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Representation of Preferences by Generalized Coherent Risk Measures

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In this paper the model of generalized coherent risk measures is considered. Within the bounds of the model the properties of acceptance set are examined. A notion of elliptic cone is introduced. It is shown that the elliptic cone can be used as an acceptance set. The properties of the elliptic acceptance cone, particularly the interrelation between the cone shape and the risk aversion value, are studied.

Keywords: preference relation, stochastic dominance, risk measure, risk aversion, generalized coherent risk measure, acceptance set, elliptic cone.

Introduction

We work in a probability space $(\Omega, \mathcal{A}, P)$, where Ω is a reference set, $\mathcal{A}$ — a σ-algebra specified on Ω, P — a probability measure, specified on the sets of $\mathcal{A}$.

Definition 0.1. *A **Risk** X on $(\Omega, \mathcal{A})$ is any measurable mapping from Ω to $\mathbf{R}$ (a random variable).*

The set of all risks on $(\Omega, \mathcal{A})$ we denote by $\mathcal{X}$.

1. Orders and Preferences on the Set of Risks

Natural Orders

We can specify order relation $\leqslant$ on the set $\mathcal{X}$:

$$X \leqslant Y \Longleftrightarrow P(\omega : X(\omega) \leqslant Y(\omega)) = 1.$$

Strict order $<$ is an order relation determined by:

$$X < Y \Longleftrightarrow P(\omega : X(\omega) < Y(\omega)) = 1.$$

Suppose $|\Omega| = n$. Then we can submit a σ-algebra $\mathcal{A}$ in the form of $\mathcal{A} = 2^{\Omega}$. Probability measures P on the measurable space can be represented as the elements of the standard simplex in $\mathbf{R}^n$:

$$S^n = \{P = (p^1, \ldots, p^n) \in \mathbf{R}^n : \\ p^1 \geqslant 0, \ldots, p^n \geqslant 0,\, p^1 + \cdots + p^n = 1\}.$$

The set of all risks $\mathcal{X}$ is isomorphic to $\mathbf{R}^n$. Renumbering the elements of Ω in some arbitrary way: $\Omega = \{\omega^1, \ldots, \omega^n\}$, we denote $P(\omega^i) = p^i$, $X(\omega^i) = X^i$, $i = 1, \ldots, n$. We identify random the variables $X \in \mathcal{X}$ with the vectors $X = (X^1, \ldots, X^n) \in \mathbf{R}^n$.

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We assume that $X \leqslant Y$ if $X^i \leqslant Y^i$ for all $i = 1, \ldots, n$.

Another way to specify an order on the set $\mathcal{X}$ is related to stochastic dominance. Denote by $\mathcal{F}$ the set of all distribution functions, by F_X — the distribution function of a random variable X:

$$F_X(x) = P(X \leqslant x).$$

Let $\mathcal{F}_k$ be a set of all distribution functions with finite values of k-th moments:

$$\mathcal{F}_k = \{F \in \mathcal{F} : \ |\mu_k^F| < \infty\}, \quad \mu_k^F = \int_{-\infty}^{\infty} t^k dF(t).$$

For a given $F \in \mathcal{F}$ specify a sequence of functions $F^{(k)}, \ k = 1, 2, \ldots$:

$$F^{(1)}(x) = F(x), \quad F^{(k+1)}(x) = \int_{-\infty}^{x} F^{(k)}(t)dt, \quad -\infty < x < \infty.$$

Suppose $F, Q \in \mathcal{F}_k$. We say that Q has ***k-order stochastic dominance*** over F ($F \leqslant_k Q$), if

$$F^{(k)}(x) \geqslant Q^{(k)}(x), \quad -\infty < x < \infty.$$

We can also introduce strict stochastic dominance. Suppose $F, Q \in \mathcal{F}_k$. We say that Q *strictly dominates F with the order k* ($F <_k Q$), if

$$F \leqslant_k Q \quad \text{and} \quad \exists\, x \in \mathbf{R} : \ F^{(k)}(x) > Q^{(k)}(x).$$

For the case of a finite reference set the orders $\leqslant$ and $\leqslant_1$ are consistent - from $X \leqslant Y$ it follows that $X \leqslant_1 Y$. This is a consequence of the fact that for all $X, Y \in \mathcal{X}$ $P(X^i) = P(Y^i), \quad i = 1, \ldots, n$.

