

УДК 517.55

On Spectral Projection for the Complex Neumann Problem

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Received 08.04.2012, received in revised form 20.06.2012, accepted 20.07.2012

We show that the L^2 -spectral kernel function of the $\bar{\partial}$ -Neumann problem on a non-compact strongly pseudoconvex manifold is smooth up to the boundary.

Keywords: $\bar{\partial}$ -Neumann problem, strongly pseudoconvex domains, spectral kernel function.

Introduction

The $\bar{\partial}$ -Neumann problem appears naturally in studying the Dirichlet form for the Dolbeault complex on a compact complex manifold Z with boundary. More precisely, one minimizes the Dirichlet norm over the space of differential forms of bidegree $(0, q)$ on Z whose complex normal parts on the boundary of Z vanish. The Euler-Lagrange equations of this variational problem just amount to the $\bar{\partial}$ -Neumann problem.

While the differential equation in Z in the $\bar{\partial}$ -Neumann problem is a generalized Laplace equation, the boundary conditions fail to satisfy the Shapiro-Lopatinskii condition. Hence, the elliptic regularity in Sobolev spaces on Z is violated. The main a priori estimate for $(0, 1)$ -forms on compact strongly pseudoconvex manifolds was proved by Morrey, see the references in [1]. When compared with a priori estimates for elliptic boundary value problems, the estimate of Morrey bears loss of 1 in the regularity. For differential forms of arbitrary bidegree $(0, q)$ with $q \geq 1$ on strongly pseudoconvex manifolds the main a priori estimate was later proved by Kohn [2] who extended in this way the theory of harmonic integrals by W. Hodge (1941) and K. Kodaira (1953) to compact strongly pseudoconvex manifolds.

The $\bar{\partial}$ -Neumann problem initiated readily the study of so-called subelliptic operators which occurred intensively in the 1970s and 1980s. In complex analysis this study was mostly focused upon the regularity of solutions of the $\bar{\partial}$ -Neumann problem in pseudoconvex domains of finite type. For a current survey in this direction we refer the reader to [3].

The most difficult part of [2] is the proof of regularity of solutions up to the boundary ∂Z . This proof was simplified by Kohn and Nirenberg in [4]. To this end, they had elaborated calculus of pseudodifferential operators which are nowadays referred to as classical ones.

The proof of regularity in the $\bar{\partial}$ -Neumann problem raised the problem of constructing explicit integral formulas for the solution. A satisfactory theory is nowadays available in [5]. We also mention an earlier paper [6] which studied estimates for the kernel function of the $\bar{\partial}$ -Neumann

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operator. First steps towards calculus of pseudodifferential operators relevant to several complex variables were summarized in [7].

For compact strongly pseudoconvex manifolds Z the $\bar{\partial}$ -Neumann operator satisfies a pseudolocal estimate with gain 1 in the Sobolev scale. Combining this estimate with a familiar argument of topological tensor products shows that the spectral kernel function of the $\bar{\partial}$ -Neumann problem is smooth up to the boundary of $Z \times Z$.

The present paper is motivated by the question of M. Shubin whether the spectral kernel function of the complex Laplacian under its natural boundary conditions is still C^∞ up to the boundary of $Z \times Z$, if Z is not compact. We answer the question in the affirmative. To this end we prove that a pseudolocal estimate holds even for non-compact strongly pseudoconvex manifolds. A close result was established in [8] using different techniques.

The spectral theory of the $\bar{\partial}$ -Neumann problem has been previously studied in [9, 10] for compact manifolds with boundary. In [11], heat kernel asymptotics are developed for the heat kernel of a general elliptic operator with non-coercive boundary conditions.

1. The $\bar{\partial}$ -Neumann Problem

Let Z be a Hermitian complex manifold of dimension n with C^∞ boundary ∂Z . We always think of Z as a closed subdomain of a larger Hermitian complex manifold Z' of the same dimension.

Suppose Z is strongly pseudoconvex, i.e., at each point of ∂Z the Levi form restricted to the tangent hyperplane has $n - 1$ positive eigenvalues. This slightly differs from the usual notation, for we don't control ∂Z at the points at infinity if there are any.

Let \mathcal{F} be a Hermitian holomorphic vector bundle over Z' . For $q = 0, 1, \dots, n$, we set $\mathcal{F}^q = \mathcal{F} \otimes_{\mathbb{C}} \bigwedge^{0,q} T^*(Z')$. This bundle is the one we are interested in, since its sections are the differential forms of type $(0, q)$ on Z' with coefficients in \mathcal{F} . The operator $\bar{\partial}$ gives rise to a differential operator $\bar{\partial}_{\mathcal{F}}$ on the \mathcal{F} -valued differential forms by $\bar{\partial}_{\mathcal{F}} = 1 \otimes \bar{\partial}$.

A Hermitian metric on Z' induces a volume element dv on Z' . When combined with a Hermitian metric on \mathcal{F} , this allows one to define a conjugate linear isomorphism of bundles $* : \mathcal{F}^q \rightarrow \mathcal{F}^{q'}$ by $*v = (\cdot, v)_z dv$. Here, $\mathcal{F}^{q'} = \mathcal{F}' \otimes_{\mathbb{C}} \bigwedge^{n, n-q} T^*(Z')$ stands for the dual bundle of \mathcal{F}^q .

Furthermore, in the space $C^\infty_{\text{comp}}(Z', \mathcal{F}^q)$ we can introduce an inner product by the formula

$$(u, v)_{L^2(Z', \mathcal{F}^q)} = \int_{Z'} (u(z), v(z))_z dv = \int_{Z'} \langle *v, u \rangle_z$$

for $u, v \in C^\infty_{\text{comp}}(Z', \mathcal{F}^q)$. We say that a form u is square integrable on Z' if the function $(u, u)_z$ is integrable with respect to dv . As usual, the space of square integrable $(0, q)$ -forms with coefficients in \mathcal{F} on Z' is denoted by $L^2(Z', \mathcal{F}^q)$. The inner product $(u, v)_{L^2(Z', \mathcal{F}^q)}$ actually turns $L^2(Z', \mathcal{F}^q)$ into a Hilbert space with norm

$$\|u\|_{L^2(Z', \mathcal{F}^q)} := \sqrt{(u, u)_{L^2(Z', \mathcal{F}^q)}},$$

as is easy to check.

