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Centralizers of Finite p-Subgroups in Simple Locally Finite Groups

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We are interested in the following questions of B. Hartley: (1) Is it true that, in an infinite, simple locally finite group, if the centralizer of a finite subgroup is linear, then G is linear? (2) For a finite subgroup F of a non-linear simple locally finite group is the order $|CG(F)|$ infinite? We prove the following: Let G be a non-linear simple locally finite group which has a Kegel sequence $\mathcal{K} = \{(G_i, 1) : i \in \mathbf{N}\}$ consisting of finite simple subgroups. Let p be a fixed prime and $s \in \mathbf{N}$. Then for any finite $p-$subgroup F of G, the centralizer $C_G(F)$ contains subgroups isomorphic to the homomorphic images of $SL(s, \mathbf{F}_q)$. In particular $C_G(F)$ is a non-linear group. We also show that if F is a finite p-subgroup of the infinite locally finite simple group G of classical type and given $s \in \mathbf{N}$ and the rank of G is sufficiently large with respect to $|F|$ and s, then $C_G(F)$ contains subgroups which are isomorphic to homomorphic images of $SL(s, K)$.

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Hartley asked the following question: Let G be a simple locally finite group containing a finite subgroup with linear centralizer. Does it follow that G is linear? He proved in [2, Theorem A] the following: Let G be any non-linear simple locally finite group and F be a finite subgroup of G. Then there exist subgroups $D \lhd C \leqslant G$ such that D contains $[C, F]$ and $C \cap F$, and C/D is a direct product of finite alternating groups of unbounded orders. So the above question is not answered positively in the general case because $[C, F]$ is not necessarily identity.

By standard methods, one may reduce the question to countable non-linear simple locally finite groups. So, we may assume that G is a non-linear countable, simple locally finite group. In [6] it is mentioned that, the structure of centralizers of simple locally finite groups which has a Kegel sequence as a union of finite simple subgroups and the ones which has no, particular type of such Kegel sequences are quite different.

Recall that an element in a simple group of Lie type is *semisimple* if its order and the characteristic of the field is relatively prime. In the alternating groups all elements are semisimple.

Definition 1. *A subgroup F of a finite non-abelian simple group G is called a* ***totally semisimple*** *subgroup if every element of F is a semisimple element in G whenever it is a simple group of Lie type. If G is alternating, then all finite subgroups are totally semisimple.*

Observe that, a simple locally finite group G has a local system consisting of finite simple subgroups if and only if G has a Kegel sequence $\mathcal{K} = \{(G_i, 1) \mid i \in \mathbf{N}\}$. For more information about Kegel sequences see [3]. Our main result is the following.

Theorem 2. *Let G be a non-linear simple locally finite group which has a Kegel sequence $\mathcal{K} = \{(G_i, 1) : i \in \mathbf{N}\}$ consisting of finite simple subgroups. Let p be a fixed prime and $s \in \mathbf{N}$. Then*

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for any finite $p-$subgroup F of G, the centralizer $C_G(F)$ contains subgroups isomorphic to the homomorphic images of $SL(s,\mathbf{F}_q)$. In particular $C_G(F)$ is a non-linear group.

Proof. Step 1. Let F be a finite p-subgroup for a fixed prime p of the simple locally finite non-linear group G which has a Kegel sequence consisting of finite simple subgroups G_i. By the classification of the finite simple groups, we may assume that G has a Kegel sequence for a fixed type of classical group. Let p_i be the characteristic of the field over which the group G_i is defined. Let $\wp = \{p_i \mid i \in \mathbf{N}\}$ be the set of all these primes. If the set $\wp$ is infinite, then we may pass to a subsequence such that each G_i is defined over a field of characteristic p_i and $p_i \neq p_{i+1}$. In this case, by deleting the fixed prime p from the infinite list and if necessary by passing to a subsequence, we may assume that our p does not belong to the set of primes which appear as a characteristic. Hence F becomes a totally semisimple subgroup in each G_i. Otherwise the set $\wp$ is finite. We may pass to a subsequence such that each G_i is a simple group over a field of fixed prime r. If $p \neq r$ then again F becomes a totally semisimple subgroup in each G_i.