By means of first-order stochastic dominance we can determine an order relation $\leqslant_1$ ($<_1$) on $\mathcal{X}$:

$$X \leqslant_1 Y \ (X <_1 Y) \Longleftrightarrow F_X \leqslant_1 F_Y \quad (F_X <_1 F_Y).$$

A ***Preference relation*** $\preceq$ on the set $\mathcal{X}$ is a complete transitive binary relation on $\mathcal{X}$. Risks X and Y are called *equivalent* if $X \preceq Y$ and $Y \preceq X$.

Suppose that a preference relation $\preceq$ reflects an individual attitude to risk of a certain investor. The relation $X \preceq Y$ means that in equal conditions the investor prefers a financial instrument with return Y to an instrument with return X(or both instruments are equally preferable if $X \sim Y$).

If to take into account that in equal conditions market participants seek to maximal profits, it is reasonable to require that a preference relation $\preceq$ on $\mathcal{X}$ should be conformed to the order $\leqslant$ on $\mathcal{X}$:

$$X \leqslant Y \Longrightarrow X \preceq Y.$$

Such preference relations are called *monotone*. They are called *strictly monotone* if

$$X < Y \Longrightarrow X \prec Y.$$

One of the ways to describe preferences on the set of all risks is to represent them by a real-valued functional.

A preference relation is represented on $\mathcal{X}$ by a measure $\rho : \mathcal{X} \to \mathbf{R}$ if one of the following conditions holds:

$$\rho(X) \leqslant \rho(Y), \text{ if } X \preceq Y, \quad X, Y \in \mathcal{X} \tag{1}$$

$$\rho(X) \leqslant \rho(Y), \text{ if } Y \preceq X, \quad X, Y \in \mathcal{X} \tag{2}$$

The functional ρ is called a **risk measure**.

Hereinafter we deal with risk measures that represent preference relations like in (1).

2. Risk Aversion

Risk aversion is a disposition of a person to accept a bargain with an uncertain payoff rather than another bargain with a more certain, but possibly lower, expected payoff. A preference $\preceq$ on $\mathcal{X}$ is called risk averse if for all nonsingular risks $\Delta : E\Delta = 0$ and any $a \in \mathbf{R}$

$$a + \Delta \prec a. \tag{3}$$

In terms of risk measure the property of risk aversion can be written as

$$\rho(a + \Delta) < \rho(a).$$

For some preferences we can get a numerical characteristic of risk aversion.

By W_a denote a degenerate risk localized at $a \in \mathbf{R}$ (P(X=a)=1). If a preference relation $\preceq$ on $\mathcal{X}$ is conformed with $\leqslant_1$ then $W_a \preceq W_b$ if $a < b$. Moreover it is logical to assume that the preference is strict $W_a \prec W_b$.

A preference $\preceq$ on $\mathcal{X}$ is called *regular* if it is conformed with stochastic dominance $\leqslant_1$, for all $a, b \in \mathbf{R} : a < b$ $W_a \prec W_b$ and in every equivalence class $K \in \mathcal{X}_\sim$ there is exactly one degenerate distribution.

It was shown in [1] that for regular preference (3) can be written as $\forall \Delta \in \mathcal{X} : \quad E\Delta = 0, a \in \mathbf{R}$ $\exists\, c > 0$

$$a + \Delta \sim a - c \tag{4}$$

If a regular preference on $\mathcal{X}$ is represented by a risk measure ρ (which is also called regular in this case) then (4) can be written as

$$\rho(a + \Delta) = \rho(a - c).$$

Value c can be interpreted as a price for which a person agrees to accept uncertainty. It can be used as a quantitative assessment of risk aversion that was introduced in [1].

A functional $\rho : \mathcal{X} \to R$ is called *canonical* if $\rho(W_a) = a$ $\forall a \in R$. Every regular risk measure has a canonical analog.

