 < R_1 \leq a_0(\phi) \leq R_2,
$$

$$
k_0^{-1} \rho_f^q \leq p_f(\rho_f) \leq k_0 \rho_f^q, \quad k_0^{-1} \rho_f^{q_2} \leq \frac{\partial p_f(\rho_f)}{\partial \rho_f} \leq k_0 \rho_f^{q_3}.
$$

2) the functions $v_s \in C^3(\Omega)$ and $v_f \in C^2(\Omega)$, and their derivatives up to the second order are bounded in \(\Omega\), and satisfy the conditions

$$
v_s^3 \leq v_s \leq v_f^3, \quad v_f^3 \leq v_f \leq v_f^3.
$$
where \( k_0, \alpha_i, R_i, i = 1, 2 \) are positive constants, \( q_1, \ldots, q_8 \) are fixed real parameters;
2) the initial functions \( \phi^0, \rho^0 \) satisfy the following smoothness conditions: \( \phi^0 \in C^{2+\alpha}(\Omega), \rho^0 \in C^{2+\alpha}(\Omega) \) and the matching conditions
\[
\frac{dp_f(\rho^0)}{dx} \bigg|_{x=0,x=1} = 0,
\]
as well as satisfy the inequalities
\[
0 < m_0 \leq \phi^0(x) \leq M_0 < 1, \quad 0 < m_1 \leq \rho^0(x) \leq M_1 < \infty, \quad x \in \bar{\Omega},
\]
where \( m_0, M_0, m_1, M_1 \) are given positive constants. Then problem (1)–(5) has a local solution, i.e., there exists a value of \( t_0 \in (0, T) \) such that
\[
v_s(x, t) \in C^{3+\alpha,1+\alpha/2}(\bar{Q}_{t_0}), \quad (\phi(x, t), p_s(x, t), p_f(x, t), \rho_f(x, t)) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_{t_0}),
\]
\[
v_f(x, t) \in C^{1+\alpha,1+\alpha/2}(\bar{Q}_{t_0}).
\]
Moreover, \( 0 < \phi(x, t) < 1, \quad \rho_f(x, t) > 0 \) \( \forall Q_{t_0} \).

2. Local solvability

Under the conditions of the theorem into force (4) we have \( p_{col} = p^0(t) \). Following [9], we rewrite the system (1)–(3). Suppose that \( \hat{x} = \hat{x}(\tau, x, t) \) is a solution of the Cauchy problem
\[
\frac{\partial \hat{x}}{\partial \tau} = v_s(\hat{x}, \tau), \quad \hat{x}|_{\tau=t} = x.
\]
We set \( \dot{x} = \dot{x}(0, x, t) \) and take \( \hat{x} \) and \( t \) for the new variables. Then \( 1 - \phi(\hat{x}, t) = (1 - \phi^0(\hat{x}))\hat{J}(\hat{x}, t) \),

where \( \hat{J}(\hat{x}, t) = \frac{\partial \hat{x}}{\partial x}(\hat{x}, t) \) is the Jacobian of the transformation. The system of equations (1)–(3) in the new variables is of the form
\[
\frac{\partial (1 - \hat{\phi})}{\partial t} + \frac{(1 - \hat{\phi})^2}{1 - \phi^0} \frac{\partial \hat{v}_s}{\partial x} = 0, \quad \frac{\partial}{\partial \tau}(\hat{\rho}_f \hat{\phi}) + \frac{(1 - \hat{\phi})}{1 - \phi^0} \frac{\partial}{\partial x}(\hat{\rho}_f \hat{\phi} \hat{v}_f) = v_s \frac{(1 - \hat{\phi})}{1 - \phi^0} \frac{\partial}{\partial x}(\hat{\rho}_f \hat{\phi}),
\]
\[
\hat{\phi}(\hat{v}_s - \hat{v}_f) = k(\hat{\phi}) \left( \frac{1 - \hat{\phi}}{1 - \phi^0} \right) \frac{\partial \hat{p}_f}{\partial x}, \quad \frac{1 - \hat{\phi}}{1 - \phi^0} \frac{\partial \hat{v}_s}{\partial x} = -a_1(\hat{\phi}) \hat{p}_s,
\]
where \( a_1(\phi) = 1/\xi(\phi) \).