By the above paragraph we may choose a subsequence such that either F is a totally semisimple subgroup in each finite simple group G_i of fixed classical type for all $i \in \mathbf{N}$ and so theorem is proved in the totally semisimple and alternating cases in [1], or F is a p-group in G_i of fixed classical type defined over a field of characteristic p.

So we are in the case that F is a p-group in G_i where G_i is of fixed classical type defined over a field of characteristic p.

Reduction of centralizer from projective special classical groups to classical groups case.

As $C_G(F)N/N \leqslant C_{G/N}(F/N)$ if $C_G(F)N/N$ contains subgroups isomorphic to $SL(s,\mathbf{F}_q)$, then so is $C_{G/N}(F/N)$.

By using this we may reduce the proof of the Theorem to Classical groups.

Step 2. Let p be a fixed prime. Let F_1 be a finite p-subgroup of the finite simple group G of classical type defined over a field of characteristic p. G is constructed from a vector space V of dimension m and $G = T/Z(T)$ where $T = SL(V)$, $Sp(V)$, $\Omega^{\pm}(V)$ or $SU(V)$. Since $(|Z(T)|,\ p) = 1$, by Schur-Zassenhaus theorem, the inverse image L of F_1 in T can be written as $L = F \times Z(T)$ where F is a finite p-subgroup of T isomorphic to F_1. Then $C_T(L) = C_T(F)$ and

$$C_T(F \times Z(T))/Z(T) = C_T(F)Z(T)/Z(T) \leqslant C_{T/Z(T)}(F_1) = C_G(F_1)$$

moreover as $C_T(F) \geqslant Z(T)$, the order

$$|(C_T(F) \cap Z(T))| = |Z(T)| = (m, |\mathbf{F}_q| - 1) < |\mathbf{F}_q| = q$$

So in order to prove the Theorem, it is enough to show that, for finite p-subgroup F of T the group $C_T(F)$ contains subgroups isomorphic to homomorphic images of $SL(s,\mathbf{F}_q)$. First we show this for $SL(n,\mathbf{F}_q)$.

Lemma 3. *Let F be a finite p-subgroup of $SL(n,\mathbf{F}_q)$ or $GL(n,\mathbf{F}_q)$ where $\mathbf{F}_q$ be a finite field of order q of characteristic p. Then $C_{SL(n,\mathbf{F}_q)}(F)$ has subgroups isomorphic to $SL(\frac{n}{|F|},\mathbf{F}_q)$ provided that n is large.*

Proof. Let V_n be an n-dimensional vector space over $\mathbf{F}_q$ on which F acts. First we show $\dim(C_{V_n}(F)) \geqslant \dfrac{n}{|F|}$. We may prove this by induction on $|F|$.

Assume that $|F| = p$. Since characteristic of the field is p and the element $x \in F$ is of order p, we have $x^p = 1$, which implies $(x-1)^p = 0$. Then F has either Jordan block of size p or it fixes the given vector. As we wish to show that F fixes a subspace of large dimension, we may assume that, F has Jordan blocks of size p on the whole space. In this case V_n can be written as a direct sum of the corresponding F invariant subspaces and in each F invariant subspace we have an eigenvector corresponding to the eigenvalue 1 and hence it is fixed. Then we have $\dim(C_{V_n}(F)) \geqslant \frac{n}{p} = \frac{n}{|F|} \geqslant$ number of Jordan blocks of F on V_n.

Now assume that $|F| > p$. Since F is a p-group, there exists a normal subgroup $H \lhd F$ such that $|F : H| = p$. By induction assumption $\dim(C_{V_n}(H)) \geqslant \frac{n}{|H|}$. Now consider $C_{V_n}(F)$. The group F acts on $C_{V_n}(H) = W$ as a cyclic group of order p as H acts trivially on W. Hence $\dim(C_{V_n}(F)) \geqslant \frac{\dim(W)}{p} = \frac{n}{|F|}$ by the first paragraph.

