

# Evaluating the Rational Generating Function for the Solution of the Cauchy Problem for a Two-Dimensional Difference Equation with Constant Coefficients

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**Abstract**—We propose an algorithm for evaluation of rational generating functions for solutions of the Cauchy problems for two-dimensional difference equations with constant coefficients. The coefficients of one-dimensional difference equations and the initial data are used to solve the corresponding Cauchy problems. The algorithm is implemented in the Maple computer algebra system.

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## 1. INTRODUCTION

Studying and solving linear difference equations are among the topical problems of modern computer algebra [1, 2].

Generating functions are a powerful tool for investigating difference equations both in the theory of discrete dynamical systems and in the enumerative combinatorial analysis; this method allows one to apply methods of complex analysis to the problems of enumerative combinatorics.

The one-dimensional case has been extensively studied and presents no difficulties [3, 4]. In [5], A. de Moivre considered the power series

$$f(0) + f(1)z + \dots + f(k)z^k + \dots$$

with the coefficients  $f(0), f(1), \dots$  that satisfy the difference equation

$$\begin{aligned} c_m f(x+m) + c_{m-1} f(x+m-1) + \dots \\ + c_0 f(x) = 0, \quad x = 0, 1, 2, \dots, \end{aligned} \quad (1)$$

where  $c_m \neq 0$  and  $c_j \in \mathbb{C}$  are certain constants. He proved that such power series always represent rational functions (De Moivre's theorem).

In a substantially less studied multidimensional case [6–9], it is important to investigate rational generating functions, which, according to Stanley's hierarchy [10], represent “the most useful” class of generating functions. In the enumerative combinatorial analysis, there is a broad class of two-dimensional sequences that lead to rational generating functions,

for example, in the problems on the number of paths on an integer lattice, on the number of nodes in labeled trees, and the arrangement of chess pieces on a chessboard [11–13].

Generating functions for the multiple sequences whose elements can be expressed in terms of rational functions, exponential functions, and gamma function, constitute a broad and important class of hypergeometric-type functions [14, 15]. Study of these functions leads to the problem of solving overdetermined systems of linear equations with polynomial coefficients [16].

A significant number of works were devoted to developing algorithms for solving difference equations of various types (see, for example, [2, 17]). In this paper, we present an algorithm that computes a generating function for the solution of a Cauchy problem for a two-dimensional difference equation with constant coefficients without finding the solution itself, i.e., by using the coefficients of the difference equation and the initial data of the Cauchy problem. Since the initial data set is unbounded, we add a condition for the rationality of the generating function of initial data in order to describe all initial data by a finite set of finite-dimensional vectors. The algorithm is based on the results obtained in [6, 18].

A multidimensional version of De Moivre's theorem has been formulated and proved in [18]. To formulate this result, we first introduce some notations and definitions.

## 2. FORMULATION OF THE PROBLEM AND KNOWN RESULTS

Let us denote the points of the  $n$ -dimensional integer lattice  $\mathbb{Z}^n = \mathbb{Z} \times \dots \times \mathbb{Z}$  (where  $\mathbb{Z}$  denotes the set of integers) by  $x = (x_1, \dots, x_n)$ , and a finite subset of points from  $\mathbb{Z}^n$  by  $A$ . By a difference equation with respect to an unknown function  $f(x)$  of integer arguments  $x = (x_1, \dots, x_n)$  with constant coefficients  $c_\alpha$ , where  $\alpha = (\alpha_1, \dots, \alpha_n)$ , we call an expression of the form

$$\sum_{\alpha \in A} c_\alpha f(x + \alpha) = 0. \tag{2}$$

Throughout the paper, we assume that the set  $A$  lies in the positive octant  $\mathbb{Z}_0^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{Z}, x_i \geq 0, i = 1, \dots, n\}$  of the integer lattice and satisfies the condition

$$\exists m = (m_1, \dots, m_n) \in A : c_m \neq 0 \text{ and } \forall \alpha \in A, \forall j = 1, \dots, n \text{ so that } \alpha_j \leq m_j \text{ holds,} \tag{3}$$

Let us denote the *characteristic polynomial* of (2) by

$$P(z) = \sum_{\alpha \in A} c_\alpha z^\alpha = \sum_{\alpha \in A} c_{\alpha_1, \dots, \alpha_n} z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

A *generating function* for the function  $f(x)$  of integer arguments  $x \in \mathbb{Z}_0^n$  is defined as

$$F(z) = \sum_{x \in \mathbb{Z}_0^n} \frac{f(x)}{z^{x+I}}, \text{ where } I = (1, \dots, 1).$$

Consider, for example, the difference equation

$$\begin{aligned} f(x_1 + 1, x_2 + 1) - f(x_1 + 1, x_2) \\ - f(x_1, x_2 + 1) + f(x_1, x_2) = 0. \end{aligned}$$

Its solution is any function of the form

$$f(x_1, x_2) = \varphi(x_1) + \psi(x_2),$$

where  $\varphi$  and  $\psi$  are arbitrary functions of integer arguments.