Since
\[
v_s \frac{\partial}{\partial x}(\hat{\rho}_f \hat{\phi}) = \frac{\partial}{\partial x}(\hat{\rho}_f \hat{\phi} v_s) - \hat{\rho}_f \hat{\phi} \frac{\partial v_s}{\partial x},
\]
it follows that the continuity equation for the liquid phase can be reduced to the form
\[
\frac{1}{(1 - \hat{\phi})} \frac{\partial}{\partial t}(\hat{\rho}_f \hat{\phi}) + \frac{1}{1 - \phi^0} \frac{\partial}{\partial x}(\hat{\rho}_f \hat{\phi} (\hat{v}_f - v_s)) + \frac{1}{1 - \phi^0} \hat{\rho}_f \hat{\phi} \frac{\partial v_s}{\partial x} = 0.
\]
Using the continuity equation for the solid phase, we find that
\[
\frac{\partial}{\partial \tau}(\hat{\rho}_f \hat{\phi}) + \frac{1}{(1 - \phi^0)} \frac{\partial}{\partial x}(\hat{\rho}_f \hat{\phi} (\hat{v}_f - \hat{v}_s)) = 0.
\]
Finally, passing from \( (\hat{x}, t) \) to the mass Lagrangian variables \( (y, t) \) by the rule
\[
(1 - \phi^0(\hat{x})) d\hat{x} = dy, \quad y(\hat{x}) = \int_0^{\hat{x}} (1 - \phi^0(\eta)) d\eta \in [0, 1]
\]
and preserving the notation \( y \) for the variable \( x \), we obtain
\[
\frac{\partial(1 - \phi)}{\partial t} + (1 - \phi)^2 \frac{\partial v_s}{\partial x} = 0, \quad \frac{\partial}{\partial t} \left( \rho_f \frac{\phi}{1 - \phi} \right) + \frac{\partial}{\partial x} (\rho_f \phi (v_f - v_s)) = 0,
\]
\[
\phi (v_s - v_f) = k(\phi) (1 - \phi) \frac{\partial \rho_f}{\partial x},
\]
\[
(1 - \phi) \frac{\partial v_s}{\partial x} = -a_1(\phi) p_c, \quad p_c = p^0(t) - p_f.
\]
Finally, we turn to the dimensionless variables
\[
t' = \frac{t}{t_1}, \quad x' = \frac{x}{L}, \quad v'_s = \frac{v_s}{v_1}, \quad v'_f = \frac{v_f}{v_1}, \quad \rho'_f = \frac{\rho_f}{\rho_s},
\]
\[
p'_f = \frac{p'_f}{p_1}, \quad p'_s = \frac{p_s}{p_1}, \quad p'_c = \frac{p_c}{p_1}, \quad p_{tot}' = \frac{p_{tot}}{p_1}, \quad a'_1(\phi) = \frac{a_1(\phi)}{a_0}, \quad k'(\phi) = \frac{k(\phi)}{k_1},
\]
where \( L = \int_0^1 (1 - \phi^0(\eta)) d\eta, \quad t_1 = \frac{L}{v_1}, \quad a_0 = \frac{v_1}{L p_1}, \quad k_1 = \frac{v_1 L}{p_1}, \) \( v_1, p_1 \) are fixed positive quantities having the dimension of velocity and pressure accordingly.

Then the domain \( x' \) is the unit interval \([0,1]\) and the system of equations will retain its structure (dashes omitted).

Using the rheological relationship, Darcy’s law and the conditions \( v_s |_{x=0,1} = 0 \), we find that
\[
p^0(t) = \int_0^1 a'_1(\phi) p'_f dx \left( \int_0^1 a_1(\phi) dx \right)^{-1} \equiv P^0(\phi, \rho_f).
\]
Taking into account Darcy’s law, the second equation of the system assumes the form
\[
\frac{\partial}{\partial t} \left( \rho_f \frac{\phi}{1 - \phi} \right) - \frac{\partial}{\partial x} \left( \rho_f k(\phi) (1 - \phi) \frac{\partial \rho_f}{\partial x} \right) = 0.
\]
From the first and fourth equations of the system follows that
\[
\frac{1}{1 - \phi} \frac{\partial \phi}{\partial t} = a_1(\phi) (p_f - p^0).
\]
This equation can be written as
\[
\frac{\partial G(\phi)}{\partial t} = p_f - p^0,
\]
where the function \( G(\phi) \) is defined by the equation
\[
\frac{dG(\phi)}{d\phi} = \frac{1}{(1 - \phi) a_1(\phi)}.
\]
Let
\[
a(\phi) = \frac{\phi}{1 - \phi}, \quad K(\phi) = k(\phi)(1 - \phi), \quad b(\rho_f) = \rho_f \frac{\partial \rho_f}{\partial p_f}.
\]
Taking into account the conditions (5), we obtain the following problem for finding functions \( \rho_f, \phi \):
\[
\frac{\partial}{\partial t} \left( a(\phi) \rho_f \right) - \frac{\partial}{\partial x} \left( K(\phi) b(\rho_f) \frac{\partial \rho_f}{\partial x} \right) = 0, \quad (6)
\]
\[
\frac{\partial G(\phi)}{\partial t} = p_f (\rho_f) - p^0(t), \quad (7)
\]
\[
\frac{\partial \rho_f}{\partial x} |_{x=0,x=1} = 0, \quad \rho_f |_{t=0} = \rho^0(x), \quad \phi |_{t=0} = \phi^0(x). \quad (8)
\]
Lemma 1. Let the data of problem (6)–(8) satisfy the conditions of the theorem. Then problem (6)–(8) has a unique local solution, i.e., there exists a value of \( t_0 \) such that