We may form a basis for V_n extending the basis of $C_{V_n}(F)$. Consider the non-singular linear transformations of $C_{V_n}(F)$ which acts trivially on the remaining basis vectors of V_n. Then $C_{V_n}(F)$ has a subgroup isomorphic to $GL(\frac{n}{|F|}, \mathbf{F}_q)$ and contained in $C_{GL(n,\mathbf{F}_q)}(F)$. □

Now we prove the above result for the other classical groups that will complete the proof of the Theorem.

Observe that in the next Lemma, classical groups over fields of characteristic 2 is also included, main reference is [2]. In particular if G is an orthogonal group over a field of characteristic 2, then we assume $\dim(V_m)$ is even as in [2].

Lemma 4. *Let $G = G(m, \mathbf{F}_q)$ be unitary, symplectic or orthogonal group over a finite field $\mathbf{F}_q$ of characteristic p. Let F be a finite p-subgroup of G. Let s be a given integer. If $m \geqslant 4|F| + 2s|F|$, then $C_G(F)$ contains subgroups isomorphic to $SL(s, \mathbf{F}_q)$.*

Proof. Let V_m be an m dimensional vector space over a finite field $\mathbf{F}_q$ of characteristic p associated to the group G and (,) be a non-degenerate symmetric, unitary or symplectic form on V_m. For the orthogonal groups over a field of characteristic 2, the vector space V_m is associated with a quadratic form $g : V_m \to \mathbf{F}_q$ together with a $\mathbf{F}_q$ valued bilinear form (,) on V_m such that $g(\lambda x + \mu y) = \lambda^2 g(x) + \lambda\mu(x, y) + \mu^2 g(y)$ where $x, y \in V_m$ and $\lambda, \mu \in \mathbf{F}_q$.

By [2, (b), p. 508] if $m \geqslant 2s|F| + 4|F|$, then F leaves invariant a totally isotropic (respectively totally singular) subspace of V_m of dimension at least $s|F|$. Then F acts on this F invariant subspace of dimension $s|F|$ and by Lemma 3, if dimension of the totally isotropic subspace $\geqslant s|F|$, then $C_{SL(m,\ \mathbf{F}_q)}(F)$ contains a subgroup isomorphic to $SL(s, \mathbf{F}_q)$. Then by Witt extension theorem we may extend the action to the isometries of the vector space V_m and hence $C_G(F)$ contains subgroups isomorphic to $SL(s, \mathbf{F}_q)$. □

Completion of the Proof of Theorem 2. Let s be a given integer. In a non-linear locally finite simple group with a given finite subgroup F, we may find a classical group G_i where the rank of G_i is sufficiently large. Then by Lemma 3 and Lemma 4, $C_{G_i}(F)$ contains a subgroup which is isomorphic to $SL(s, \mathbf{F}_q)$. Hence $C_G(F)$ contains subgroups isomorphic to homomorphic images $SL(s, \mathbf{F}_q)$ for any $s \in \mathbf{N}$. In particular $C_G(F)$ is a non-linear group. □

Theorem 5. *If F is a finite p-subgroup of the infinite locally finite simple group G of classical type and the rank of G is sufficiently large with respect to $|F|$, then $C_G(F)$ contains subgroup isomorphic to homomorphic images of $SL(s, \mathbf{K})$ where $\mathbf{K}$ is a locally finite field. In particular $C_G(F)$ is an infinite group.*

Proof. If F is a totaly semisimple element of G, then the result can be extracted from [1, Theorem 1.11] and the proof of [1, Theorem 2.1]. If F is a p-subgroup of a locally finite simple group defined over a field of characteristic p, then Lemma 3 and Lemma 4 give the result. □

Definition 6. *Let $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{I}\}$ be a Kegel cover of a simple locally finite group G. A finite p-subgroup F of G is called a $\mathcal{K}$-p-subgroup if $(p, |N_i|) = 1$ for all $i \in I$.*

In general, for any non-linear, simple locally finite group, it is not true that, every finite p-subgroup is a $\mathcal{K}$-p-subgroup. In [7] Meierfrankenfeld showed that, there exists a simple non-linear locally finite group G such that $C_G(x)$ is a p-group for an element x of order p. But by [1], if G has a Kegel sequence as above, then $C_G(x)$ involves an infinite non-linear simple subgroup. So the groups in [7] do not have such a nice Kegel sequence. Our results can be used in this direction to decide whether G has a nice Kegel sequence or not; provided that we know the structure of the centralizers of its elements.