The set on which we define the ‘‘initial data’’ of the difference equation (2), which satisfies condition (3), is defined by

$$X_0 = \{\tau \in \mathbb{Z}_0^n : \tau \not\geq m\},$$

where the symbol  $\not\geq$  means that the point  $\tau$  lies in the complement of the set defined by the inequalities  $\tau_j \geq m_j, j = 1, \dots, n$ .

The *Cauchy problem* is formulated as a problem of finding the solution  $f(x)$  of (2) that, on the set  $X_0$ , coincides with a given function  $\varphi(x)$ :

$$f(x) = \varphi(x), \quad x \in X_0. \tag{4}$$

It is easy to show (see, for example, [8]) that, if condition (3) is satisfied, then problem (2), (4) has a unique solution in the positive octant  $\mathbb{Z}_0^n$ . A question on solv-

ability of problem (2), (4) without imposing any constraints of the form (3) was considered in [6]. An explicit formula for computing the generating function  $F(z)$  in the case of arbitrary dimension was given in [18].

The further discussion (for  $n = 2$ ) will require some notation. Let us split the rectangle

$$\Pi_m = \{x \in \mathbb{Z}_0^2 : x_k \leq m_k, k = 1, 2\}$$

into the following four subsets:

$$\Gamma_{0,0} = \{x \in \mathbb{Z}_0^2 : x_1 < m_1, x_2 < m_2\},$$

$$\Gamma_{0,1} = \{x \in \mathbb{Z}_0^2 : x_1 < m_1, x_2 = m_2\},$$

$$\Gamma_{1,0} = \{x \in \mathbb{Z}_0^2 : x_1 = m_1, x_2 < m_2\},$$

$$\Gamma_{1,1} = \{x \in \mathbb{Z}_0^2 : x_1 = m_1, x_2 = m_2\}.$$

The mapping  $J$ , which associates each point of the rectangle  $\Pi_m$  to a certain element in the set of pairs  $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ , is defined as

$$J(\tau) = \begin{cases} (0, 0), & \tau \in \Gamma_{0,0}; \\ (0, 1), & \tau \in \Gamma_{0,1}; \\ (1, 0), & \tau \in \Gamma_{1,0}; \\ (1, 1), & \tau \in \Gamma_{1,1}. \end{cases}$$

The generating function  $\Phi(z) = \sum_{x \in X_0} \varphi(x) z^{x+I}$  of the initial data for the problem (2), (4) can be represented as the sum

$$\Phi(z) = \sum_{\tau \in \Pi_m} \Phi_\tau(z),$$

where

$$\Phi_\tau(z) = \sum_{y=0}^{\infty} \frac{\varphi(\tau + J(\tau)y)}{z^{\tau+J(\tau)y+I}}$$

are one-dimensional generating functions ( $\tau \in \Pi_m$ ).

**Theorem 1** [17]. *The generating function  $F(z)$  for the solution of problem (2), (4) under condition (3) and the generating function of initial data  $\Phi(z)$  are related as follows:*

$$P(z)F(z) = \sum_{\tau \in \Pi_m} \Phi_\tau(z)P_\tau(z), \tag{5}$$

where the polynomials  $P_\tau(z)$  have the form

$$P_\tau(z) = \sum_{\substack{\alpha \leq m \\ \alpha \leq \tau}} c_\alpha z^\alpha. \tag{6}$$

Let us consider some particular cases. For  $n = 1$ , the generating function

$$F(z) = \sum_{x=0}^{\infty} \frac{f(x)}{z^x}$$

for the solution  $f(x)$  of (1) under the condition  $f(x) = \varphi(x)$  (where  $x = 0, \dots, m - 1$  and  $\varphi(x)$  defines the initial data) is

$$F(z) = \frac{\sum_{\alpha=1}^m \sum_{x=0}^{\alpha-1} \frac{c_\alpha \varphi(x)}{z^{x-\alpha}}}{\sum_{\alpha=0}^m c_\alpha z^\alpha}. \tag{7}$$

For  $n = 2$ , Theorem 1 was proved in [19] in connection with the study of rational Riordan arrays. For arbitrary  $n > 1$ , the proof was given in [18]. The properties of the generating function for the solution of a difference equation in rational cones of an integer lattice were studied by T. Nekrasova (see, e.g. [20]).

