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Kolmogorov System with Explicit Hyperbolic Limit Cycle

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A class of Kolmogorov differential system is introduced. We show that under suitable assumptions on parameters, an algebraic hyprbolic limit cycle can occur, the explicit expression of this limit cycle is given.

Keywords: Kolmogorov differential system, Invariant curve, Singular point, Periodic solution, Algebraic limit cycle.
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Introduction

The analysis of the existence, number and stability of limit cycles of the non linear differential system:

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x,y), \\ \dot{y} = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1}$$

where P and Q are polynomials, has long been a topic of interests.

Many mathematical models in biology science and population dynamics, frequently involve the systems of ordinary differential equations having the form

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = xF(x,y), \\ \dot{y} = \dfrac{dy}{dt} = yG(x,y), \end{cases} \tag{2}$$

where $x(t)$ and $y(t)$ represent the population density of two species at time t, and $F(x,y)$, $G(x,y)$ are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems.

Kolmogorov models are widely used in ecology to describe the interaction between two populations, and a limit cycle corresponds to an equilibrium state of the system.

When $F(x,y)$ and $G(x,y)$ are polynomials of degrees $\geqslant 2$, limit cycles can occur and there is an extensive literature dealing with their existence, number and stability (see for instance May [13], Lloyd, Pearson,Saèz and Szántó [11, 12], Huang [8], Huang, Wang, Cheng [9], Huang, Zhu [10], Boqian and Demeng [4], Cheng [5], and references therein), but to our knowledge, the

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exact analytic expressions of the limit cycles for a given kolmogorov system is still unknown except in simplest and specific cases.

This paper is a contribution in that direction, motivated by the recent publication of some research papers exhibiting planar polynomial systems with one or more algebraic limit cycles analytically given (see for instance Bendjeddou and Cheurfa [1, 2], Benyoucef, Bendjeddou [3], Chengbin, Boqian [4], Peng Yue-hui [15] and references therein), we will prove the existence of a limit cycle of a class of Kolmogorov system, and give its explicit form.

1. Some useful notions

Let us recall some useful notions.

For $U \in \mathbb{R}[x,y]$, the algebraic curve $U=0$ is called an invariant curve of the polynomial system (2), if for some polynomial $K \in \mathbb{R}[x,y]$ called the cofactor of the algebraic curve, we have

$$xF(x,y)\frac{\partial U}{\partial x} + yG(x,y)\frac{\partial U}{\partial y} = KU. \tag{3}$$

The curve $\Gamma = \left\{(x,y) \in \mathbb{R}^2 : U(x,y) = 0\right\}$ is non-singular of system (2) if the equilibrium points of the system that satisfy

$$\begin{cases} xF(x,y) = 0, \\ yG(x,y) = 0 \end{cases} \tag{4}$$

are not contained on the curve Γ.

A limit cycle $\gamma = \{(x(t), y(t)), t \in [0,T]\}$, is a T-periodic solution of system (2), isolated with respect to all other possible periodic solutions of the system.

Let $\gamma(t)$ be periodic orbit of system(2) of period T, then γ is an hyperbolic limit cycle if $\int_0^T \operatorname{div}(\gamma)dt$ is different from zero.

We construct here a multi-parameter Kolmogorov system admitting a limit cycle if some conditions on the parameters are satisfied.

2. The main result

As a main result, we have the following theorem,

Theorem 1. *The polynomial differential system*

$$\begin{cases} \dot{x} = x\left(\left(ax^{n+1} + bx^n + x\left(cy^m + dy^{m-1} + h\right)\right) - (x+y)\left(mcy^m + d\,(m-1)\,y^{m-1}\right)\right), \\ \dot{y} = y\left(\left(y(ax^n + bx^{n-1} + h) + cy^{m+1} + dy^m\right) + (x+y)\left(nax^n + b(n-1)x^{n-1}\right)\right), \end{cases} \tag{5}$$

where a, c are positive real, b, d are negative real, and h satisfied

$$\begin{aligned} &\max\left\{(-1)^n\left(\frac{n-1}{a}\right)^{n-1}\left(\frac{b}{n}\right)^n, (-1)^m\left(\frac{m-1}{c}\right)^{m-1}\left(\frac{d}{m}\right)^m\right\} < h < \\ &< (-1)^n\left(\frac{n-1}{a}\right)^{n-1}\left(\frac{b}{n}\right)^n + (-1)^m\left(\frac{m-1}{c}\right)^{m-1}\left(\frac{d}{m}\right)^m, \quad n \geqslant 2, m \geqslant 2 \end{aligned} \tag{6}$$

admits an hyperbolic limit cycle in realistic quadrant.The limit cycle is represented by the closed trajectory of the curve Γ.