For the canonical risk measure ρ value of risk aversion $c_{a,\rho}(\Delta)$ is a solution of

$$\rho(a + \Delta) = \rho(a - c). \tag{5}$$

3. Generalized Coherent Risk Measures

The term "generalized coherent risk measure" was introduced in [2] and presented a modification of classical coherent risk measures introduced in [3].

We consider another modification. It is also called generalized coherent risk measures because it defines a broader class of functionals then classic coherent measures.

The axiomatics of generalized coherent risk measures is bases on the axiomatics of classical coherent risk measures with some distinctions.

A risk is called *acceptable* if investor agrees to work with it without investing any capital.

The set of all acceptable to an investor risks is called an ***acceptance set*** and is denoted by A ($A \subset \mathcal{X}$).

Suppose that $|\Omega| = n$. Then $X = (x_1, \ldots, x_n)$ is a vector from $\mathbf{R}^n$.

Also we introduce a norm $\|\cdot\|$ in $\mathbf{R}^n$. It can be for example $\|\cdot\|_p$ $(1 \leqslant p < \infty)$, given by

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \|X\|_\infty = \lim_{p\to\infty} \|X\|_p = \max\{|x_1|, \ldots, |x_n|\}$$

We postulate that any acceptance set A satisfies the following axioms:
A1: $C_+ \subset A,\ C_+ = \{X \in \mathcal{X} : X \geqslant 0\}$
A2: $A \bigcap C_- = \emptyset,\ C_- = \{X \in \mathcal{X} : X < 0\}$
A3: A is a convex cone (if $X \in A,\ Y \in A$, then $\alpha_1 X + \alpha_2 Y \in A,\ \alpha_1, \alpha_2 \geqslant 0$).

A ***generalized coherent risk measure*** f_A, associated with A is determined by

$$f_A(X) = f_{A,\|\cdot\|}(X) = \delta_A(X) \inf_{Y \in \partial A} \|X - Y\|,$$
$$\delta_A(X) = \begin{cases} 1, & X \in A, \\ -1, & X \in A^c \end{cases}, \tag{6}$$

where ∂A is a boundary of A.

The functional $f_A(X)$ exhibits the following properties:

M) monotonicity:

$$f_A(X) \leqslant f_A(Y),\ \ \forall X, Y \in \mathcal{X}, X \leqslant Y;$$

PH) positive homogeneity:

$$f_A(\lambda X) = \lambda f_A(X),\ \forall \lambda \geqslant 0, X \in \mathcal{X};$$

S) superadditivity:

$$f_A(X + Y) \geqslant f_A(X) + f_A(Y),\ \ \forall X, Y \in \mathcal{X};$$

Sh) shortcut property:
$\forall X \in \mathcal{X}\ \exists\ X'(X) \in \partial A$ that $\|X - X'(X)\| = \inf_{Y \in \partial A} \|X - Y\|$ and

$$f_A(X + \lambda u(X)) = f_A(X) + \lambda, \quad -\infty < \lambda \leqslant \lambda_A(X),$$

$$\text{where } \lambda_A(X) > 0,\ u(X) = \delta_A(X) \frac{X - X'_A(X)}{\|X - X'_A(X)\|}$$

It is obvious that the classical coherent risk measure is a particular case of generalized coherent risk measures corresponding to the norm $|| \cdot || = || \cdot ||_\infty$.

In [4] there is given a representation theorem for a generalized coherent risk measure, associated with an acceptance set A.

By $\mathcal{X}^*$ denote the dual space (the space of linear continuous functionals on $\mathcal{X}$), the dual cone A^* is defined as

$$A^* = \{g \in \mathcal{X}^* :\ g(X) \geqslant 0,\ X \in A\}. \tag{7}$$

Distinguish the subset of functionals with unit norm:

$$A_1^* = \{g \in A^* :\ \|g\|_* = 1\}.$$

Theorem 3.1. *Let f be a generalized coherent risk measure, defined by an acceptance set A. Then the following representation is valid:*

$$f_A(X) = \inf_{g \in A_1^*} g(X), \quad X \in \mathcal{X}. \tag{8}$$

Value of Risk Aversion for Generalized Coherent Risk Measures

For a classical coherent risk measure ρ we can easily find the value of risk aversion. Since this measure of risk is canonical and possesses the property of translation invariance [3], we have:

$$c_\rho(\Delta) = -\rho(\Delta).$$

Consider a generalized coherent risk measure $f(x)$. As it is not canonical, first we find its canonical analog f_A^K:

$$f_A^K(X) = \begin{cases} \dfrac{f(X)}{f(I)}, & X \in A, \\ \dfrac{-f(X)}{f(-I)}, & X \notin A. \end{cases}$$

