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A Multidimensional Analog of the Weierstrass ζ-function in the Problem of the Number of Integer Points in a Domain

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A multidimensional analog of the Weierstrass ζ-function in $\mathbb{C}^n$ is a differential $(0, n-1)$-form with singularities in the points of the integer lattice $\Gamma \subset \mathbb{C}^n$. Using this form we construct a Γ-invariant $(n, n-1)$-form $\tau(z) \wedge dz$. The integral of this form over a domain's boundary is equal to difference between the number of integer points in the domain and its volume.

Keywords: Weierstrass ζ-function, integer lattice, Bochner-Martinelli kernel, Gauss circle problem.

The number of integer lattice points in a circle centered at the origin equals its area plus some error term. There is a problem of estimating the growth of the error as the radius of the circle increases. This question was posed and answered by C.F. Gauss who proved that the difference of the number I of integer points in the circle $x^2 + y^2 \leqslant r^2$ and its area V is equal to $O(r)$. Some of the great mathematicians of the last century spent time trying to improve this estimate: Sierpiński, Hardy, Littlewood, Vinogradov to name but a few. The best known result is due to M. Huxley stating that the error is $O(r^{\frac{46}{73}})$, although it is conjectured that it must be $O(r^{\frac{1}{2}+\varepsilon})$ (see [1] or [2] for the discussion). The same question can be obviously asked about any domain and not only planar ones (see e.g. [3–5]).

If D is a domain in $\mathbb{C} = \mathbb{R}^2$ then to count its integer points one can use the Weierstrass ζ-function for the lattice Γ of Gaussian integers

$$\zeta(z) = \frac{1}{z} + \sum_{\gamma \in \Gamma'} \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right).$$

If ∂D is piecewise smooth and does not intersect Γ then the number of the lattice points in D is

$$I(D) = \frac{1}{2\pi i} \int\limits_{\partial D} \zeta(z)\, dz.$$

It was shown in [6] that while $\zeta(z)$ is not Γ-invariant this may be corrected by adding a linear term, and its integral over ∂D gives then the desired difference

$$I\,(D) - \mathrm{Vol}\,(D) = \frac{1}{2\pi i} \int\limits_{\partial D} (\zeta\,(z) - \pi \bar{z})\, dz. \tag{1}$$

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To follow this idea in higher complex dimensions, one should use then the representation of the Dirac δ-function by the Bochner-Martinelli integral and an analog of the Weierstrass ζ-function [7].

Let $\Gamma = (\mathbb{Z} + i\mathbb{Z})^n$ be an integer lattice in $\mathbb{C}^n$, denote by ω_{BM} the $(0,\, n-1)$ differential form

$$\omega_{BM}(z) = \frac{\sum\limits_{k=1}^{n} (-1)^{k-1} \bar{z}_k d[\bar{z}_k]}{\|z\|^{2n}}.$$

Following [7], we define an analog of the Weierstrass ζ-function as the differential form

$$\zeta(z) = \omega_{BM}(z) + \sum_{\gamma \in \Gamma'} \left(\omega_{BM}(z-\gamma) + \omega_{BM}(\gamma) + \sum_{i=1}^{n} \left(\frac{\partial \omega_{BM}}{\partial z_i}(\gamma) z_i + \frac{\partial \omega_{BM}}{\partial \bar{z}_i}(\gamma) \bar{z}_i \right) \right),$$

where $\Gamma' = \Gamma \setminus \{0\}$. This series converges absolutely and uniformly on compact subsets of $\mathbb{C}^n \setminus \Gamma$. As the classical ζ-function, this form is not Γ-invariant, but we can correct this.

Lemma. *There exists a differential $(0,\, n-1)$-form $l(z)$ with linear coefficients such that $\tau(z) = \zeta(z) - l(z)$ is Γ-invariant.*

Proof. Let $w = (w_1, w_2, \ldots, w_n) = (x_1 + iy_1,\, x_2 + iy_2, \ldots, x_n + iy_n)$ be an element of the lattice Γ, for $z \notin \Gamma$ we consider the difference

$$\zeta(z+w) - \zeta(z). \tag{2}$$

While it is not Γ-invariant, the derivatives of its coefficients with respect to z_i and $\bar{z}_i$ are, as is shown by a simple computation. Therefore, this differential form depends only on w and $\bar{w}$. To compute (2) we treat each its coefficient separately. Write

$$\zeta(z) = \sum_{k=1}^{n} \zeta_k d\bar{z}[k],\ k = \overline{1,n},$$

then

$$\zeta_k(z+w) - \zeta_k(z) = \int\limits_{z}^{z+w} d\zeta_k,\ k = \overline{1,n}.$$