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 : ax^n + bx^{n-1} + cy^m + dy^{m-1} + h = 0, \ \ (n \geqslant 2, m \geqslant 2)\right\}. \tag{7}$$

Proof. We will prove that Γ is composed of closed trajectory, it is nonsingular and an invariant curve of system (5), and $\int_0^T \operatorname{div}(\Gamma)dt \neq 0$ (see for instance Perko [14]).

i) The curve Γ is composed of closed trajectory.

We consider $U(x,y)$ as a function

$$f_x(y) = cy^m + dy^{m-1} + ax^n + bx^{n-1} + h,$$

where x is a real parameter, a, c positive real, b, d negative real, $\frac{df_x}{dy} = y^{m-2}\left(dm - d + cmy\right)$, $\frac{df_x}{dy} = 0 \Rightarrow y = 0$ or $y = \frac{-d\,(m-1)}{cm}$, $f_x(0) = ax^n + bx^{n-1} + h$, and

$$f_x(\frac{-d\,(m-1)}{cm}) = (-1)^{m-1}\left(\frac{m-1}{c}\right)^{m-1}\left(\frac{d}{m}\right)^m + ax^n + bx^{n-1} + h.$$

Let $p(x) = f_x(0)$, and $q(x) = f_x(\frac{-d\,(m-1)}{cm})$. We distinguish the following cases.

1. Numbers m and n are even.

Function $y \to f_x(y)$ is decreasing for $y \in \left]-\infty, \frac{-d(m-1)}{cm}\right[$ and increasing for $y \in \left]\frac{-d(m-1)}{cm}, +\infty\right[$.

If $p(x)$ is positive and $q(x)$ is negative, by applying the theorem of intermediate values we conclude that the equation $f_x(y) = 0$ admits two solutions,

$$y_1(x) \in \left]0, \frac{-d(m-1)}{cm}\right] \text{ and } y_2(x) \in \left[\frac{-d(m-1)}{cm}, +\infty\right[.$$

Function $\frac{dp}{dx} = x^{n-2}\left(b(n-1) + anx\right)$, $x \to p(x)$ is decreasing for $x \in \left]-\infty, \frac{-b(n-1)}{an}\right[$ and increasing for $x \in \left]\frac{-b(n-1)}{an}, +\infty\right[$, for $p(x)$ to be positive it is sufficient that $p\left(\frac{-b(n-1)}{an}\right) > 0$,

$$p\left(\frac{-b(n-1)}{an}\right) = (-1)^{n-1}\left(\frac{n-1}{a}\right)^{n-1}\left(\frac{b}{n}\right)^n + h,$$

$$p\left(\frac{-b(n-1)}{an}\right) > 0 \Rightarrow h > (-1)^n\left(\frac{n-1}{a}\right)^{n-1}\left(\frac{b}{n}\right)^n.$$

Function $\frac{dq}{dx} = x^{n-2}\left(b(n-1) + anx\right)$, $x \to q(x)$ is decreasing for $x \in \left]-\infty, \frac{-b(n-1)}{an}\right[$ and increasing for $x \in \left]\frac{-b(n-1)}{an}, +\infty\right[$.

If $q(0) > 0$ and $q\left(\frac{-b(n-1)}{an}\right) < 0$, then there exist $x_1 \in \left]0, \frac{-b(n-1)}{an}\right[$ and $x_2 \in \left]\frac{-b(n-1)}{an}, +\infty\right[$ that satisfied $q(x_1) = q(x_2) = 0$, and $\forall\ x \in]x_1, x_2[\ \ q(x) < 0$,

$$q(0) > 0 \Leftrightarrow (-1)^{m-1}\left(\frac{m-1}{c}\right)^{m-1}\left(\frac{d}{m}\right)^m + h > 0 \ \Leftrightarrow\ h > (-1)^m\left(\frac{m-1}{c}\right)^{m-1}\left(\frac{d}{m}\right)^m,$$

$$q\left(\frac{-b(n-1)}{an}\right) < 0 \Leftrightarrow h < (-1)^m\left(\frac{m-1}{c}\right)^{m-1}\left(\frac{d}{m}\right)^m + (-1)^n\left(\frac{n-1}{a}\right)^{n-1}\left(\frac{b}{n}\right)^n.$$

We conclude that if h satisfies the relationship (6) then for $x \in [x_1, x_2]$, there exist $y_1(x) \in \left]0, \dfrac{-d(m-1)}{cm}\right]$ and $y_2(x) \in \left[\dfrac{-d(m-1)}{cm}, +\infty\right[$ that satisfied $y_1(x_1) = y_2(x_1) = \dfrac{-d(m-1)}{cm}$, $y_1(x_2) = y_2(x_2) = \dfrac{-d(m-1)}{cm}$ and $\forall x \in]x_1, x_2[\ \ y_1(x) \neq y_2(x)$.