\[
(\varphi(x,t), \rho_f(x,t)) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_{t_0})..
\]

Furthermore, \( 0 < \phi(x,t) < 1, \rho_f(x,t) > 0 \) in \( \bar{Q}_{t_0} \).

The solvability of problem (6)–(8) is established by using the Tikhonov- Schauder fixed-point theorem: if \( V \) is a compact convex closed set of Banach space \( B \) and the operator \( \Lambda \) maps \( V \) into itself continuously in the norm of \( B \), then there is a fixed point on \( V \). [10, pp. 227].

Since the function \( \psi = G(\phi) \) is strictly monotone, at \( \phi \in (0, 1) \), that the inverse function is exist: \( \phi = G^{-1}(\psi) \). Assuming that \( \rho(x,t) = \rho_f(x,t) - \rho^0(x), \omega(x,t) = G(\phi) - G(\phi^0) \).

We represent the equations (6),(7) in the form

\[
\begin{align*}
\frac{\partial}{\partial t} (a(\omega)(\rho + \rho^0)) &= \frac{\partial}{\partial x} \left( K(\omega)b(\rho + \rho^0) \frac{\partial(\rho + \rho^0)}{\partial x} \right), \\
\frac{\partial \omega}{\partial t} &= p_f(\rho + \rho^0) - p^0(t).
\end{align*}
\]

Here \( a(\omega) = \frac{\phi(\omega)}{1 - \phi(\omega)}, K(\omega) = k(\phi(\omega))(1 - \phi(\omega)), \phi(\omega) = G^{-1}(\omega + G(\phi^0)) \). Moreover,

\[
\rho \mid_{t=0} = \omega \mid_{t=0} = \frac{\partial(\rho + \rho^0)}{\partial x} \bigg|_{x=0,x=1} = 0.
\]

For the Banach space, we choose the space \( C^{2+\beta,1+\beta/2}(\bar{Q}_{t_0}) \), where \( \beta \) is any number from the interval \( (0, \alpha) \), \( \alpha \in [0, 1] \). Let

\[
V = \left\{ \bar{\rho}(x,t), \bar{\omega}(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_{t_0}) \mid \bar{\rho} \mid_{t=0} = \bar{\omega} \mid_{t=0} = \frac{\partial \bar{\rho}}{\partial x} \bigg|_{x=0,x=1} = 0, \right. \\
\frac{m_1}{2} - \rho^0(x) \leq \bar{\rho}(x,t) \leq 2M_1 - \rho^0(x) < \infty, \\
G(m_0/2) - G(\phi^0) \leq \bar{\omega}(x,t) \leq G\left(\frac{M_0 + 1}{2}\right) - G(\phi^0) < \infty, \quad (x,t) \in Q_{t_0}.
\]

\[
(\bar{\omega})_{1+\alpha,(1+\alpha)/2, Q_{t_0}}, |\bar{\rho}|_{1+\alpha,(1+\alpha)/2, Q_{t_0}}, |\bar{\rho}|_{2+\alpha,(2+\alpha)/2, Q_{t_0}}, |\bar{\rho}|_{2+\alpha,(2+\alpha)/2, Q_{t_0}} \leq K_1 + K_2.
\]

where \( K_1 \) is an arbitrary positive constant, while the positive constant \( K_2 \) will be given later. We note that on the set \( V \) following inequalities hold: \( 0 < \frac{m_0}{2} \leq \phi(\bar{\omega}) \leq \frac{M_0 + 1}{2} < 1, a(\bar{\omega}) > 0, K(\bar{\omega}) > 0 \).