Corollary 7. *Let G be a non-linear simple locally finite group and $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbf{N}\}$ be a Kegel sequence of G. Then for any finite $\mathcal{K}$-p-subgroup F, the centralizer $C_G(F)$ contains subgroups isomorphic to the homomorphic images of $SL(s, \mathbf{K})$. In particular $C_G(F)$ is an infinite group.*

Proof. This result is an easy application of splitting of the centralizer

$$C_{G_i/N_i}(FN_i/N_i) = C_{G_i}(F)N_i/N_i.$$

□

Lemma 8. *Let G be an infinite simple linear group over a locally finite field K of characteristic p and F be a p-subgroup of G (F could be infinite). Then $|C_G(F)| \geqslant |K|$. Moreover if $|K|$ is infinite, then $|C_G(F)|$ is infinite.*

Proof. By Zorn's lemma every p-subgroup of G is contained in a maximal p-subgroup P of G. As maximal p-subgroups of G are nilpotent; P is nilpotent and hence $Z(P) \leqslant C_G(F)$. As $Z(P)$ contains a subgroup isomorphic to the additive group of K we have whenever $|K|$ is infinite, then $|C_G(F)|$ is infinite. □

Definition 9 ([1])**.** *Let $\bar{G}$ be a simple linear algebraic group. A finite abelian subgroup A consisting of semisimple elements in $\bar{G}$ is called a d-abelian subgroup if one of the following holds:*

1. *The root system associated with $\bar{G}$ has type A_l, and the Hall π-subgroup of A is cyclic where π is the set of primes dividing $l+1$.*
2. *The root system associated with $\bar{G}$ has type B_l, C_l, D_l or G_2 and the Sylow 2-subgroup of A is cyclic.*
3. *The root system associated with $\bar{G}$ has type E_6, E_7 or F_4 and the Hall-$\{2,3\}$-subgroup of A is cyclic.*
4. *The root system associated with $\bar{G}$ has type E_8 and the Hall-$\{2,3,5\}$-subgroup of A is cyclic.*

Theorem 10. *Let G be an infinite simple classical group of rank l over a field of characteristic p and F be a finite subgroup of G with $F = P \times Q$ where P is a p-subgroup and Q is p'-part of F. Let s be a given integer. If Q is d-abelian and l is sufficiently large with respect to $|F|$ and s, then $C_G(F)$ contains subgroup isomorphic to homomorphic images of $SL(s, \mathbf{K})$. In particular $C_G(F)$ is an infinite group.*

Proof. Let G be an infinite simple locally finite group over a field K of characteristic p. Let $\bar{G}$ be the simple linear algebraic group of adjoint type over an algebraically closed field $\bar{K}$ which we obtain G as a union of fixed points of Frobenius automorphisms σ^{n_i} where $n_i | n_{i+1},\ i \in \mathbf{N}$, for details see [1]. Then Q is a d-abelian subgroup of $\bar{G}$ and by [8, Theorem 5.8(c) and Exercise 5.11], Q is contained in a maximal torus fixed by σ^{n_i} for all i. The group $C_{\bar{G}}(Q)^\circ$ is generated by T and the U_a with $a \in \Sigma_1$ where Σ_1 be the system of roots vanishing on Q and $C_{\bar{G}}(Q)^\circ$ is reductive group by [8, 4.1 (b) E35] with Σ_1 as its root system.

Since the order of $C_{\bar{G}}(Q)/C_{\bar{G}}(Q)^\circ$ is a finite fixed number it is enough to consider the theorem for $C_{\bar{G}}(Q)^\circ$.