Theorem 1 easily yields the following multidimensional analog of De Moivre’s theorem, which is essential for developing the computational algorithm.

**Theorem 2** [18]. *If the generating function of initial data  $\Phi(z)$  is rational, then the generating function  $F(z)$  for the solution of problem (2), (4) under condition (3) is also rational.*

This theorem and its converse (stating the equivalence of rationality for the generating functions of initial data and the solution of problem (2), (4) under condition (3)) were proved in [18].

### 3. DESCRIPTION OF THE ALGORITHM

For  $n = 1$ , formula (7) for the generating function consists of a finite number of summands, so the algorithm for its computing is rather simple. The input data are two finite sets of numbers: the coefficients of the difference equation  $\{c_\alpha\}_{\alpha=0}^m$  and the initial data  $\{\varphi_\alpha\}_{\alpha=0}^m$ ; the algorithm yields a fractional-rational function of the form (7).

In the general case, for  $n > 1$ , the initial data set  $X_0$  is infinite. For  $n = 2$ , the algorithm computing the generating function  $F(z)$  is reduced to computing a finite number of one-dimensional generating functions for the sequences whose elements lie along the coordinate axes and are uniquely defined by the coefficients of the one-dimensional difference equation and by a finite amount of initial data (which are actually different for each sequence).

Initial data of a two-dimensional difference equation correspond to a finite set of one-dimensional sequences lying along the coordinate axes. In this paper, we consider a case where each such sequence is given by a one-dimensional difference equation and, therefore, can be “encoded” by finite one-dimensional sets of coefficients of the difference equation and by the set of its initial data.

The coefficients of one-dimensional difference equations are given by a matrix  $C$  in which the ele-

ments of the first column define the rays on which initial data lie, while the corresponding elements of the second column define the coefficients of the difference equations that, define the initial data lying on these rays.

The initial data of the one-dimensional difference equations are given by the matrix *InData*, which contains a finite initial data subset of the Cauchy problem for the given two-dimensional equation. Since the initial data sets of the one-dimensional difference equations may actually differ (e.g., in their sizes) and should be made consistent, only some elements of this matrix need to be set for the algorithm. Let us illustrate the procedure of setting the initial data by the following example.

Let the initial data subsets of the two-dimensional difference equation that lie along the horizontal axis be given by the three one-dimensional difference equations

$$\begin{aligned} c_3^0 \varphi(x + 3, 0) + c_2^0 \varphi(x + 2, 0) \\ + c_1^0 \varphi(x + 1, 0) + c_0^0 \varphi(x, 0) &= 0, \\ c_2^1 \varphi(x + 2, 1) + c_1^1 \varphi(x + 1, 1) + c_0^1 \varphi(x, 1) &= 0, \\ c_2^2 \varphi(x + 2, 2) + c_1^2 \varphi(x + 1, 2) + c_0^2 \varphi(x, 2) &= 0, \end{aligned}$$

and the initial data subsets of the two-dimensional difference equation that lie along the vertical axis be given by the two one-dimensional difference equations

$$\begin{aligned} d_2^0 \varphi(0, y + 2) + d_1^0 \varphi(0, y + 1) + d_0^0 \varphi(0, y) &= 0, \\ d_3^1 \varphi(1, y + 3) + d_2^1 \varphi(1, y + 2) \\ + d_1^1 \varphi(1, y + 1) + d_0^1 \varphi(1, y) &= 0. \end{aligned}$$

Then the initial data of the one-dimensional difference equations can be given by the matrix

$$\begin{pmatrix} * & \varphi(1, 2) & * \\ \varphi(0, 1) & \varphi(1, 1) & * \\ \varphi(0, 0) & \varphi(1, 0) & \varphi(2, 0) \end{pmatrix}.$$

The algorithm skips the elements denoted by  $*$ ; for instance,  $\varphi(0, 2)$  is evaluated by using the corresponding difference equation and the initial data of  $\varphi(0, 0)$  and  $\varphi(0, 1)$ .