Let us construct an operator \( \Lambda \) mapping \( V \) in \( V \). Suppose that \( \bar{\omega}, \bar{\rho} \in V \). Using (10), we define the function \( \omega \) by the equality

\[
\omega = \int_0^t \left( p_f(\bar{\rho}(x,\tau) + \rho^0(x)) - \int_0^1 \frac{a_1(\phi(\bar{\omega}))}{1 - \phi(\bar{\omega})} p_f(\bar{\rho}(x,\tau) + \rho^0(x)) dx \right) \left( \int_0^1 \frac{a_1(\phi(\bar{\omega}))}{1 - \phi(\bar{\omega})} dx \right)^{-1} d\tau.
\]

From the representation (12) it follows that smoothness \( \omega \) is determined by the smoothness of functions \( \bar{\rho}, \rho^0 \) and \( \rho^0 \). In particular, we have an estimate

\[
|\omega|_{2+\alpha,1+\alpha/2, Q_{t_0}} = C_1(m_0, M_0, m_1, M_1, K_1, T, |\rho^0|_{2+\alpha,\Omega})(1 + t_0|\bar{\rho}_{xx}|_{2+\alpha/2,\Omega}).
\]
Lemma 2. Let function \( a_1(\phi), \phi \in (0,1) \) satisfies the following condition
\[
(1 - \phi)a_1(\phi) = a_0(\phi)\phi^{\alpha_1}(1-\phi)^{\alpha_2}, \quad 0 < R_1 \leq a_0(\phi) \leq R_2,
\]
where \( R_i > 0, \alpha_i > 0, i = 1, 2. \) Then we have the estimate of the form
\[
R_2|G(\phi_1) - G(\phi_2)| \geq |\phi_1 - \phi_2|.
\]

Proof. Assume without loss of generality that \( 0 < \phi_1 \leq \phi_2 < 1. \) From the definition of functions \( G(\phi) \) and \( a_1(\phi) \), we have
\[
0 < \Delta G \equiv G(\phi_2) - G(\phi_1) = \int_{\phi_1}^{\phi_2} \frac{ds}{(1-s)a_1(s)} \geq \frac{1}{R_2}(\phi_2 - \phi_1).
\]
Lemma 2 is proved.

In this way, we have estimate
\[
|\phi(x,t) - \phi^0(x)| \leq \delta(t), \quad \delta(t) \to 0 \quad \text{as} \quad t \to 0,
\]
which implies, that there exists a value \( t_1 = t_1(m_0, M_0, m_1, M_1) \), such that for all \( t_0 \leq t_1 \) the following inequality holds
\[
0 < \frac{m_0}{2} \leq \phi(x,t) \leq \frac{M_0 + 1}{2}, \quad (x,t) \in Q_{t_0}, \quad \text{(13)}
\]

Taking into account (13) we also have the estimate for function \( \omega(x,t) \):
\[
G\left(\frac{m_0}{2}\right) \leq \omega(x,t) \leq G\left(\frac{M_0 + 1}{2}\right).
\]

Using (9), (11) and \( \omega(x,t) \) we find the function \( \rho(x,t) \) as a solution of the problem (here and elsewhere, we assume that the initial and boundary conditions are matched):
\[
\rho \bigg|_{t=0} = \frac{\partial \rho^0}{\partial x} \bigg|_{x=0, x=1} = 0, \quad \frac{\partial \rho^0}{\partial x} \bigg|_{x=0, x=1} = 0.
\]

The equation for \( \rho(x,t) \) is uniformly parabolic. In view of the properties of \( \omega(x,t) \) and \( \rho^0(x) \) problem (14) has a classical solution [8]. In addition, we have the following estimate:
\[
\left| \frac{\partial a(\omega)}{\partial t} \right| \leq C_0(m_0, M_0, m_1, M_1, \max_{0 \leq t \leq T} |\rho^0(t)|).
\]

Under the additional condition smallness for the value of the time interval the following statement holds [9].