We now restrict our section spaces and operators thereon to the manifold Z , thus obtaining

$$\begin{aligned} \mathcal{E}^q &:= C^\infty(Z, \mathcal{F}^q), \\ \mathcal{L}^q &:= L^2(Z, \mathcal{F}^q), \end{aligned}$$

etc. It is obvious that \mathcal{L}^q just amounts to the completion of $\{u \in \mathcal{E}^q : \|u\|_{\mathcal{L}^q} < \infty\}$ in the norm $\|\cdot\|_{\mathcal{L}^q}$.

Let \mathcal{D}_T^q be the set of all sections $u \in \mathcal{L}^q$, for which there is a sequence $\{u_\nu\}$ with the following properties:

- 1) $u_\nu \in \mathcal{L}^q \cap \mathcal{E}^q$;
- 2) $\{u_\nu\}$ converges to u in \mathcal{L}^q ; and
- 3) $\{\bar{\partial}_{\mathcal{F}} u_\nu\}$ is a Cauchy sequence in \mathcal{L}^{q+1} .

The mapping $T : \mathcal{D}_T^q \rightarrow \mathcal{L}^{q+1}$ defined by $Tu = \lim \bar{\partial}_{\mathcal{F}} u_\nu$, where $\{u_\nu\}$ is a sequence with properties 1)–3), is called the maximal operator generated by $\bar{\partial}_{\mathcal{F}}$.

Note that T is well defined. Indeed, if $\{u'_\nu\}$ is another sequence satisfying 1)–3), and $f = \lim \bar{\partial}_{\mathcal{F}} u'_\nu$, then for all $g \in C^\infty(Z, \mathcal{F}^{q+1'})$ with a compact support in the interior of Z we get

$$\begin{aligned} \langle Tu - f, g \rangle &= \lim \langle \bar{\partial}_{\mathcal{F}} u_\nu - \bar{\partial}_{\mathcal{F}} u'_\nu, g \rangle \\ &= \lim \langle u_\nu - u'_\nu, \bar{\partial}'_{\mathcal{F}} g \rangle = 0, \end{aligned}$$

whence $Tu = f$.

We will think of T as an unbounded operator from \mathcal{L}^q to \mathcal{L}^{q+1} , whose domain is \mathcal{D}_T^q . Since \mathcal{D}_T^q contains $\mathcal{L}^q \cap \mathcal{E}^q$ the operator T is densely defined and closed.

From the lemma of du Bois-Reymond and the uniqueness of a weak limit it follows that if $u \in \mathcal{D}_T^q$ then $Tu = \bar{\partial}_{\mathcal{F}} u$ in the sense of distributions in the interior of Z .

Lemma 1.1. *As defined above, T satisfies $T\mathcal{D}_T^q \subset \mathcal{D}_T^{q+1}$ and $T^2 = 0$.*

Proof. Assume that $u \in \mathcal{D}_T^q$ and $\{u_\nu\}$ is a sequence with properties 1)–3). We set $f_\nu = \bar{\partial}_{\mathcal{F}} u_\nu$. Then $Tu = \lim f_\nu$. And since $\bar{\partial}_{\mathcal{F}} f_\nu = 0$, we obtain that $Tu \in \mathcal{D}_T^{q+1}$ and $T(Tu) = 0$. \square

Thus we have the following complex of Hilbert spaces and their closed linear mappings:

$$\mathcal{L} : 0 \longrightarrow \mathcal{L}^0 \xrightarrow{T} \mathcal{L}^1 \xrightarrow{T} \dots \xrightarrow{T} \mathcal{L}^n \longrightarrow 0. \quad (1)$$

The L^2 -cohomology of the Dolbeault complex on Z with coefficients in \mathcal{F} is just the cohomology of complex (1.1). More precisely, the cohomology at step q denoted by $H^q(\mathcal{L}^\cdot)$ is defined to be the quotient of the null-space of $T : \mathcal{D}_T^q \rightarrow \mathcal{L}^{q+1}$ over the range of $T : \mathcal{D}_T^{q-1} \rightarrow \mathcal{L}^q$.

We now define T^* , the adjoint of T , as usual for unbounded operators. Namely, let $\mathcal{D}_{T^*}^q$ be the set of all forms $g \in \mathcal{L}^q$ with the property that there is $v \in \mathcal{L}^{q-1}$ satisfying $(Tu, g)_{\mathcal{L}^q} = (u, v)_{\mathcal{L}^{q-1}}$ for all $u \in \mathcal{D}_T^{q-1}$. We define $T^* : \mathcal{D}_{T^*}^q \rightarrow \mathcal{L}^{q-1}$ by $T^*g = v$.

The operator T^* is well defined because the domain $\mathcal{D}_{T^*}^{q-1}$ is dense in \mathcal{L}^{q-1} . It is easy to see that if $g \in \mathcal{D}_{T^*}^q \cap \mathcal{E}^q$ then $T^*g = \bar{\partial}_{\mathcal{F}}^* g$, where $\bar{\partial}_{\mathcal{F}}^* = *^{-1} \bar{\partial}'_{\mathcal{F}} *$ is the formal adjoint of $\bar{\partial}_{\mathcal{F}}$.