Generally speaking, suppose that, for each  $\xi_1$  from 0 to  $m_1 - 1$ , there exist numbers  $\mu_{\xi_1} \in \mathbb{N}$  and  $c_{\mu_{\xi_1}}^{\xi_1}, c_{\mu_{\xi_1}-1}^{\xi_1}, \dots, c_0^{\xi_1} \in \mathbb{C}$  such that the initial data subset  $\{\varphi(\xi_1, x_2)\}_{x_2=0}^\infty$  lying along the vertical axis of an integer lattice  $\mathbb{Z}_+^2$  satisfies the one-dimensional Cauchy problem of the form

$$\begin{aligned} \sum_{i=0}^{\mu_{\xi_1}} c_{\mu_{\xi_1}-i}^{\xi_1} f(\xi_1, x_2 + \mu_{\xi_1} - i) &= 0, \\ f(\xi_1, x_2) &= \varphi(\xi_1, x_2), \quad x_2 = 0, \dots, \mu_{\xi_1} - 1. \end{aligned}$$

Then, according to formula (7), for each  $\xi_1$  from 0 to  $m_1 - 1$ , the one-dimensional generating function for the sequence  $\{\varphi(\xi_1, x_2)\}_{x_2=0}^\infty$  is a fractional-rational function of the form

$$\Phi^{\xi_1}(z_2) = \frac{\sum_{\alpha=1}^{\mu_{\xi_1}} \sum_{x_2=0}^{\alpha-1} \frac{c_{\alpha}^{\xi_1} \varphi(\xi_1, x_2)}{z_2^{x_2-\alpha}}}{\sum_{\alpha=0}^{\mu_{\xi_1}} c_{\alpha}^{\xi_1} z_2^{\alpha}}.$$

Similarly, for each  $\xi_2$  from 0 to  $m_2 - 1$ , there are numbers  $v_{\xi_2} \in \mathbb{N}$  and  $d_{v_{\xi_2}}^{\xi_2}, d_{v_{\xi_2}-1}^{\xi_2}, \dots, d_0^{\xi_2} \in \mathbb{C}$  such that the initial data subset  $\{\varphi(x_1, \xi_2)\}_{x_1=0}^\infty$  lying along the horizontal axis of the integer lattice  $\mathbb{Z}_+^2$  satisfies the one-dimensional Cauchy problem of the form

$$\sum_{i=0}^{v_{\xi_2}} d_{v_{\xi_2}-i}^{\xi_2} f(x_1 + v_{\xi_2} - i, \xi_2) = 0, \\ f(x_1, \xi_2) = \varphi(x_1, \xi_2), \quad x_1 = 0, \dots, v_{\xi_2} - 1.$$

Therefore, for each  $\xi_2$  from 0 to  $m_2 - 1$ , the one-dimensional generating function for the sequence  $\{\varphi(x_1, \xi_2)\}_{x_1=0}^\infty$  is a fractional-rational function of the form

$$\Psi^{\xi_2}(z_1) = \frac{\sum_{\alpha=1}^{v_{\xi_2}} \sum_{x_1=0}^{\alpha-1} \frac{d_{\alpha}^{\xi_2} \varphi(x_1, \xi_2)}{z_1^{x_1-\alpha}}}{\sum_{\alpha=0}^{v_{\xi_2}} d_{\alpha}^{\xi_2} z_1^{\alpha}}.$$

Thus, the input data are finite and is given by

- the  $(m_1 + 1) \times (m_2 + 1)$  matrix  $c = (c_{\alpha_1, \alpha_2})$ ,  $\alpha_1 = 0, \dots, m_1$ ,  $\alpha_2 = 0, \dots, m_2$  comprising the coefficients  $c_{\alpha_1, \alpha_2}$  of the two-dimensional difference equation of form (2);
- the  $M_1 \times M_2$  matrix  $InData = (d_{x_1, x_2})$  containing the initial data of the two-dimensional Cauchy problem, where  $M_1 = \max\{v_0, \dots, v_{m_2-1}, m_1 - 1\}$ ,  $M_2 = \max\{\mu_0, \dots, \mu_{m_1-1}, m_2 - 1\}$ , and  $d_{x_1, x_2} = \varphi(x_1, x_2)$  if  $(x_1, x_2) \in X_0$  and  $d_{x_1, x_2} = 0$  for the other  $(x_1, x_2)$ ;
- the  $(m_1 + m_2) \times 2$  matrix  $C$  of tuples that determine the coefficients of the difference equations, which, in turn, describe the initial data lying along the coordinate axes. Here the first element of the row is a pair of the form  $[\xi_1 + 1, 0]$  or  $[0, \xi_2 + 1]$ ,  $\xi_1, \xi_2 \in \mathbb{Z}_+$ . The pair of the form  $[\xi_1 + 1, 0]$  selects the subsequence of initial data  $\varphi(\xi_1, y)$ ,  $y \geq 0$  along the vertical axis, while the pair of the form  $[0, \xi_2 + 1]$  selects the subsequence of initial data  $\varphi(y, \xi_2)$ ,  $y \geq 0$  along the horizontal axis. The second element in the row represents the coefficients of the one-dimensional difference equa-

tion, which describe the corresponding initial data. The number of the coefficients may differ for each set.