Then Γ is composed by closed trajectory in first quadrant, more precisely in domain

$$D = \left\{(x, y) \in \mathbb{R}^2,\ x_1 \leqslant x \leqslant x_2,\ \ y_1(\frac{-b(n-1)}{an}) \leqslant y \leqslant y_2(\frac{-b(n-1)}{an})\right\}.$$

2. Number m is even and n is odd number.

Function $x \to p(x)$ is increasing for $x \in]-\infty, 0[\cup \left]\dfrac{-b(n-1)}{an}, +\infty\right[$ and decreasing for $x \in \left]0, \dfrac{-b(n-1)}{an}\right[$, for $p(x)$ to be positive it is sufficient that $p\left(\dfrac{-b(n-1)}{an}\right) > 0$, and this is verified if $h > \ (-1)^n \left(\dfrac{n-1}{a}\right)^{n-1} \left(\dfrac{b}{n}\right)^n$.

Function $x \to q(x)$ is increasing for $x \in]-\infty, 0[\cup \left]\dfrac{-b(n-1)}{an}, +\infty\right[$ and decreasing for $x \in \left]0, \dfrac{-b(n-1)}{an}\right[$, when $q(0) > 0$ and $q\left(\dfrac{-b(n-1)}{an}\right) < 0$, there exist $x_1 \in \left]0, \dfrac{-b(n-1)}{an}\right[$ and $x_2 \in \left]\dfrac{-b(n-1)}{an}, +\infty\right[$ such that $q(x_1) = q(x_2) = 0$, and $\forall\ x \in]x_1, x_2[\ \ q(x) < 0$.

We have

$$q(0) > 0 \Leftrightarrow h > (-1)^m \left(\frac{m-1}{c}\right)^{m-1} \left(\frac{d}{m}\right)^m,$$

$$q\left(\frac{-b(n-1)}{an}\right) < 0 \Leftrightarrow h < (-1)^m \left(\frac{m-1}{c}\right)^{m-1} \left(\frac{d}{m}\right)^m + (-1)^n \left(\frac{n-1}{a}\right)^{n-1} \left(\frac{b}{n}\right)^n.$$

As before we conclude that Γ is composed by closed trajectory in first quadrant.

3. Number m is odd and n is even number.

Function $y \to f_x(y)$ is increasing for $y \in]-\infty, 0[\cup \left]\dfrac{-d(m-1)}{cm}, +\infty\right[$ and decreasing for $y \in \left]0, \dfrac{-d(m-1)}{cm}\right[$.

If $p(x)$ is positive and $q(x)$ is negative, by applying the theorem of intermediate values the equation $f_x(y) = 0$ admits two solutions,

$$y_1(x) \in \left]0, \frac{-d(m-1)}{cm}\right[\text{ and } y_2(x) \in \left]\frac{-d(m-1)}{cm}, +\infty\right[.$$

As before we conclude that if h satisfied the relationship (6), the curve Γ represents a closed trajectory in the first quadrant.

4. Numbers m and n are odd.

Simply combine between the case 3, when m is odd number and the case 2, when n is odd number, we conclude also, if h satisfied the relationship(6), the curve Γ represents a closed trajectory in the first quadrant.

ii) The curve Γ is nonsingular of system (5).

We recall that the curve Γ is nonsingular of system(5) if the following system has no real solution

$$\begin{cases} ax^n + bx^{n-1} + cy^m + y^{m-1} + h = 0, \\ x^{n-1}(x+y)(anx + b(n-1)) = 0, \\ y^{m-1}(x+y)(cmy + d(m-1)) = 0. \end{cases} \tag{8}$$

Note that the closed trajectory of the curve Γ is in realistic quadrant then it does not intersect the axes, and does not intersect the line $y=-x$, the pair $\left(\dfrac{-b(n-1)}{an}, \dfrac{-d(m-1)}{cm}\right)$ cancels the second and the third equations but not the first, because $h \neq \ (-1)^n\left(\dfrac{n-1}{a}\right)^{n-1}\left(\dfrac{b}{n}\right)^n +$ $(-1)^m\left(\dfrac{m-1}{c}\right)^{m-1}\left(\dfrac{d}{m}\right)^m$, then the curve Γ is nonsingular.