Moreover, the Stokes theorem tells us that the elements of $\mathcal{D}_{T^*}^q$, which are smooth up to the boundary of Z , satisfy certain conditions on ∂Z . We write these in the form $n(g) = 0$ on ∂Z , where $n(g)$ is the complex normal component of g , cf. Section 3.2.2 in [12]. The equality $n(g) = 0$ means that the coefficients of g at each point of ∂Z satisfy a homogeneous system of linear equations, the latter varying smoothly over ∂Z .

Lemma 1.2. *$T^*\mathcal{D}_{T^*}^q \subset \mathcal{D}_{T^*}^{q-1}$ and $T^{*2} = 0$.*

Proof. Indeed, if $g \in \mathcal{D}_{T^*}^q$ and $u \in \mathcal{D}_T^{q-2}$ then by the very definition and Lemma 1.1 we get

$$(Tu, T^*g)_{\mathcal{L}^{q-1}} = (T(Tu), g)_{\mathcal{L}^q} = 0.$$

Therefore, $T^*g \in \mathcal{D}_{T^*}^{q-1}$ and $T^*(T^*g) = 0$, as desired. \square

Let us introduce an operator L on \mathcal{L}^q with a domain \mathcal{D}_L^q , which has the property that if $u \in \mathcal{D}_L^q \cap \mathcal{E}^q$ then $Lu = \Delta u$, where $\Delta = \bar{\partial}_{\mathcal{F}}^* \bar{\partial}_{\mathcal{F}} + \bar{\partial}_{\mathcal{F}} \bar{\partial}_{\mathcal{F}}^*$ is the Laplacian of the Dolbeault complex on Z' with coefficients in \mathcal{F} . Namely, write \mathcal{D}_L^q for the set of all $u \in \mathcal{D}_T^q \cap \mathcal{D}_{T^*}^q$ with the property that $Tu \in \mathcal{D}_{T^*}^{q+1}$ and $T^*u \in \mathcal{D}_T^{q-1}$. Then the operator $L: \mathcal{D}_L^q \rightarrow \mathcal{L}^q$ is defined by

$$Lu = T^*Tu + TT^*u,$$

cf. § 4.2 in [12].

The $\bar{\partial}_{\mathcal{F}}$ -Neumann problem on the manifold Z in the L^2 setting consists in the following: Given a section $f \in \mathcal{L}^q$, when is there $u \in \mathcal{D}_L^q$ such that $Lu = f$, and how does u depend on f ?

The weak orthogonal decomposition is actually the first step in solving the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem. Set

$$\mathcal{H}^q = \{u \in \mathcal{D}_T^q \cap \mathcal{D}_{T^*}^q : Tu = T^*u = 0\},$$

for $q = 0, 1, \dots, n$. Since the operators T and T^* are closed, \mathcal{H}^q is a closed subspace of \mathcal{L}^q . Denote by $H: \mathcal{L}^q \rightarrow \mathcal{H}^q$ the orthogonal projection of \mathcal{L}^q onto \mathcal{H}^q .

Lemma 1.3. $u \in \mathcal{H}^q$ if and only if $u \in \mathcal{D}_L^q$ and $Lu = 0$.

Proof. If $u \in \mathcal{H}^q$ then obviously $u \in \mathcal{D}_L^q$ and $Lu = 0$. If $Lu = 0$ then $(Lu, u)_{\mathcal{L}^q} = 0$, and since

$$(Lu, u)_{\mathcal{L}^q} = \|Tu\|_{\mathcal{L}^{q+1}}^2 + \|T^*u\|_{\mathcal{L}^{q-1}}^2$$

we have $u \in \mathcal{H}^q$. □

Lemma 1.4. *The operator L is selfadjoint, and $(L + 1)^{-1}$ exists, is bounded, and is everywhere in \mathcal{L}^q defined.*

Proof. Since T is a closed operator and the domain of T is dense, the same is also true for T^* , and $(T^*)^* = T$.

It follows that the operators $(TT^* + 1)^{-1}$ and $(T^*T + 1)^{-1}$ exist, are bounded, selfadjoint and defined everywhere in \mathcal{L}^q , cf. [13].

We now easily verify that $(L + 1)^{-1}$ exists, is bounded, is everywhere defined, and is given by the formula

$$(L + 1)^{-1} = (TT^* + 1)^{-1} + (T^*T + 1)^{-1} - 1,$$

which completes the proof. □

Corollary 1.1 (weak orthogonal decomposition). *The range of L is orthogonal to \mathcal{H}^q , and*

$$\mathcal{L}^q = \mathcal{H}^q \oplus \overline{LD_L^q}, \tag{2}$$

where $\overline{LD_L^q}$ denotes the closure of LD_L^q in \mathcal{L}^q .

Proof. This follows immediately from the selfadjointness of L and Lemma 1.3. □

In particular, if LD_L^q is closed then we arrive at the "strong orthogonal decomposition"

$$\mathcal{L}^q = \mathcal{H}^q \oplus T^*TD_L^q \oplus TT^*\mathcal{D}_L^q. \tag{3}$$

2. The $\bar{\partial}$ -Neumann Operator

The results of this and the next section go back at least as far as [2]. We bring them only for completeness.

Definition 2.1. Let LD_L^q be closed and $f \in \mathcal{L}^q$, then $f = Hf + Lu$ where $u \in \mathcal{D}_L^q$. The $\bar{\partial}_{\mathcal{F}}$ -Neumann operator $N: \mathcal{L}^q \rightarrow \mathcal{D}_L^q$ is defined by $Nf = u - Hu$.

Note that N is well defined. Indeed, if also $f = Hf + Lu'$ where $u' \in \mathcal{D}_L^q$ then $L(u - u') = 0$ whence

$$(u - Hu) - (u' - Hu') = (u - u') - H(u - u') = 0.$$

We summarize the properties of the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator. They generalize those of the Green operator from the Hodge theory, for the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem itself stems from the desire to extend the Hodge theory to the case of manifolds with boundary.