In the general case, the matrix  $C$  has the following structure:

$$C = \begin{pmatrix} (1, 0) & (c_{\mu_0}^0, c_{\mu_0-1}^0, \dots, c_0^0) \\ (2, 0) & (c_{\mu_1}^1, c_{\mu_1-1}^1, \dots, c_0^1) \\ \dots & \dots \\ (m_1, 0) & (c_{\mu_{m_1-1}}^{m_1-1}, c_{\mu_{m_1-1}-1}^{m_1-1}, \dots, c_0^{m_1-1}) \\ (0, 1) & (d_{v_0}^0, d_{v_0-1}^0, \dots, d_0^0) \\ (0, 2) & (d_{v_1}^1, d_{v_1-1}^1, \dots, d_0^1) \\ \dots & \dots \\ (0, m_2) & (d_{v_{m_2-1}}^{m_2-1}, d_{v_{m_2-1}-1}^{m_2-1}, \dots, d_0^{m_2-1}) \end{pmatrix}.$$

For  $n = 2$ , formula (5) can be rewritten as follows (which considerably simplifies the development of the algorithm):

$$F(z_1, z_2) = \left( \sum_{\xi_1=0}^{m_1-1} \sum_{\xi_2=0}^{m_2-1} \frac{P_{\xi_1, \xi_2}(z_1, z_2)}{z_1^{\xi_1+1} z_2^{\xi_2+1}} \varphi(\xi_1, \xi_2) + \sum_{\xi_1=0}^{m_1-1} \frac{P_{\xi_1, m_2}(z_1, z_2)}{z_1^{\xi_1+1}} \Phi_{m_2}^{\xi_1}(z_2) + \sum_{\xi_2=0}^{m_2-1} \frac{P_{m_1, \xi_2}(z_1, z_2)}{z_2^{\xi_2+1}} \Psi_{m_1}^{\xi_2}(z_1) \right) / P(z), \tag{8}$$

where

$$\Phi_{m_2}^{\xi_1}(z_2) = \frac{z^{m_2} \sum_{\alpha=1}^{\mu_{\xi_1}} \sum_{x_2=0}^{\alpha-1} \frac{c_{\alpha}^{\xi_1} \varphi(\xi_1, x_2 + m_2)}{z_2^{x_2-\alpha}}}{\sum_{\alpha=0}^{\mu_{\xi_1}} c_{\alpha}^{\xi_1} z_2^{\alpha}} \tag{9}$$

and

$$\Psi_{m_1}^{\xi_2}(z_1) = \frac{z^{m_1} \sum_{\alpha=1}^{v_{\xi_2}} \sum_{x_1=0}^{\alpha-1} \frac{d_{\alpha}^{\xi_2} \varphi(x_1 + m_1, \xi_2)}{z_1^{x_1-\alpha}}}{\sum_{\alpha=0}^{v_{\xi_2}} d_{\alpha}^{\xi_2} z_1^{\alpha}}. \tag{10}$$

All the summands on the right-hand side of the equality are divided into three groups. In the first group, the number of summands is finite; in the second and third groups, the generating function of one variable is assumed to be computed. In the last two formulas, the summation of generating series starts from the elements  $m_2$  and  $m_1$ , respectively.

Below, we propose an implementation of the algorithm that computes the rational generating function for

the solution of the Cauchy problem formulated for the two-dimensional difference equation with constant coefficients on the condition that the initial data of this problem are described by one-dimensional difference equations with constant coefficients. Note that the condition for rationality of the generating function is essential as it ensures the finiteness of the input data for the proposed algorithm.

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**Algorithm 1.** An algorithm for computing the initial data  $\varphi(x_0, y_0)$  at arbitrary point  $(x_0, y_0) \in X_0$ .

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**Input:** Matrices  $c$ ,  $C$ , and  $InData$ , and coordinates  $(x_0, y_0) \in X_0$ .