Lemma 3. For \( t_0 \leq \min(t_1, t_2) \), \( t_2 = \ln 2/C_0(m_0, M_0, m_1, M_1) \), the classical solution of problem (14) satisfies the following inequality in \( Q_{t_0} \):
\[
0 < \frac{m_1}{2} \leq \rho(x,t) + \rho^0(x) \leq 2M_1 < \infty.
\]
Proof. Further, setting \( U(x, t) = \rho(x, t) + \rho^0(x) \), we can express problem (14) in the form
\[
\frac{\partial}{\partial t} (a(\omega)U) = \frac{\partial}{\partial x} \left( K(\omega) b(\rho) \frac{\partial U}{\partial x} \right), \quad \frac{\partial U}{\partial x} \bigg|_{x=0,x=1} = 0, \quad U|_{t=0} = \rho^0. \tag{15}
\]
First, we show that \( U(x, t) \geq 0, \; (x, t) \in \Omega_n \). In equation (15), let us make the change \( U(x, t) = -z(x, t) \). Then
\[
z \frac{\partial a}{\partial t} + a \frac{\partial z}{\partial t} = \frac{\partial}{\partial x} (K b \frac{\partial z}{\partial x}).
\]
Let
\[
z^{(0)}(x, t) = \max \{ z, 0 \}, \quad z^{(0)}(x, t) \big|_{t=0} = \max \{ -\rho^0, 0 \} = 0,
\]
\[\sigma(x, t) = z^{(0)}(x, t) \left( |z^{(0)}(x, t)|^2 + \varepsilon \right)^{-1/2}, \quad \varepsilon > 0.
\]
Let us multiply the equation for the function \( z \) by \( \sigma \) and then integrate over \( \Omega \). We obtain the equality
\[
\frac{d}{dt} \int_0^1 a(\sigma(z - (|z^{(0)}|^2 + \varepsilon)^{1/2})) - \varepsilon \int_0^1 a(\sigma(z - (|z^{(0)}|^2 + \varepsilon)^{1/2})) dx + \varepsilon \int_0^1 K b \frac{\partial z}{\partial x} (|z^{(0)}|^2 + \varepsilon)^{-3/2} dx = 0. \tag{16}
\]
Let \( A^+(t) = \{ x \in \Omega \mid z(x, t) > 0 \}, \quad A^-(t) = \{ x \in \Omega \mid z(x, t) \leq 0 \}. \) Then
\[
\int_0^1 a(\sigma(z - (|z^{(0)}|^2 + \varepsilon)^{1/2})) dx = -\varepsilon \int_{A^+(t)} a(\sigma(z^2 + \varepsilon)^{1/2}) dx - \varepsilon \int_{A^-(t)} \frac{\partial a}{\partial t} dx,
\]
\[
\int_0^1 a(|z^{(0)}|^2 + \varepsilon)^{1/2} dx = \int_{A^+(t)} a(|z|^2 + \varepsilon)^{1/2} dx + \varepsilon \int_{A^-(t)} a dx,
\]
\[
\int_0^1 a(|z^{(0)}|^2 + \varepsilon)^{1/2} \big|_{t=0} dx = \varepsilon \int_0^1 a \big|_{t=0} dx,
\]
\[
\int_{A^+(t)} a|z| dx = \int_{A^+(t)} a^{(0)} dx.
\]
Integrating relation (16) with respect to time, we obtain
\[
\int_{A^+(\tau)} a(|z|^2 + \varepsilon)^{1/2} dx + \varepsilon \int_{0}^{t} a \frac{\partial a}{\partial t} dx + \varepsilon \int_{0}^{t} K b \frac{\partial z}{\partial x} (z^2 + \varepsilon)^{-3/2} dx d \tau = \\
\varepsilon \int_{0}^{t} a \frac{\partial a}{\partial t} dx + \varepsilon \int_{0}^{t} a \frac{\partial a}{\partial t} dx + \varepsilon \int_{0}^{t} a \frac{\partial z}{\partial x} dx d \tau + \varepsilon \int_{0}^{t} a \big|_{t=0} dx.
\]
Therefore,
\[
\int_0^1 a z^{(0)} dx \leq \varepsilon \int_0^1 \left| \frac{\partial a}{\partial t} \right| dx d \tau + \varepsilon \int_{0}^{t} a \big|_{t=0} dx.
\]
Passing to the limit as \( \varepsilon \to 0 \), we find that \( z^{(0)} = 0 \), i.e. \( U \geq 0 \).