iii) Γ is an invariant curve of system (5).

$$\frac{\partial U}{\partial x}\dot{x}+\frac{\partial U}{\partial y}\dot{y}=\left(x^n\left(nax+b(n-1)\right)+y^m\left(mcy+d\left(m-1\right)\right)\right)U, \tag{9}$$

the cofactor is $K(x,y)=x^n\left(nax+b(n-1)\right)+y^m\left(mcy+d\left(m-1\right)\right)$.

iv) $\displaystyle\int_0^T \operatorname{div}(\Gamma)dt \neq 0$. Note that

$$\int_0^T \operatorname{div}(\Gamma)dt=\int_0^T K(x(t),y(t))dt, \tag{10}$$

see for instance Giacomini & Grau [12].

$$\int_0^T K(x(t),y(t))dt=\int_0^T x^n\left(nax+b(n-1)\right)dt+\int_0^T y^m\left(mcy+d\left(m-1\right)\right)dt=$$
$$={}_{\Gamma}\frac{x^n\left(nax+b(n-1)\right)}{xy\left(x+y\right)\left(nax^{n-1}+b(n-1)x^{n-2}\right)}dy-\frac{y^m\left(mcy+d\left(m-1\right)\right)}{xy\left(x+y\right)\left(mcy^{m-1}+d\left(m-1\right)y^{m-2}\right)}dx=$$
$$={}_{\Gamma}\frac{x}{y\left(x+y\right)}dy-\frac{y}{x\left(x+y\right)}dx,$$

by applying the Green formula,

$${}_{\Gamma}\frac{x}{y\left(x+y\right)}dy-\frac{y}{x\left(x+y\right)}dx=2\int\int_{(\Gamma)}\frac{1}{\left(x+y\right)^2}dxdy, \tag{11}$$

where $int(\Gamma)$ denotes the interior of Γ.

As the factor $\dfrac{1}{\left(x+y\right)^2}$ is positive then $\displaystyle\int_0^T K(x(t),y(t))dt$ is nonzero. □

3. Examples

Example 1. The system

$$\begin{cases} \dot{x}=x\left(x^4-2x^3-2xy^2+2x-4y^3+3y^2\right), \\ \dot{y}=y\left(3x^4+4x^3y-4x^3-6x^2y+2y^3-3y^2+2y\right), \end{cases} \tag{12}$$

admits one limit cycle represented by the curve $x^3-2x^2+2y^2-3y+2=0$, and it has six singular points, the limit cycle around a stable focus (Fig. 1).

Example 2. Let $a=3$, $b=-5$, $c=1$, $d=-3$, $h=5$,

Note that, h satisfied $\max\left\{\dfrac{27b^4}{256a^3}, \dfrac{-4d^3}{27c^2}\right\}<h<\dfrac{27b^4}{256a^3}-\dfrac{4d^3}{27c^2}$.

The system

$$\begin{cases} \dot{x}=x\left(3x^5-5x^4-2xy^3+3xy^2+5x-3y^4+6y^3\right), \\ \dot{y}=y\left(12x^5+15x^4y-15x^4-20x^3y+y^4-3y^3+5y\right) \end{cases} \tag{13}$$

admits one limit cycle represented by the curve $3x^4-5x^3+y^3-3y^2+5=0$, and it has six singular points, the limit cycle encloses a stable focus (Fig. 2).