**Output:** The value of the function of initial data  $\varphi(x_0, y_0)$ .

**Procedure**  $\varphi(c, C, InData, (x_0, y_0))$

$m_1 :=$  the number of rows in the matrix  $c$

$m_2 :=$  the number of columns in the matrix  $c$

**if**  $x_0 < m_1$  **then**

**for**  $i$  from 1 to  $m_1 + m_2$  **do**

**if**  $c[i, 1] = (x_0 + 1, 0)$  **then**

$c_0 := C[i, 2]$

**end if**

**end for**

$\varphi_0 :=$   $x_0$ th column of the matrix  $InData$

$l_0 := y_0$

**end if**

**if**  $y_0 < m_2$  **then**

**for**  $i$  from 1 to  $m_1 + m_2$  **do**

**if**  $c[i, 1] = (0, y_0 + 1)$  **then**

$c_0 := C[i, 2]$

**end if**

**end for;**

$\varphi_0 :=$   $y_0$ th row of the matrix  $InData$

$l_0 := x_0$

**end if**

$l :=$  the length of the vector  $c_0$

**for**  $i$  from  $l$  to  $l_0$  **do**

$$\varphi_0(i) := -\sum_{j=0}^{l-1} \frac{c_0[j]}{c_0[m]} \varphi_0(i + j - l)$$

**end for**

**return**  $\varphi_0(l_0)$

**end procedure**

This algorithm is used to compute the missing multipliers  $(\xi_1; \xi_2)$  in the first sum of (8) and to compute the generating functions in the second and third sums of (8).

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**Algorithm 2.** An algorithm for computing a one-dimensional generating function according to formulas (9) and (10).

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**Input:** A vector  $M$  of coefficients of a difference equation (in ascending order of subscripts); a vector  $\Phi$  of initial data (in ascending order of subscripts) whose length is lesser than that of  $M$  by one; an integer number  $start$  from which the summation begins; a number  $t$  that takes value 1 or 2; and a variable  $x_t$  that is used to find the generating function.

**Output:** The generating function determined by formulas (9) or (10) depending on the value of  $t$ .

**Procedure**  $GF(M, \Phi, start, t)$

$F := 0$

$n :=$  the length of the vector  $M$  without one

**for**  $\alpha$  from 1 to  $n + 1$  **do**

$c_{\alpha-1} := M[\alpha]$

**end for**

**for**  $\alpha$  from 1 to  $n$  **do**

$\varphi[\alpha - 1] := \Phi[\alpha]$

**end for**

**for**  $\alpha$  from  $n$  to  $start + n - 1$  **do**

$\varphi[\alpha] := 0$

**for**  $j$  from 1 to  $n$  **do**

$$\varphi[\alpha] := \varphi[\alpha] - \frac{c_{j-1}}{c_n} \cdot \varphi[\alpha - n + j - 1]$$

**end for**

**end for**

**for**  $\alpha$  from 0 to  $n - 1$  **do**

$\varphi[\alpha] := \varphi[start + \alpha]$

**end for**

**for**  $\alpha$  from 1 to  $n$  **do**

**for**  $x$  from 0 to  $\alpha - 1$  **do**

$$F := F + \frac{c_\alpha \cdot \varphi[x]}{z_t^{x-\alpha+1}}$$

**end for**

**end for**

$Q := 0$

**for**  $\alpha$  from 0 to  $n$  **do**

$$Q := Q + c_\alpha \cdot z_t^\alpha$$

**end for**

$$F := F \cdot z_t^{start} / Q$$

**return**  $F$

**end procedure**

**Algorithm 3.** The algorithm for computing two-dimensional generating function of the solution of the initial value problem (2), (4)