After multiplication by \( U^{l-1}(x, t), l > 2 \), equation (15) can be expressed as
\[
\frac{1}{l} \frac{\partial (aU^l)}{\partial t} + (l - 1) K b U^{l-2} \left( \frac{\partial U}{\partial x} \right)^2 + \frac{l - 1}{l} U^{l} \frac{\partial a}{\partial t} = \frac{\partial}{\partial x} \left( K b U^{l-1} \frac{\partial U}{\partial x} \right).
\]
Then
\[ \frac{1}{l} \int_0^1 aU^l dx \leq \frac{l - 1}{l} \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| \int_0^1 aU^l dx. \]
Therefore,
\[ y'(t) \leq \frac{l - 1}{l} \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| y(t), \quad y'(t) = \int_0^1 (a^{1/l})^l dx, \]
\[ y(t) \leq y(0) \exp \left\{ \frac{l - 1}{l} \int_0^t \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| dt \right\}. \]
After passing to the limit as \( l \to \infty \), we obtain
\[ \max_{0 \leq x \leq 1} U(x, t) \leq \max_{0 \leq x \leq 1} \rho(0) \exp \left\{ \int_0^t \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| dt \right\}. \]
Taking into account the inequality \( \max_{0 \leq x \leq 1} \rho(0) \leq M_1 \) and choosing \( t \) from the condition
\[ t \leq t_2, \quad \exp \left\{ \int_0^{t_2} \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| dt \right\} \leq 2, \]
we obtain upper bound for \( \rho \). To obtain a lower estimate we represent equation (15) in the form
\[ \frac{1}{l} \frac{\partial (az^l)}{\partial t} + (l + 1)Kbz^{l-1}(\frac{\partial z}{\partial x})^2 = \frac{l + 1}{l} \frac{z}{a} \frac{\partial a}{\partial t} = \frac{\partial}{\partial x} \left( Kbz^{l-1} \frac{\partial z}{\partial x} \right). \]
Then we obtain inequality
\[ \frac{1}{l} \frac{d}{dt} \int_0^1 az^l dx \leq \frac{l + 1}{l} \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| \int_0^1 az^l dx, \]
then we obtain the estimate
\[ \max_{0 \leq x \leq 1} U(x, t) \leq \max_{0 \leq x \leq 1} \frac{1}{\rho(x)} \exp \left\{ \int_0^t \max_{0 \leq x \leq 1} \left| \frac{1}{a} \frac{\partial a}{\partial t} \right| dx \right\} \leq \frac{2}{m_1}. \]
Lemma 3 is proved.

In view of Lemma 3 and the properties of \( \tilde{\omega} \), we have the following estimates [8, Sec. 3]:
\[ |\rho|_{a,a/2,Q_{t_0}} \leq C_2, \]
\[ |\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_3 \left( 1 + |\rho|^2_{2+\alpha,\Omega} + |\tilde{\omega}|_{a,a/2,Q_{t_0}} + |\tilde{\omega}|_{\alpha,a/2,Q_{t_0}} + |\omega|_{a,a/2,Q_{t_0}} \right), \]
in which the constant \( C_2, C_3 \) depends on \( K_1, m_0, m_1, M_0, M_1 \). Therefore
\[ |\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq C_4(K_{1,0}, m_0, m_1, M_0, M_1). \]
Let \( C_5 = \max\{C_1, C_4\} \). Choose \( K_2 \) so that \( C_5 \leq \frac{K_1 + K_2}{2} \). Then, for \( t_0 < \min(t_1, t_2, (K_1 + K_2)^{-1}) \) we obtain
\[ |\rho|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq K_1 + K_2, \quad |\omega|_{2+\alpha,1+\alpha/2,Q_{t_0}} \leq K_1 + K_2. \]
It remains to verify conditions

\[ |\rho|_{1+\alpha, (1+\alpha)/2, Q_{t_0}} \leq K_1, \quad |\omega|_{1+\alpha, (1+\alpha)/2, Q_{t_0}} \leq K_1. \]

Integrating equation (14) with respect to time, we obtain \( |\rho|_{0, Q_{t_0}} \leq C_\alpha t_0 \). From the equation (12) we obtain \( |\omega|_{0, Q_{t_0}} \leq C_\gamma t_0 \). Further, using for \( \rho, \omega \) an inequality of the form [11, pp. 35]

\[ |u|_{1+\alpha, (1+\alpha)/2, Q_{t_0}} \leq C|u|_{2+\alpha, 1+\alpha/2, Q_{t_0}}^c |u|_{0, Q_{t_0}}^{1-c}, \quad c = (1+\alpha)(2+\alpha)^{-1}, \]

we find that there exists a sufficiently small value of \( t_0 \), depending on \( K_1 \) and \( K_2 \), such that the required estimates hold: \( |\rho|_{1+\alpha, (1+\alpha)/2, Q_{t_0}} \leq K_1, \quad |\omega|_{1+\alpha, (1+\alpha)/2, Q_{t_0}} \leq K_1. \)

Thus, the operator \( \Lambda \) maps the set \( V \) into itself for sufficiently small values of \( t_0 \). Using the estimates obtained above, we can easily show the continuity of the operator \( \Lambda \) in the norm of the space \( C^{2+\beta, 1+\beta/2}(\overline{Q}_{t_0}) \). By the Tikhonov-Schauder theorem, there exists a fixed point \( (\rho, \omega) \in V \) of the operator \( \Lambda \).