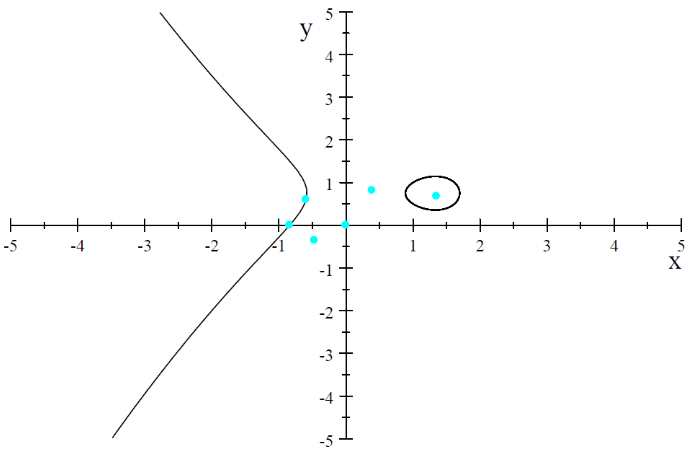

Fig. 1. Limit cycle of system (13) with singular points

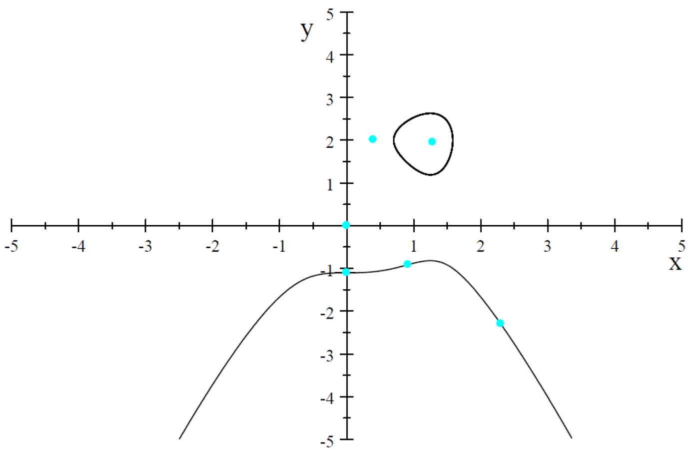

Fig. 2. Limit cycle of system (13) with singular points.

Conclusion

We proposed in this paper a polynomial Kolmogorov system, where just choose the parameters satisfying the conditions of the theorem (1), we conclude directly that the system has a limit cycle in the realistic quadrant, and we give it explicitly.

References

[1] A.Bendjeddou, R.Cheurfa, On the exact limit cycle for some class of planar differential systems, *Nonlinear differ. equ. appl.*, **14**(2007), 491–498.

[2] A.Bendjeddou, R.Cheurfa, Cubic and quartic planar differential systems with exact algebraic limit cycles, *Elect. J. of Diff. Equ.*, (2011), no. 15, 1–12.

[3] S.Benyoucef, A.Bendjeddou, A class of Kolmogorov system with exact limit cycles, *International journal of pure and applied mathematics*, **103**(2015), no. 4, 439–451.

[4] Shen Boqian,Liu Demeng, Existence of limit cycles for a cubic Kolmogorov system with an hyperbolic solution, *Northwest Math.*, **16**(2000), no. 1, 91–95.

[5] K.S.Cheng, Uniqueness of a limit cycle for a predator-prey system, *SIAM J. Math. Anal.*, **12**(1981), no. 4, 541–548.

[6] SI Chengbin, Shen Boqian, The existence of limit cycles for the Kolmogorov cubic system with a quartic curve solution, *J. Sys. Sci. & Math. Scis.*, **28**(2008), no. 3, 334–339.

[7] H.Giacomini, M.Grau, On the stability of limit cycles for planar differential systems, *J. of Diff. Equ.*, **213**(2005), no. 2, 368–388.

[8] X.C.Huang, Limit cycle in a Kolmogorov-type model, *Internat. J. Math. & Math Sci.*, **13**(1990), no. 3, 555–566.

[9] X.Huang, Y Wang, A Cheng, Limit cycles in a cubic predator–prey differential system, *J. Korean Math. Soc.*, **43**(2006), no. 4, 829–843.

[10] X.C Huang and Lemin Zhu, Limit cycles in a general Kolmogorov model, *Nonlin. Anal. Theo. Meth. and Appl.* **60** (2005), 1393–1414.

[11] N.G.Lloyd, J.M.Pearson, E.Sáez, I.Szántó, Limit cycles of a Cubic Kolmogorov System, *Appl. Math. Lett.*, **9**(1996), no. 1, 15–18.

[12] N.G.Lloyd, J.M.Pearson, E.Sáez, I.Szántó, A cubic Kolmogorov system with six limit cycles, *International Journal Computers and Mathematics with Applications*, **44**(2002), 445–455.

[13] R.M. May, Limit cycles in predator-prey communities, *Science*, **177**(1972), 900–902.

[14] L.Perko, Differential equations and dynamical systems, Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.

[15] Peng Yue-hui, Limit Cycles in a Class of Kolmogorov Model with Two Positive equilibrium Points, *Natural Science journal of Xiangtan University*, **32**(2010), no. 4, 10–15.

Система Колмогорова с явными гиперболическими предельными циклами

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Вводится класс дифференциальных систем Колмогорова. Показано, что при подходящих предположениях относительно параметров для алгебраического предельного гиперболического цикла можно получить явное выражение.

Ключевые слова: Колмогорова дифференциальная система, инвариантная кривая, особая точка, периодическое решение, предельный алгебраический цикл.