These integrals do not depend on the path connecting z and $z + w$ as long as it does not pass through the points of the lattice Γ. So we take z such that none of $z_1,\ z_2, \ldots, z_n$ is a Gaussian integer, and a polygonal chain consisting of line segments connecting points $(z_1,\, z_2, \ldots, z_n)$, $(z_1 + x_1,\, z_2, ..., z_n)$, $(z_1 + w_1,\, z_2, ..., z_n)$, $(z_1 + w_1,\, z_2 + x_2,\, z_3, ..., z_n)$, $(z_1 + w_1,\, z_2 + w_2,\, z_3, ..., z_n), \ldots, (z_1 + w_1, ..., z_n + x_n)$, $(z_1 + w_1, ..., z_n + w_n)$ as the integration path.

The integral of $d\zeta_1$ along the first segment of the path equals

$$\int\limits_{0}^{x_1} \left(\frac{\partial \zeta_1}{\partial z_1}(z_1 + t, z_2, ..., z_n)\, dt + \frac{\partial \zeta_1}{\partial \bar{z}_1}(z_1 + t, z_2, ..., z_n) \right) dt,$$

which is equal to

$$x_1 \int\limits_{0}^{1} \left(\frac{\partial \zeta_1}{\partial z_1}(z_1 + t, z_2, ..., z_n) + \frac{\partial \zeta_1}{\partial \bar{z}_1}(z_1 + t, z_2, ..., z_n) \right) dt,$$

due to Γ-invariance of the derivatives of ζ_1. The similar holds for the rest.

Thus, we can write

$$\zeta(z+w) - \zeta(z) = \sum_{k=1}^{n} \sum_{m=1}^{n} x_k \alpha_{mk} d\bar{z}[m] + i y_k \beta_{mk} d\bar{z}[m],$$

where

$$\alpha_{mk} = \int_0^1 \left(\frac{\partial \zeta_m}{\partial z_k}(z_1, ..., z_k + t, ..., z_n) + \frac{\partial \zeta_m}{\partial \bar{z}_k}(z_1, ..., z_k + t, ..., z_n) \right) dt,$$

$$\beta_{mk} = \int_0^1 \left(\frac{\partial \zeta_m}{\partial z_k}(z_1, ..., z_k + it, ..., z_n) - \frac{\partial \zeta_m}{\partial \bar{z}_k}(z_1, ..., z_k + it, ..., z_n) \right) dt.$$

Computing the derivatives of ζ_k , $k = \overline{1,n}$, we see that the coefficients α_{mk} and β_{mk} are related as follows

$$\beta_{mk} = -i\alpha_{mk},\ k \neq m,$$

$$\beta_{mk} = -\alpha_{mk},\ k = m.$$

Therefore we have

$$\zeta(z+w) - \zeta(z) = \sum_{k=1}^{n} \sum_{m=1}^{n} \left[\delta_{mk} \bar{w}_k \alpha_{mk} + (1 - \delta_{mk})(x_k + y_k) \alpha_{mk} \right] d\bar{z}[m],$$

where δ_{mk} is the Kronecker delta. It is clear that we may take

$$l(z) = \sum_{k=1}^{n} \sum_{m=1}^{n} \left[\delta_{mk} \bar{z}_k \alpha_{mk} + (1 - \delta_{mk})(\mathrm{Re} z_k + \mathrm{Im} z_k) \alpha_{mk} \right] d\bar{z}[m].$$

□

The form $\zeta(z) - l(z)$ is Γ-invariant, we shall denote it by $\tau(z)$. Similarly to the one-dimensional (planar) case, there is a version of formula (1).

Theorem. *Let D be a domain in $\mathbb{C}^n$ with a piecewise smooth boundary ∂D without lattice points on it. Then the number of lattice points inside D is related to the volume* Vol (D) *by the formula*

$$I - \mathrm{Vol}(D) = \frac{1}{(2\pi i)^n} \int_{\partial D} \tau(z) \wedge dz.$$

Proof. The integration set is obviously homologous to the sum of spheres of sufficiently small radius with centers in the lattice points inside D. Let S be one of such spheres. The integral

$$\frac{1}{(2\pi i)^n} \int_S \zeta(z) \wedge dz$$

over the sphere with the center $w = (w_1, w_2, ..., w_n) \in \Gamma$ equals to 1. Indeed, the interior of $S = \partial B$ contains only one term of $\zeta(z) \wedge dz$ with singularity: $\omega_{BM}(z - w) \wedge dz$. Due to Stokes' theorem, the remaining terms are either zero or cancel each other. Therefore,

$$\frac{1}{(2\pi i)^n} \int_{\partial D} \zeta(z) \wedge dz \tag{3}$$

is equal to the number of the lattice points in the interior of D.