Let us establish uniqueness of the solution of problem (6)–(8).

Suppose that \((\rho_f^{(1)}, \phi^{(1)})\) and \((\rho_f^{(2)}, \phi^{(2)})\) are two different solutions of problem. Their difference \( \rho = \rho_f^{(1)} - \rho_f^{(2)}, \phi = \phi^{(1)} - \phi^{(2)} \) is the solution of the linear homogeneous system

\[
\begin{align*}
\frac{\partial}{\partial t}(d_0 \rho + d_1 \phi) - \frac{\partial}{\partial x} \left( d_2 \frac{\partial \rho}{\partial x} + d_3 \rho + d_4 \phi \right) &= 0, \\
\frac{\partial}{\partial t}(h_0 \phi) - h_1 \rho + V(t) &= 0,
\end{align*}
\]

with zero initial and boundary conditions \( \phi|_{t=0} = \rho|_{t=0} = \left. \frac{\partial \rho}{\partial x} \right|_{x=0, x=1} = 0. \)

The coefficients

\[
\begin{align*}
d_0 &= a(\phi^{(1)}) > 0, \quad d_1 = \frac{(a(\phi^{(1)}) - a(\phi^{(2)}))\rho_f^{(2)}}{\phi^{(1)} - \phi^{(2)}} > 0, \quad d_2 = K(\phi^{(2)})b(\rho_f^{(2)}) > 0, \\
d_3 &= K(\phi^{(1)})b(\rho_f^{(1)}) - b(\phi^{(2)}) \frac{\partial \rho_f^{(1)}}{\partial x}, \quad d_4 = b(\rho_f^{(2)}) \frac{\partial (\phi^{(1)} - \phi^{(2)})}{\partial x}, \\
h_0 &= \frac{G(\phi^{(1)}) - G(\phi^{(2)})}{\phi^{(1)} - \phi^{(2)}} > 0, \quad h_1 = \frac{p(\rho_f^{(1)}) - p(\rho_f^{(2)})}{\rho_f^{(1)} - \rho_f^{(2)}}, \\
h_2 &= \frac{a_1(\phi^{(1)}) p_f(\rho_f^{(1)}) - p_f(\rho_f^{(2)})}{1 - \phi^{(1)}} a_1(\phi^{(1)}) \left( \int_0^1 a_1(\phi^{(1)}) \frac{dx}{1 - \phi^{(1)}} \right)^{-1}, \\
h_3 &= \frac{a_1(\phi^{(1)}) - a_1(\phi^{(2)})}{1 - \phi^{(1)}} \frac{a_1(\phi^{(2)})}{1 - \phi^{(2)}} (\phi^{(1)} - \phi^{(2)})^{-1} \times \\
&\left( p_f(\rho_f^{(2)}) \left( \int_0^1 a_1(\phi^{(2)}) \frac{dx}{1 - \phi^{(2)}} \right)^{-1} - \int_0^1 a_1(\phi^{(2)}) p_f(\rho_f^{(2)}) dx \left( \int_0^1 a_1(\phi^{(1)}) \frac{dx}{1 - \phi^{(1)}} \int_0^1 a_1(\phi^{(2)}) \frac{dx}{1 - \phi^{(2)}} \right)^{-1} \right)
\end{align*}
\]

are bounded for all \( x \in [0, 1], \ t \in [0, T] \).
Taking into account (18), equation (17) can be represented as
\[
\frac{\partial}{\partial t}(d_0 \rho) + \frac{d_1}{h_0} (h_1 \rho - V(t)) + h_0 \phi \frac{\partial}{\partial x} \left( \frac{d_1}{h_0} - \frac{\partial \rho}{\partial x} (d_2 \rho / \partial x + d_3 \rho + d_4 \phi) \right) = 0.
\] (19)