**Input:** Matrices  $c$ ,  $C$ , and  $InData$ .

**Output:** The generating function  $F(z)$  of the solution to the difference equation.

```

1: Procedure GenFunc( $c$ ,  $C$ ,  $InData$ )
2:    $m_1 :=$  the number of rows in the matrix  $c$ 
3:    $m_2 :=$  the number of columns in the matrix  $c$ 
4:    $F := 0$ 
5:   for  $x_1$  from 1 to  $m_1 - 1$  do
6:     for  $x_2$  from 1 to  $m_2 - 1$  do
7:        $F := F + P_{x_1, x_2}(z_1, z_2) z_1^{x_1} z_2^{x_2}$ 
          $\cdot \varphi(c, C, InData, (x_0, y_0))$ 
8:     end for
9:   end for
10:  for  $\xi_1$  from 0 to  $m_1 - 1$  do
11:     $\Phi :=$  ( $\xi_1 + 1$ )th column of the matrix  $InData$ 
12:     $M := C[i, 2]$ , such that  $C[i, 1] = [\xi_1 + 1, 0]$ 
13:    for  $i$  from 1 to  $\xi_1 + \xi_2$  do
14:      if  $C[i, 1] = (\xi_1 + 1, 0)$  then
15:         $M := C[i, 2]$ 
16:      end if
17:    end for
18:     $F := F + \frac{P_{\xi_1, m_2}(z_1, z_2)}{z_1^{\xi_1 + 1}} \cdot GF(M, \Phi, m_2, 2)$ 
19:  end for
20:  for  $\xi_2$  from 0 to  $m_2 - 1$  do
21:     $\Phi :=$  ( $\xi_2 + 1$ )th row of the matrix  $InData$ 
22:    for  $i$  from 1 to  $\xi_1 + \xi_2$  do
23:      if  $C[i, 1] = (0, \xi_2 + 1)$  then
24:         $M := C[i, 2]$ 
25:      end if
26:    end for
27:     $F := F + \frac{P_{m_1, \xi_2}(z_1, z_2)}{z_2^{\xi_2 + 1}} \cdot GF(M, \Phi, m_1, 1)$ 
28:  end for
29:   $F := F / P(z_1, z_2)$ 
30:  return  $F$ 
31: end procedure

```

4. EXAMPLES AND CONCLUSIONS

The algorithm is implemented in Maple 2015 64bit. The complete code of the program is available at <https://github.com/lyapinap/KLS2016/blob/master>

/algorithm2016.mw. The computations were carried out on Intel Core i7-4790 (3.6 GHz) with 32 GB RAM under Microsoft Windows 7 Enterprise x64 SP1. For the examples given below, the running time was less than one second.

**Example 1.** The binomial coefficients

$$C_n^k = \frac{n!}{k!(n-k)!}$$

can be represented as a solution of the difference equation

$$f(x+1, y+1) - f(x, y+1) - f(x, y) = 0$$

with the initial data

$$\phi(x, y) = \begin{cases} 1, & \text{if } x \geq 0, \quad y = 0; \\ 0, & \text{if } x = 0, \quad y \geq 1. \end{cases}$$

In this example,  $m_1 = m_2 = 1$  and the initial data set has the form  $X_0 = \{(x_1, x_2) \in \mathbb{Z}_+^2 : (x_1, x_2) \neq (1, 1)\}$  (see Fig. 1).

The input data for the algorithm are given by the matrices

$$c = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

$$InData = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} (0,1) & (-1,1) \\ (1,0) & (1,0) \end{pmatrix}.$$

The algorithm yields the following generating function for the solution of the corresponding Cauchy problem:

$$F(z, w) = \frac{1}{zw - w - 1}.$$

**Example 2.** Consider certain sequences of  $x$  elements  $a_1 a_2 \dots a_x$  where  $a_1 = 0$  and  $a_j \in \{0, 1\}$  for  $2 \leq j \leq x$ . The element  $a_j$  is called isolated if it differs from all neighbor elements. The number of the sequences containing  $y$  isolated elements is denoted by  $r(x, y)$ . Obviously,  $r(x, y) = 0$  for  $x < y$ . So,  $r(x, y)$  will take the following values:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & \dots \\ 0 & 0 & 0 & 1 & 0 & 4 & 4 & \dots \\ 0 & 0 & 1 & 0 & 3 & 3 & 9 & \dots \\ 0 & 1 & 0 & 2 & 2 & 5 & 8 & \dots \\ 1 & 0 & 1 & 1 & 2 & 3 & 5 & \dots \end{pmatrix},$$

where the bottom-left corner corresponds to the value  $r(0, 0)$ .

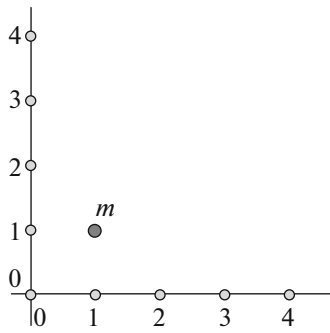


Fig. 1. Initial data set  $X_0$ .