Lemma 2.1. *Suppose LD_L^q is closed. Then the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator N has the following properties:*

1) N is bounded, selfadjoint, $HN = NH = 0$, and we have the orthogonal decomposition

$$f = Hf + T^*TNf + TT^*Nf \tag{4}$$

for all $f \in \mathcal{L}^q$.

2) If $f \in \mathcal{D}_T^q$ and $Tf = 0$ then $TNf = 0$. If moreover LD_L^{q+1} is closed then $TNf = NTf$.

3) If $f \in \mathcal{D}_{T^*}^q$ and $T^*f = 0$ then $T^*Nf = 0$. If moreover LD_L^{q-1} is closed then $T^*Nf = NT^*f$.

Proof. See [12, 4.2.5] and elsewhere. □

The Laplacian Δ is well known to be an elliptic differential operator on Z' . Hence it follows that the harmonic differential forms $u \in \mathcal{H}^q$ are infinitely differentiable in the interior of Z , and the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator N , if exists, preserves the interior regularity.

Beginning with its classical forms, the Dirichlet norm has been an important technical tool in studying the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem. Given any $u, v \in \mathcal{D}_T^q \cap \mathcal{D}_{T^*}^q$, the Dirichlet inner product of these differential forms is defined by

$$D(u, v) = (Tu, Tv)_{\mathcal{L}^{q+1}} + (T^*u, T^*v)_{\mathcal{L}^{q-1}} + (u, v)_{\mathcal{L}^q},$$

and the Dirichlet norm is $D(u) = \sqrt{D(u, u)}$.

The space $\mathcal{D}_T^q \cap \mathcal{D}_{T^*}^q$ with the Dirichlet norm is a complete (Hilbert) space. It is denoted by \mathcal{D}^q .

Since $D(u) \geq \|u\|_{\mathcal{L}^q}$ for all $u \in \mathcal{D}^q$ there exists only one selfadjoint operator S with a domain $\mathcal{D}_S^q \subset \mathcal{D}^q$, such that if $u \in \mathcal{D}_S^q$ and $v \in \mathcal{D}^q$ then

$$D(u, v) = (Su, v)_{\mathcal{L}^q}. \tag{5}$$

The following lemma gives a useful description of the operator L because our estimates will be in the norm $D(u)$.

Lemma 2.2. *The equalities hold $\mathcal{D}_L^q = \mathcal{D}_S^q$ and $L = S - 1$, where the operator S is defined by (5).*

Proof. If $u \in \mathcal{D}_L^q$ and $v \in \mathcal{D}^q$, then $D(u, v) = ((L+1)u, v)_{\mathcal{L}^q}$ is fulfilled. Hence by the uniqueness of S , we have $S = L + 1$. □

3. Completely Continuous Norms

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space \mathcal{L} . We will say that the norm $\|\cdot\|_1$ is completely continuous with respect to the norm $\|\cdot\|_2$ if every sequence in \mathcal{L} which is bounded in the norm $\|\cdot\|_1$ has a convergent subsequence in the norm $\|\cdot\|_2$.

Lemma 3.1. *If the norm D on \mathcal{D}^q is completely continuous with respect to $\|\cdot\|_{\mathcal{L}^q}$ then \mathcal{H}^q is finite dimensional.*

Proof. Observe that if $u, v \in \mathcal{H}^q$ then $D(u, v) = (u, v)_{\mathcal{L}^q}$. Suppose that the dimension of \mathcal{H}^q is infinite. Then there exists an infinite sequence $\{u_\nu\}$ of orthonormal elements in \mathcal{H}^q . Since $D(u_\nu) = \|u_\nu\|_{\mathcal{L}^q} = 1$ the sequence $\{u_\nu\}$ contains a convergent subsequence. But this is at variance with the fact that if $\nu \neq \mu$ then $\|u_\nu - u_\mu\|_{\mathcal{L}^q} = \sqrt{2}$. \square

Lemma 3.2. *If the norm D on \mathcal{D}^q is completely continuous with respect to $\|\cdot\|_{\mathcal{L}^q}$ then there exists a constant $c > 0$ such that for all $u \in \mathcal{D}^q$ orthogonal to \mathcal{H}^q , we have*

$$\|Tu\|_{\mathcal{L}^{q+1}}^2 + \|T^*u\|_{\mathcal{L}^{q-1}}^2 \geq c\|u\|_{\mathcal{L}^q}^2.$$

Proof. Consider the Hilbert space $\mathcal{L}^{q+1} \times \mathcal{L}^{q-1}$ which is equipped with the norm

$$\|\{f, v\}\| = (\|f\|_{\mathcal{L}^{q+1}}^2 + \|v\|_{\mathcal{L}^{q-1}}^2)^{1/2}.$$

Let $M : \mathcal{D}^q \rightarrow \mathcal{L}^{q+1} \times \mathcal{L}^{q-1}$ be the mapping defined by $Mu = \{Tu, T^*u\}$. Note that M is a closed operator.

We will prove that the range of M is closed. Suppose that $M\mathcal{D}^q$ is not closed. Then there exists a sequence $\{u_\nu\}$ in \mathcal{D}^q , such that $\lim Mu_\nu = \{f, v\}$ and $\{f, v\} \notin M\mathcal{D}^q$.

Set $u'_\nu = u_\nu - Hu_\nu$, then u'_ν are orthogonal to \mathcal{H}^q and $\lim Mu'_\nu = \{f, v\}$. If $\|u'_\nu\|_{\mathcal{L}^q}$ are bounded then $D(u'_\nu) = (\|Mu'_\nu\|^2 + \|u'_\nu\|_{\mathcal{L}^q}^2)^{1/2}$ are bounded, too. Then by hypothesis $\{u'_\nu\}$ has a convergent subsequence with a limit u , and since M is closed then $Mu = \{f, v\}$ which contradicts the assumption that $\{f, v\} \notin M\mathcal{D}^q$. Thus by choosing a subsequence, if necessary, we may actually assume that $\lim \|u'_\nu\|_{\mathcal{L}^q} = \infty$.