Multiplying the equation (19) by \( \rho(x,t) \) and consequently integrating by \( x \) from 0 to 1, we obtain
\[
\frac{d}{dt} \int_0^1 \rho_1^2(x,t)dx \leq C \left( \int_0^1 \rho_1^2(x,t)dx + \int_0^1 u^2(x,t)dx + V^2(t) \right),
\] (20)
where \( \rho_1(x,t) = d_0^{1/2} |\rho(x,t)|, u(x,t) = h_0 \phi(x,t) \). Here the constant \( C \) depends on \( T \) and quantities
\[
\max_{(x,t) \in Q_T} \frac{1}{\phi^{(i)}(x,t)}, \quad \max_{(x,t) \in Q_T} 1 - \phi^{(i)}(x,t), \quad \max_{(x,t) \in Q_T} \rho_f^{(i)}(x,t), \quad \max_{(x,t) \in Q_T} \frac{1}{\rho_f^{(i)}(x,t)},
\]
\[
\max_{(x,t) \in Q_T} \left\| \frac{\partial \phi^{(i)}(x,t)}{\partial t} \right\|, \quad \max_{(x,t) \in Q_T} \left\| \frac{\partial \rho_f^{(i)}(x,t)}{\partial t} \right\|, \quad \max_{(x,t) \in Q_T} \left\| \frac{\partial \rho_f^{(i)}(x,t)}{\partial x} \right\|, \quad i = 1, 2.
\]

For \( V(t) \), we also have \( V(t) \leq C \int_0^1 (\rho_1(x,t) + |u(x,t)|)dx \).

Integrating equation (18) by time and taking into account the estimate for \( V(t) \), we obtain
\[
|u(x,t)| \leq C \int_0^t \left( \rho_1(x,\tau) + |V(\tau)| \right) d\tau \leq C \left( \int_0^1 \rho_1(x,\tau)d\tau + \int_0^t \int_0^1 \rho_1(x,\tau)d\tau d\tau + \int_0^t \int_0^1 |u(x,\tau)|d\tau d\tau \right).
\]
Integrating last inequality by \( x \) from 0 to 1, we obtain Gronwall inequality for function
\[
\int_0^1 |u(x,t)|dx:
\]
\[
\int_0^1 |u(x,t)|dx \leq C \left( \int_0^1 \int_0^1 \rho_1(x,\tau)d\tau d\tau + \int_0^t \int_0^1 |u(x,\tau)|d\tau d\tau \right).
\]
Therefore
\[
\int_0^1 |u(x,t)|dx \leq C \int_0^t \int_0^1 \rho_1(x,\tau)d\tau d\tau, \quad |V(t)| \leq C \left( \int_0^1 \rho_1(x,t)dx + \int_0^1 \int_0^1 \rho_1(x,\tau)d\tau d\tau \right),
\]
and consequently \( |u(x,t)| \leq C \left( \int_0^t \rho_1(x,\tau)d\tau + \int_0^1 \int_0^1 \rho_1(x,\tau)d\tau d\tau \right) \). Hence we obtain from (20):
\[
\frac{d}{dt} \| \rho_1(t) \|^2 \leq C \left( \| \rho_1(t) \|^2 + \int_0^t \| \rho_1(\tau) \|^2 d\tau \right).
\] (21)
We set \( w(t) = \int_0^t \| \rho_1(\tau) \|^2 d\tau \), then from (21) we obtain \( \frac{d^2 w}{dt^2} \leq C \left( \frac{dw}{dt} + w(t) \right) \). This yields
\[
\frac{d}{dt} \left( e^t \left( \frac{dw}{dt} - (C + 1)w \right) \right) \leq 0, \quad \text{so we have inequality} \quad \frac{dw}{dt} \leq (C + 1)w. \quad \text{Therefore} \quad w(t) = 0 \quad \text{in} \quad \rho = 0, \quad \phi = 0. \quad \text{Lemma 1 is proved.}
\]
After finding \( \phi \) and \( \rho_f \), we find \( p_{tot} = P^0(\rho_f, \phi) \). Then we find \( p_s = (\rho^0 - \phi p_f)(1 - \phi)^{-1} \). We find \( v_s \) from the equation
\[
\frac{\partial v_s}{\partial x} = -a_1(\phi)(1 - \phi)^{-1}(p_{tot} - p_f),
\]
and from the Darcy’s law we obtain
\[
v_f = v_s - k(\phi)(1 - \phi)^{-1} \frac{\partial p_f}{\partial x}.
\]
Since $(\phi, p_f) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$, then we have: $v_s \in C^{3+\alpha,1+\frac{\alpha}{2}}(Q_{t_0}),$
$v_f \in C^{1+\alpha,1+\frac{\alpha}{2}}(Q_{t_0}),$ $(p_f, p_s) \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_0})$. \hfill \Box

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References