In [13], the values  $r(x, y)$  were shown to be a solution of the Cauchy problem for the difference equation

$$r(x+2, y+1) - r(x+1, y+1) - r(x+1, y) - r(x, y+1) + r(x, y) = 0$$

with the initial data

$$\begin{aligned} \varphi(0, 0) &= 1, & \varphi(1, 0) &= 0, \\ \varphi(x, 0) &= \varphi(x-1, 0) + \varphi(x-2, 0), & x &\geq 2, \\ \varphi(0, y) &= 0, & y &\geq 1, \\ \varphi(1, 1) &= 1, & \varphi(1, y) &= 0, \quad y \geq 2. \end{aligned}$$

In this case, the input data for the algorithm are given by the matrices

$$c = \begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\text{InData} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} (0, 1) & (-1, -1, 1) \\ (1, 0) & (0, 1) \\ (2, 0) & (0, 0, 1) \end{pmatrix}.$$

The algorithm yields the following generating function for the solution of the corresponding Cauchy problem:

$$F(z, w) = \frac{z-1}{z^2w - zw - w - z + 1}.$$

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#### REFERENCES

1. Abramov, S.A., Barkatou, M.A., van Hoeij, M., and Petkovsek, M., Subanalytic solutions of linear difference equations and multidimensional hypergeometric sequences, *J. Symbolic Comput.*, 2011, vol. 46, pp. 1205–1228.
2. Abramov, S.A., Gheffar, A., and Khmel'nov, D.E., Rational solutions of linear difference equations: Universal denominators and denominator bounds, *Program. Comput. Software*, 2011, vol. 37, no. 2, pp. 78–86.
3. Gel'fond, A.O., *Ischislenie konechnykh raznostei* (Evaluation of Finite Differences), Moscow: KomKniga, 2006.
4. Isaacs, R.F., A finite difference function theory, *Univ. Nac. Tucuman. Rev.*, 1941, vol. 2, pp. 177–201.
5. Moivre, A. de, De fractionibus algebraicis radicalitate immunibus ad fractiones simpliciores reducendis, deque summandis terminis quarumdam serierum aequali intervallo a se distantibus, *Philos. Trans.*, vol. 32, pp. 162–178.
6. Bousquet-Melou, M. and Petkovsek, M., Linear recurrences with constant coefficients: The multivariate case, *Discrete Math.*, 2000, vol. 225, pp. 51–75.
7. Leinartas, E.K., Multiple Laurent series and difference equations, *Sib. Math. J.*, 2004, vol. 45, no. 2, pp. 321–326.
8. Leinartas, E.K., Multiple Laurent series and fundamental solutions of linear difference equations, *Sib. Math. J.*, 2007, vol. 48, no. 2, pp. 268–272.
9. Leinartas, E.K., Passare, M., and Tsikh, A.K., Multidimensional versions of the Poincaré theorem for difference equations, *Mat. Sb.*, 2008, vol. 199, no. 10, pp. 87–104.
10. Stanley, R.P., *Enumerative Combinatorics*, Cambridge Univ. Press, 2011.
11. Egorychev, G.P., *Integral'noe predstavlenie i vychislenie kombinatornykh summ* (Integral Representation and Evaluation of Combinatorial Sums), Novosibirsk: Nauka, 1977.
12. Baccherini, D., Merlini, D., and Sprugnoli, R., Level generation trees and proper Riordan arrays, *Appl. Anal. Discrete Math.*, 2008, no. 2, pp. 69–91.
13. Bloom, D.M., Singles in a sequence of coin tosses, *College Math. J.*, 1998, pp. 307–344.
14. Sadykov, T.M., On a multidimensional system of hypergeometric differential equations, *Sib. Math. J.*, 1998, vol. 39, no. 5, pp. 986–997.
15. Dickenstein, A., Sadykov, T.M., Bases in the solution space of the Mellin system, *Sbornik Mathematics*, 2007, vol. 198, no. 9-10, pp. 1277–1298.
16. Dickenstein, A. and Sadykov, T.M., Algebraicity of solutions to the Mellin system and its monodromy, *Dokl. Math.*, 2007, vol. 75, no. 1, pp. 80–82.
17. Abramov, S.A., Search of rational solutions to differential and difference systems by means of formal series, *Program. Comput. Software*, 2015, vol. 41, no. 2, pp. 65–73.
18. Leinartas, E.K. and Lyapin, A.P., On rationality of multidimensional reciprocal power series, *Zh. Sib. Fed. Univ. Ser. Mat. Fiz.*, 2009, vol. 2, no. 4, pp. 499–455.
19. Lyapin, A.P., Riordan arrays and two-dimensional difference equations, *Zh. Sib. Fed. Univ. Ser. Mat. Fiz.*, 2009, vol. 2, no. 2, pp. 210–220.
20. Nekrasova, T.I., On the hierarchy of generating functions for solutions of multidimensional difference equations, *Izv. Irkutsk. Gos. Univ.*, 2014, vol. 9, pp. 91–102.

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