Now set $U_\nu = u'_\nu / \|u'_\nu\|_{\mathcal{L}^q}$. Then $\lim \|MU_\nu\| = 0$ and $D(U_\nu)$ are bounded. Therefore $\{U_\nu\}$ has a convergent subsequence $\{U_{\nu_k}\}$ such that

$$\begin{aligned} \lim U_{\nu_k} &= U, \\ \lim MU_{\nu_k} &= \{0, 0\}. \end{aligned}$$

Hence $MU = 0$ so that $U \in \mathcal{H}^q$. Since U_ν is orthogonal to \mathcal{H}^q we have $U = 0$, but $\|U_\nu\|_{\mathcal{L}^q} = 1$. This contradiction proves that the range $M\mathcal{D}^q$ is closed in $\mathcal{L}^{q+1} \times \mathcal{L}^{q-1}$.

Let R be the restriction of M to the orthogonal complement of \mathcal{H}^q in \mathcal{D}^q . Then R is one-to-one and has a closed range. By the closed graph theorem, the inverse R^{-1} is bounded. Hence there is $c > 0$ such that $\|Ru\|^2 \geq c\|u\|_{\mathcal{L}^q}^2$. This proves the lemma. \square

Theorem 3.1. *If the norm D on \mathcal{D}^q is completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^q}$, then LD_L^q is closed.*

Proof. By Lemma 3.2, there exists $c > 0$ with the property that for all $u \in \mathcal{D}_L^q$ which are orthogonal to \mathcal{H}^q we have

$$(Lu, u)_{\mathcal{L}^q} \geq c\|u\|_{\mathcal{L}^q}^2,$$

so that $\|Lu\|_{\mathcal{L}^q} \geq c\|u\|_{\mathcal{L}^q}$.

Set $f = \lim Lu_\nu$. We may assume that u_ν are orthogonal to \mathcal{H}^q , and then $\|u_\nu\|_{\mathcal{L}^q}$ are uniformly bounded. Therefore, $\{u_\nu\}$ has a subsequence whose arithmetic means converge, cf. [13]. Denoting this limit by u , we get $f = Lu$, which completes the proof. \square

The question of when the norm D on \mathcal{D}^q is completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^q}$, is very difficult in the general case and it requires special consideration. We present some consequences here.

Corollary 3.1. *Suppose the norm D on \mathcal{D}^q is completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^q}$. Then the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem is solvable at step q in the sense that there exist operators H and N in \mathcal{L}^q with properties 1)–3) of Lemma 2.1.*

Proof. This follows immediately from Lemma 2.1 and Theorem 3.1. \square

For compact manifolds with boundary Z the subspace \mathcal{H}^0 is usually of infinite dimension. So by Lemma 3.1 the Dirichlet norm D may not be completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^0}$ on \mathcal{D}^0 . However, the following result holds.

Theorem 3.2. *If the norm D on \mathcal{D}^1 is completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^1}$ then LD_L^0 is closed.*

Proof. See for instance [12, 4.2.6]. \square

The next result immediately follows from Lemma 2.1 and Theorem 3.1. Recall that $\mathcal{H}^0 = \ker T^0$.

Corollary 3.2. *Suppose the norm D on \mathcal{D}^1 is completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^1}$. Then $f = Hf + T^*NTf$ for any section $f \in \mathcal{D}_T^0$, where $H: \mathcal{L}^0 \rightarrow \mathcal{H}^0$ is the orthogonal projection.*

When acting on sections of $\mathcal{F}^0 = \mathcal{F}$, the differential operator $\bar{\partial}_{\mathcal{F}}$ has injective symbol. Since

$$\mathcal{H}^0 = \{u \in \mathcal{L}^0 \cap C_{\text{loc}}^\infty(Z^\circ, \mathcal{F}) : \bar{\partial}_{\mathcal{F}}u = 0\},$$

where Z° stands for the interior of Z , the operator H^0 is a generalisation of the classical Bergman projector. Corollary 3.2 gives $H^0 = 1 - T^*NT$.

4. Pseudolocal Estimates

The regularity of the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator near the boundary of Z is a much more delicate problem. It initiated the study of non-elliptic boundary value problems, thus motivating a development of pseudodifferential theory, cf. [4]. Kohn proved in [2] that if Z is a compact strongly pseudoconvex manifold then the norm D on \mathcal{D}^q is completely continuous with respect to the norm $\|\cdot\|_{\mathcal{L}^q}$ for all $q = 1, \dots, n$. Moreover, the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator preserves the regularity up to the boundary in the scale of Sobolev spaces $H^s(Z, \mathcal{F}^q)$, with $s = 0, 1, \dots$, in the sense that $f \in H^s(Z, \mathcal{F}^q)$ implies $Nf \in H^s(Z, \mathcal{F}^q)$. Kohn's original approach was considerably simplified in [4] in a very general framework via elliptic regularisation.

One says that a subelliptic estimate of order $\varepsilon > 0$ holds for the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem at step q in a neighbourhood U of a boundary point $z^0 \in \partial Z$ if there is a constant c such that

$$\|u\|_{H^\varepsilon(Z, \mathcal{F}^q)} \leq cD(u) \tag{6}$$

for every smooth form u which is supported in $Z \cap U$ and satisfies the boundary condition $n(u) = 0$ on $\partial Z \cap U$.

The systematic study of subelliptic estimates in [4] provides the following "pseudolocal estimate."