By [4, p 20, Proposition] if $s \in G$ is semisimple, then $s \in C_G(s)^\circ$ and every unipotent element of $C_G(s)$ lies in $C_G(s)^\circ$ hence the p-part P of F also lies in $C_G(Q)^\circ$. The group $(C_G(Q)^\circ)'$ is a semisimple subgroup and P is a finite p-subgroup which is fixed by the Frobenius automorphisms σ^{n_i}. The automorphisms σ^{n_i} are automorphisms of $C_G(Q)^\circ$ see [8, 3.2, E.10.] and so automorphisms of $(C_G(Q)^\circ)'$. Our group P lies in the semisimple part of $C_{\bar{G}}(Q)^\circ$. Let $(C_{\bar{G}}(Q)^\circ)' = H_1 H_2 \dots H_k$ be the product of simple algebraic groups H_i. Then the Frobenius automorphism acts on the components and choose an orbit of σ^{n_1} containing a component of large rank. Since the rank of $C_{\bar{G}}(Q)^\circ$ is sufficiently large and this rank is the sum of the ranks of the simple components H_i, we have such a component. For the existence of this large rank see [5]. Then the rank of $C_{C_{\bar{G}(Q)^\circ}}(P)$ is sufficiently large. Then the fixed points of the automorphisms σ^{n_i} on this component gives the centralizers of large cardinality as the fixed points contains the classical groups of large cardinalities and certainly they are in the centralizer of F.

Then the fixed points of the automorphisms on this component gives the centralizers which proves the theorem. □

Dedicated to V. M. Levchuk and A. Yu. Ol'shanskii on the occasion of their 70th birthday.

References

[1] K. Ersoy, M. Kuzucuoğlu, Centralizers of subgroups in simple locally finite groups, *J. Group Theory*, **15**(2012), no. 1, 9–22.

[2] B. Hartley, Centralizing Properties in Simple Locally Finite Groups and Large Finite Classical Groups, *J. Austral. Math. Soc. (Series A)*, **49**(1990), 502–513.

[3] B. Hartley, M. Kuzucuoğlu, Centralizers of elements in locally finite simple groups, *Proc. London Math. Soc.*, **62** (1991), no. 3, 301–324.

[4] J. Humphreys, Conjugacy Classes in Semisimple Algebraic Groups, AMS, Mathematical Surveys and Monographs, 1995.

[5] M. Kuzucuoğlu, Centralizers of semisimple subgroups in locally finite simple groups, *Rend. Sem. Mat. Univ. Padova*, **92**(1994), 79-90.

[6] M. Kuzucuoğlu, Centralizers in simple locally finite groups, *International Journal of Group Theor.*, **02**(2012), no. 1, 1-10.

[7] U. Meierfrankenfeld, Locally Finite Simple Group with a p-group as centralizer, *Turkish J. Math.*, **31**(2007), 95–103.

[8] T.A.Springer, R.Steinberg, Conjugacy Classes in Seminar on Algebraic Groups and Related Finite Groups, *Lecture Notes in Math.*, **131**, 1970, Springer-Verlag, Berlin.

Централизаторы конечных p-подгрупп в простых локально конечных группах

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Нас интересуют следующие вопросы Б.Хартли: (1) Правда ли, что в бесконечной простой локально конечной группе, если централизатор конечной подгруппы линейный, то G является линейной? (2) Для конечной подгруппы F нелинейной простой локально конечной группы порядок $|CG(F)|$ бесконечен?

Доказывается следующее: пусть G — нелинейная простая локально конечная группа, имеющая последовательность Кегеля $\mathcal{K} = \{(G_i, 1) : I \in \mathbf{N}\}$, состоящую из конечных простых подгрупп. Пусть p — фиксированное простое число, $s \in \mathbf{N}$. Тогда для любой конечной p-подгруппы F группы G централизатор $C_G(F)$ содержит подгруппы, изоморфные гомоморфному образу $SL(s, \mathbf{F}_q)$. В частности, $C_G(F)$ является нелинейной группой. Мы также показываем, что если F — конечная p-подгруппа бесконечной локально конечной простой группы G задачи классического типа и заданных $s \in \mathbf{N}$, и ранг G достаточно большой относительно $|F|$ и s, то $C_G(F)$ содержит подгруппы, изоморфные гомоморфным образам $SL(s, K)$.

Ключевые слова: централизатор, простая локально конечная, нелинейная группа.