Solving equation (5) for the measure f_A^K we find:

$$c_{a,A}(\Delta) = \begin{cases} a - \dfrac{f_A(aI+\Delta)}{f_A(I)}, & aI+\Delta \in A, \\ a + \dfrac{f_A(aI+\Delta)}{f_A(-I)}, & aI+\Delta \notin A \end{cases} \tag{9}$$

We obtain that in the model of general coherent risk measures risk aversion depends on a.

In particular,

$$c_{0,\Delta} = \frac{f(\Delta)}{f(-I)} \tag{10}$$

4. The Properties of Acceptance Sets

Consider two probability spaces $(\Omega, \mathcal{A}, P)$ and $(\Omega, \mathcal{A}, Q)$. The sets of all risks, defined on them, we denote by $\mathcal{X}_P$ and $\mathcal{X}_Q$, the acceptance sets - by A_P and A_Q.

Suppose $X_P = (X_1, \ldots, X_n) \in \mathcal{X}_P$ and $X_Q = (X_1, \ldots, X_n) \in \mathcal{X}_Q$.

We assume $A_P = A_Q$ if $\forall X_P \in A_P \Rightarrow \ X_Q \in A_Q$ and vice versa, $\forall X_Q \in A_Q \Rightarrow \ X_P \in A_P$

Theorem 4.1. *Let the preference relation $\preceq$ on $\mathcal{X}_P$ and $\mathcal{X}_Q$ be consistent with the stochastic dominance $\leqslant_1$. If $P \neq Q$ and $\exists\, X_P$ such that $X_P <_1 X_Q$ (or $X_Q <_1 X_P$), then $A_P \neq A_Q$.*

Theorem 4.2. *Let the preference relation $\preceq$ be risk averse. Then for all $X \in A, X \neq 0$ is true that $EX > 0$.* [†].

Theorem 4.3. *Let the preference relation $\preceq$ be consistent with stochastic dominance $\leqslant_1$ and $P = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. Then for every $X = (X^1, X^2, \ldots, X^n) \in A$, the vector Y which components are obtained by interchanging the components of X also lies in the cone A (the cone is symmetric about the coordinate axes).*

Proof. Consider $X = (X^1, X^2, \ldots, X^n) \in A_P$ and a vector $Y = (X^{i_1}, X^{i_2}, \ldots, X^{i_n})$, obtained by interchanging the components of X.

Since $P(X^j) = P(Y^j) = 1/n$, it follows that $F_X(x) = F_Y(x)\ \forall x \in R$, then $X \sim Y$ and $f_{A_P}(X) = f_{A_P}(Y)$. Therefore, X and Y belong or do not belong to the cone A_P contemporaneously. □

Theorem 4.4. *Let the preference relation $\preceq$ be consistent with stochastic dominance. Then for the acceptance cone A_P $(P = (p_1, \ldots, p_{k-1}, 0, p_{k+1}, \ldots, p_n))$ it is true that*

$$X = (X^1, \ldots, X^{k-1}, X^k, X^{k+1}, \ldots, X^n) \in A_P \Rightarrow$$
$$Y = (X^1, \ldots, X^{k-1}, y, X^{k+1}, \ldots, X^n) \in A_P \quad \forall y \in \mathbf{R}.$$

[†]Theorems 4.1 and 4.2 were proved in [5]

Proof. Suppose $X \in A_P$.

$$F_X(x) = F_Y(x) \quad \forall x \in \mathbf{R}$$
$$F_X \leqslant_1 F_Y \ \Rightarrow f(X) \leqslant f(Y);$$
$$F_Y \leqslant_1 F_X \ \Rightarrow f(Y) \leqslant f(X).$$