Theorem 4.1. *Let Z be a compact pseudoconvex manifold with C^∞ boundary. Suppose a subelliptic estimate (6) holds in a neighbourhood U of a boundary point z^0 . Pick arbitrary functions $\varphi, \psi \in C_{\text{comp}}^\infty(U)$, such that $\psi \equiv 1$ in a neighbourhood of the support of φ . Then, for every non-negative s there is a constant C with the property that*

$$\|\varphi Nf\|_{H^{s+2\varepsilon}(Z, \mathcal{F}^q)} \leq C (\|\psi f\|_{H^s(Z, \mathcal{F}^q)} + \|f\|_{\mathcal{L}^q})$$

for all $f \in \mathcal{L}^q \cap H^s(U, \mathcal{F}^q)$.

Proof. See Theorem 4 and Remark 6.2 in [4]. This result is actually mentioned in [3], cf. Theorem 8. □

For a compact strongly pseudoconvex manifold Z , a subelliptic estimate (6) with $\varepsilon = 1/2$ holds in a neighbourhood of every boundary point, provided $1 \leq q \leq n$. It follows that for such manifolds the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator is continuous from $H^s(Z, \mathcal{F}^q)$ to $H^{s+1}(Z, \mathcal{F}^q)$.

If Z is not compact then the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem on Z need not be solvable in the sense that the range LD_L^q is closed in \mathcal{L}^q . In order to guarantee the normal solvability one has to arrange the problem with the points at infinity. As usual, this would require pseudodifferential analysis in weighted Sobolev spaces. Still, we may try to maintain the pseudolocal estimate of Theorem 4.1, thus showing the local regularity for the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem on a non-compact strongly pseudoconvex manifold Z .

Corollary 4.1. *Assume that U is a neighbourhood of a boundary point z^0 , V a relatively compact open subset of U , and s a non-negative integer. If $u \in \mathcal{D}_L^q$ satisfies $Lu \in H^s(U, \mathcal{F}^q)$ then $u \in H^{s+1}(V, \mathcal{F}^q)$ and*

$$\|u\|_{H^{s+1}(V, \mathcal{F}^q)} \leq C (\|Lu\|_{H^s(U, \mathcal{F}^q)} + \|u\|_{L^2(U, \mathcal{F}^q)}),$$

where C depends on U, V and s but not on u .

Proof. In case the closure of V does not meet ∂Z the assertion follows from the interior regularity of the $\bar{\partial}_{\mathcal{F}}$ -Neumann. Hence we can assume that V is small enough, for if not, we shrink it. Since each boundary point of Z possesses a neighbourhood whose closure is a compact strongly pseudoconvex manifold, we can assume without loss of generality that \bar{U} is a compact strongly pseudoconvex manifold with C^∞ boundary. It is convenient to choose U sufficiently small, so that the harmonic spaces on U be trivial.

For every $q = 0, 1, \dots, n$, choose a parametrix G^q of the Laplacian Δ^q on Z' , by a parametrix is meant an inverse modulo smoothing operators, see [12, 2.1.4] and elsewhere. This is a classical pseudodifferential operator of order -2 and type $\mathcal{F}^q \rightarrow \mathcal{F}^q$ on Z' . The Schwartz kernel K_{G^q} of G^q is a C^∞ section of the bundle $\mathcal{F}^q \boxtimes \mathcal{F}^{q'}$ away from the diagonal of $Z' \times Z'$.

Fix any z in the interior of U and denote by $C^q(z, \cdot)$ the unique solution of the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem

$$\begin{aligned} \Delta C^q(z, \cdot) &= 0 && \text{in } U, \\ n(C^q(z, \cdot)) &= n(*^{-1}K_{G^q}(z, \cdot)) && \text{on } \partial U, \\ n(\bar{\partial}_{\mathcal{F}}C^q(z, \cdot)) &= n(\bar{\partial}_{\mathcal{F}}*^{-1}K_{G^q}(z, \cdot)) && \text{on } \partial U \end{aligned} \tag{7}$$

in \bar{U} . The kernel

$$K^q(z, \cdot) := *^{-1}K_{G^q}(z, \cdot) - C^q(z, \cdot)$$

gives a parametrix of the $\bar{\partial}_{\mathcal{F}}$ -Neumann problem at step q in \bar{U} in the sense that the Green formula

$$u(z) = \int_{\partial U} (n(u), t(\bar{\partial}_{\mathcal{F}}^* K^q(z, \cdot)))_{\zeta} + (n(\bar{\partial}_{\mathcal{F}} u), t(K^q(z, \cdot)))_{\zeta} ds + \int_U (\Delta u, K^q(z, \cdot))_{\zeta} dv \quad (8)$$

holds for all $u \in H^2(U, \mathcal{F}^q)$ up to a term $\mathcal{S}u$, where \mathcal{S} is a smoothing operator on \bar{U} . By $t(f)$ is meant the complex tangential component of f on ∂U , cf. Section 3.2.2 in [12].

Formula (8) is actually valid for all $u \in L^2(U, \mathcal{F}^q)$ with $\Delta u \in L^2(U, \mathcal{F}^q)$. In this case the values $n(u)$ and $n(\bar{\partial}_{\mathcal{F}} u)$ on ∂U are interpreted in a weak sense. To make it more precise it suffices to assume that the neighbourhood U is small enough. Using a local fundamental solution of Δ on Z' we find a differential form $u_0 \in H^2(U, \mathcal{F}^q)$ which satisfies $\Delta u_0 = \Delta u$ in U . Obviously, the traces of $n(u_0)$ and $n(\bar{\partial}_{\mathcal{F}} u_0)$ on ∂U are well defined. Furthermore, the difference $v = u - u_0$ lies in $L^2(U, \mathcal{F}^q)$ and satisfies $\Delta v = 0$ in U . Hence both $n(v)$ and $n(\bar{\partial}_{\mathcal{F}} v)$ possess weak limit values on ∂U , cf. [14] and elsewhere. We set $n(u) = n(u_0) + n(u - u_0)$ on ∂U , and similarly for $n(\bar{\partial}_{\mathcal{F}} u)$.