Consider now

$$\frac{1}{(2\pi i)^n}\int\limits_{\partial D} l\left(z\right)\wedge dz.$$

The integrand does not have any singularity in D, therefore again by the Stokes' theorem we have

$$\frac{1}{(2\pi i)^n}\int\limits_{\partial D} l\left(z\right)\wedge dz = \frac{1}{(2\pi i)^n}\int\limits_{D}\sum_{k=1}^{n}\sum_{m=1}^{n} d\left[\delta_{mk}\bar{w}_k\alpha_{mk} + (1-\delta_{mk})\left(\mathrm{Re}w_k + \mathrm{Im}w_k\right)\alpha_{mk}\right] d\bar{z}\left[m\right] =$$

$$= \frac{1}{(2\pi i)^n}\int\limits_{D}\sum_{m=1}^{n}\left(-1\right)^{m-1}\alpha_{mm} d\bar{z}\wedge dz.$$

Eventually,

$$\frac{1}{(2\pi i)^n}\int\limits_{\partial D} l\left(z\right)\wedge dz = \frac{\mathrm{Vol}\left(D\right)}{\pi^n}\sum_{m=1}^{n}\left(-1\right)^{m-1}\alpha_{mm}.$$

To determine the value of $\sum_{m=1}^{n}\left(-1\right)^{m-1}\alpha_{mm}$ let us consider D to be a $2n$-dimensional hypercube Q with the center at the origin with edges of length 1 parallel to coordinate axes. When integrating $\tau\left(z\right)$ along boundary of D, the integrals over opposite faces of the hypercube cancel each other due to the Γ-invariance of $\tau(z)$. Thus,

$$\frac{1}{(2\pi i)^n}\int\limits_{\partial D} \tau\left(z\right)\wedge dz = 0.$$

On the other hand, there is only one integer point, the origin, inside of Q. Therefore,

$$1 - \frac{\mathrm{Vol}\left(D\right)}{\pi^n}\sum_{m=1}^{n}\left(-1\right)^{m-1}\alpha_{mm} = 0,$$

and

$$\sum_{m=1}^{n}\left(-1\right)^{m-1}\alpha_{mm} = \pi^n.$$

This proves the theorem. □

Remark 1. Denote the basis vectors of the fundamental parallelepiped of Γ by γ_j, $j = \overline{1, 2n}$: for $j \leqslant n$ $\gamma_j = (\gamma_{1j}, ..., \gamma_{nj})$ such that $\gamma_{kj} = \delta_{kj}$, $k = \overline{1, n}$ and for $n < j \leqslant 2n$ $\gamma_j = i\gamma_{j-n}$. Then in the notation of [7, p. 92] we have

$$\eta_{kj} = \begin{cases} \alpha_{kj}, & j \leqslant n, \\ i\beta_{kj}, & n < j \leqslant 2n. \end{cases}$$

In the case of $\Gamma = \left(\mathbb{Z} + i\mathbb{Z}\right)^2 \subset \mathbb{C}^2$ we have $\alpha_{11} - \alpha_{22} = \pi^2$. The computer experiments show that

$$\alpha_{11} = -\beta_{11} = -\alpha_{22} = \beta_{22} = \frac{\pi^2}{2}, \quad \alpha_{12} = \beta_{12} = \alpha_{21} = \beta_{21} = 0.$$

Remark 2. Notice that formula (3) gives the number of integer lattice points in a domain D in the even dimensional space $\mathbb{R}^{2n} = \mathbb{C}^n$. However, it can be used for domains in the space of odd dimension $\mathbb{R}^{2n+1}$ too. Let G be a domain in $\mathbb{R}^{2n+1}$, in order to apply (3), we complexify $\mathbb{R}^{2n+1}$ and define D to be the domain $D = \left\{G + iI_\varepsilon^{2n+1}\right\}$, where $I_\varepsilon = [-\varepsilon, \varepsilon] \subset \mathbb{R}$.

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Многомерный аналог ζ-функции Вейерштрасса в задаче о числе целых точек в области

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Многомерный аналог ζ-функции Вейерштрасса в $\mathbb{C}^n$ есть дифференциальная $(0, n-1)$-форма с особенностями в узлах целочисленной решётки $\Gamma \subset \mathbb{C}^n$. В статье при помощи этой формы строится Γ-инвариантная $(n, n-1)$-форма $\tau(z) \wedge dz$, интеграл которой по границе области $D \subset \mathbb{C}^n$ равен разности числа целых точек в области и её объёма.

Ключевые слова: ζ-функция Вейерштрасса, целочисленная решётка, ядро Бохнера - Мартинелли.