Having disposed of this preliminary step, we can now return to the proof of the estimate. Let $u \in \mathcal{D}_L^q$ be an arbitrary form with $Lu \in H^s(U, \mathcal{F}^q)$. Write N_U for the $\bar{\partial}_{\mathcal{F}}$ -Neumann operator on the manifold \bar{U} . By Theorem 4.1, $u' = N_U Lu$ belongs to $H^{s+1}(U, \mathcal{F}^q)$ and satisfies

$$\|u'\|_{H^{s+1}(U, \mathcal{F}^q)} \leq C' \|Lu\|_{H^s(U, \mathcal{F}^q)}, \quad (9)$$

where C' is a constant independent of u . Since

$$\begin{aligned} \Delta u' &= Lu & \text{in } U, \\ n(u') &= 0 & \text{on } \partial U, \\ n(\bar{\partial}_{\mathcal{F}} u') &= 0 & \text{on } \partial U, \end{aligned}$$

the difference $u'' = u - u'$ lies in $L^2(U, \mathcal{F}^q)$ and fulfills $\Delta u'' = 0$ weakly in the interior of U . Moreover, both $n(u'') = n(u)$ and $n(\bar{\partial}_{\mathcal{F}} u'') = n(\bar{\partial}_{\mathcal{F}} u)$ vanish on $\partial Z \cap U$. Applying (8) yields

$$u''(z) = \int_{\partial U} (n(u''), t(\bar{\partial}_{\mathcal{F}}^* K^q(z, \cdot)))_{\zeta} + (n(\bar{\partial}_{\mathcal{F}} u''), t(K^q(z, \cdot)))_{\zeta} ds$$

up to a term $\mathcal{S}u''$. It follows that $u'' \in C^\infty(\bar{V}, \mathcal{F}^q)$.

Since $V \subset\subset U$, there is a function $\chi \in C^\infty(\bar{U})$ which is equal to 1 in a neighbourhood of $\partial U \setminus \partial Z$ and vanishes near \bar{V} . By the Stokes theorem, the above formula transforms to

$$u''(z) = \int_U (u'', \Delta(\chi K(z, \cdot)))_{\zeta} dv + \mathcal{S}u''$$

for all $z \in V$. Hence

$$\|u''\|_{H^{s+1}(V, \mathcal{F}^q)} \leq C'' (\|u\|_{L^2(U, \mathcal{F}^q)} + \|u'\|_{L^2(U, \mathcal{F}^q)}) \quad (10)$$

with C'' a constant independent of u . Combining (9) and (10) completes the proof. \square

Perhaps, there is a direct proof of Corollary 4.1 using Theorem 4.1 but we have not been able to do this.

5. Spectral Projection

By Lemma 1.4, the operator L in \mathcal{L}^q is selfadjoint, and $(L + 1)^{-1}$ is defined on all of \mathcal{L}^q . If the operator $(L + 1)^{-1}$ is compact then the spectrum of L consists of at most countable many eigenvalues $\lambda_j \geq 0$ which have no accumulation point but $+\infty$. However, $(L + 1)^{-1}$ fails to be compact for non-compact strongly pseudoconvex manifolds Z .

By the spectral theorem, for L there exists a unique orthogonal resolution E_t , $t \geq 0$, of the identity on \mathcal{L}^q , such that

$$\varphi(L) = \int_{0-}^{\infty} \varphi(t) dE_t$$

for all admissible functions φ on \mathbb{R} . It is easy to see from this that the spectrum of L coincides with the union of the sets of points of increase of all functions $(E_t u, u)_{\mathcal{L}^q}$, where $u \in \mathcal{L}^q$.

The operator $P_\lambda := E_{\lambda+0} - E_\lambda$ is an orthogonal projection of \mathcal{L}^q onto the corresponding eigenspace of L . The spectral function E_t of L is thus a ‘sum’ of P_λ over all $\lambda < t$, i.e.,

$$E_t = \int_{0-}^t P_\lambda dm(\lambda) \tag{11}$$

where $m(\lambda)$ is a non-decreasing function on \mathbb{R} .

For any interval $I = [a, b]$, the operator $E_I := E_b - E_a$ is an orthogonal projection in \mathcal{L}^q . It commutes with L , i.e., the equality $E_I L = L E_I$ holds on the domain of L . We see that L keeps invariant the range of E_I .

By the spectral kernel function of the operator L^q is meant the Schwartz kernel $K_{E_t^q}$ of the operator E_t^q . This is an element of $\mathcal{D}'(Z' \times Z', \mathcal{F}^q \boxtimes \mathcal{F}^{q'})$ with support in $Z \times Z$, such that

$$\langle E_t^q u, v \rangle_Z = \langle K_{E_t^q}, v \otimes u \rangle_{Z \times Z}$$

for all $u \in C_{\text{comp}}^\infty(Z, \mathcal{F}^q)$ and $v \in C_{\text{comp}}^\infty(Z, \mathcal{F}^{q'})$.

Taking liberties one writes

$$E_t^q u(z) = \int_Z \langle K_{E_t^q}(z, \cdot), u \rangle_\zeta \tag{12}$$

for $u \in C_{\text{comp}}^\infty(Z, \mathcal{F}^q)$. We next show that the integral makes sense for all distributions $u \in \mathcal{E}'_Z(Z', \mathcal{F}^q)$, i.e., for all generalized sections of \mathcal{F}^q with compact support in Z .

Theorem 5.1. *The spectral kernel function of L^q is infinitely differentiable up to ∂Z , i.e., $K_{E_t^q} \in C_{\text{loc}}^\infty(Z \times Z, \mathcal{F}^q \boxtimes \mathcal{F}^{q'})$.*

Proof. Since

$$C_{\text{loc}}^\infty(Z \times Z, \mathcal{F}^q \boxtimes \mathcal{F}^{q'}) = C_{\text{loc}}^\infty(Z, \mathcal{F}^q) \hat{\otimes}_\pi C_{\text{loc}}^\infty(Z, \mathcal{F}^{q'}) \stackrel{\text{top.}}{\cong} \mathcal{L}_b(\mathcal{E}'_Z(Z', \mathcal{F}^q), C_{\text{loc}}^\infty(Z, \mathcal{F}^q)),$$

the last equality being a consequence of the Schwartz kernel theorem, cf. for instance [12, § 1.5.1], it suffices to show that E_t extends to a continuous map of $\mathcal{E}'_Z(Z', \mathcal{F}^q)$ to $C_{\text{loc}}^\infty(Z, \mathcal{F}^q)$. If we prove that E_t extends to a continuous map of $H_{\text{comp}}^{-s}(Z, \mathcal{F}^q)$ to $H_{\text{loc}}^s(Z, \mathcal{F}^q)$ for each non-negative integer s , the assertion readily follows.

As mentioned, E_t is an orthogonal projection in \mathcal{L}^q . It follows that $E_t E_t = E_t$ and $E_t^* = E_t$. Using the connection between the adjoint and transposed operators, we arrive at the formula

$$E_t = E_t *^{-1} E_t' * . \tag{13}$$

If E_t maps $L^2(Z, \mathcal{F}^q)$ continuously to $H_{\text{loc}}^s(Z, \mathcal{F}^q)$, then the transpose E_t' maps $H_{\text{comp}}^{-s}(Z, \mathcal{F}^{q'})$ continuously to $L^2(Z, \mathcal{F}^{q'})$. Hence the equality (13) allows one to extend E_t to a continuous map of $H_{\text{comp}}^{-s}(Z, \mathcal{F}^q)$ to $H_{\text{loc}}^s(Z, \mathcal{F}^q)$. We are thus reduced to proving that E_t maps $L^2(Z, \mathcal{F}^q)$ continuously to $H_{\text{loc}}^s(Z, \mathcal{F}^q)$ for each non-negative integer s .

To this end, pick an arbitrary form $u \in \mathcal{L}^q$. Using formula (11) for E_t , we easily find

$$L^s E_t u = \int_{0-}^t \lambda^s P_\lambda u \, dm(\lambda)$$

and

$$\|L^s E_t u\|_{\mathcal{L}^q}^2 = \int_{0-}^t \|\lambda^s P_\lambda u\|_{\mathcal{L}^q}^2 \, dm(\lambda) \leq t^{2s} \|E_t u\|_{\mathcal{L}^q}^2,$$

which is due to the Pythagor theorem. Applying Corollary 4.1 we conclude that $E_t u \in H_{\text{loc}}^s(Z, \mathcal{F}^q)$.

To estimate a seminorm of $E_t u$ in $H_{\text{loc}}^s(Z, \mathcal{F}^q)$, we fix a relatively compact open set $V \subset Z$. Choose relatively compact open sets V_1, \dots, V_s in Z with the property that

$$V \subset\subset U_1 \subset\subset \dots \subset\subset U_s.$$

We now appeal to Corollary 4.1 to successively estimate the norm of $P_\lambda u$ in $H^s(V, \mathcal{F}^q)$, namely

$$\begin{aligned} \|P_\lambda u\|_{H^s(V, \mathcal{F}^q)} &\leq C_s (\|LP_\lambda u\|_{H^{s-1}(U_1, \mathcal{F}^q)} + \|P_\lambda u\|_{L^2(U_1, \mathcal{F}^q)}) \leq \\ &\leq C_s (\lambda \|P_\lambda u\|_{H^{s-1}(U_1, \mathcal{F}^q)} + \|P_\lambda u\|_{L^2(U_1, \mathcal{F}^q)}), \end{aligned}$$

and similarly

$$\|P_\lambda u\|_{H^{s-j}(U_j, \mathcal{F}^q)} \leq C_{s-j} (\lambda \|P_\lambda u\|_{H^{s-j-1}(U_{j+1}, \mathcal{F}^q)} + \|P_\lambda u\|_{L^2(U_{j+1}, \mathcal{F}^q)})$$

for each $j = 1, \dots, s-1$. Substituting these estimates into each other, we easily obtain

$$\begin{aligned} \|P_\lambda u\|_{H^s(V, \mathcal{F}^q)} &\leq \text{const}(s) \left(\sum_{j=0}^s \lambda^j \right) \|P_\lambda u\|_{L^2(U_s, \mathcal{F}^q)} \leq \\ &\leq \text{const}(s) \left(\sum_{j=0}^s \lambda^j \right) \|P_\lambda u\|_{\mathcal{L}^q} \end{aligned}$$

whence

$$\|E_t u\|_{H^s(V, \mathcal{F}^q)} \leq \int_{0-}^t \|P_\lambda u\|_{H^s(V, \mathcal{F}^q)} \, dm(\lambda) \leq \text{const}(s) \int_{0-}^t \left(\sum_{j=0}^s \lambda^j \right) \|P_\lambda u\|_{\mathcal{L}^q} \, dm(\lambda) \leq C \|u\|_{\mathcal{L}^q}.$$

Here, the constant C depends on s, V and t but not on u . This completes the proof. \square

Acknowledgements. The research of the first author was supported by the German Academic Exchange Service (DAAD).

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О спектральной проекции для комплексной задачи Неймана

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Мы показываем, что L^2 -спектральное ядро для решения $\bar{\partial}$ -задачи Неймана на некомпактном строго псевдовыпуклом многообразии является гладким вплоть до границы.

Ключевые слова: $\bar{\partial}$ -задача Неймана, строго псевдовыпуклые области, спектральное ядро.