Therefore, $f(Y) = f(X)$, thus we have $X \in A_P \ \Rightarrow Y \in A_P$. □

Theorem 4.5. *Let the preference relation $\preceq$ be consistent with stochastic dominance. Then the acceptance cone A_P, corresponding to $P = (p_1, \dots, p_n)$, where $p_k = 1; \quad p_i = 0, i \neq k$ can be defined by the inequality:*

$$X^k \geqslant 0 \tag{11}$$

Proof. Assume that there exists such vector

$$X = (X^1, \dots, X^{k-1}, X^k, X^{k+1}, \dots, X^n) \in A_P,$$

that $X^k < 0$, but then by Theorem 4.4 the vector

$$Z = (-1, \dots, -1, X^k, -1, \dots, -1) \in A_P,$$

but this is impossible since $Z \in C_-$, and $C_- \cap A_P = \emptyset$ by the axiom **A2**. So, if $X : \ X^k < 0$, then $X \notin A_P$.

Suppose $X^k \geqslant 0$. Then by Axiom **A1** $X = (0, \dots, 0, X^k, 0, \dots, 0) \in A_P$, hence, by Theorem 4.4

$$Y = (Y^1, \dots, Y^{k-1}, X^k, Y^{k+1}, \dots, Y^n) \in A_P$$
$$\forall\, Y^1, \dots, Y^{k-1}, Y^{k+1}, \dots, Y^n \in \mathbf{R}.$$

We obtain that if X is such that $X^k \geqslant 0$, then $X \in A_P$. Hence, (11) actually determines the acceptance cone for the given risk measure. □

5. Elliptic Acceptance Set

Consider a set

$$\sum_{i=1}^{n} \frac{(X^i - (P,X)np_i)^2}{r^2(p_i)} \leqslant (P,X)^2, \quad X = (X^1, \dots, X^n) \in \mathbf{R}^n \tag{12}$$

and suppose that it satisfies the following conditions:

$$\begin{aligned} &1. \quad (P,X) \geqslant 0; \\ &2. \quad r(p) \geqslant \sqrt{\frac{n}{p^2} + n^3 p^2} \end{aligned} \tag{13}$$

Theorem 5.1. *The set A, determined by inequality (12), is an acceptance set for some preference.*

Proof. For A to be an acceptance set it is sufficient to satisfy Axioms **A1**–**A3**.

1. First we prove that A satisfies **A2**: $A \cap C_- = \emptyset$. Consider an arbitrary $X \in C_-$ $X^i < 0 \quad \forall i = 1, 2, \dots, n \quad \Rightarrow (P,X) < 0$, therefore, X doesn't satisfy (13).

2. Then we prove **A3**: A is a convex set.

Let $X \in A$. Then $(P,X) = a > 0$ and

$$\sum_{i=1}^{n} \frac{(X^i - anp_i)^2}{r^2(p_i)} \leqslant a^2 \tag{14}$$

The set $\{Y: (P,Y)=a\}$ also satisfies (14) and it forms a n-dimensional ellipsoid E_a, which is a convex set.

Suppose that $X'=\lambda X, \quad \lambda \geqslant 0$. Then $(P,X')=\lambda a$. X' also belongs to A (It can be verified by substituting in(12)).

Moreover, it belongs to the ellipsoid $E_{\lambda a}$

$$\frac{(X^1-\lambda anp_1)^2}{r^2(p_1)}+\cdots+\frac{(X^n-\lambda anp_n)^2}{r^2(p_n)} \leqslant \lambda a^2, \tag{15}$$

like all other vectors $Y'=\lambda Y, \quad Y\in E_a$. Hence, A is a cone and all its hyperplane sections $(P,X)=a,\ a>0$ are ellipsoids.

Therefore, A is a convex set.

3. At last we prove **A1**: $C_+\subset A$.

Consider the basis $e=\{e_i,\ i=1,\dots,n:\ e_i^i=1,\ e_i^j=0\}$.

Any vector $X\in C_+$ can be represented as a convex linear combination of the elements of the basis e:

$$X=X^1e_1+X^2e_2+\cdots+X^ne_n, \quad X^i\geqslant 0,\ i=1,\dots,n$$

Since Axiom **A3** is satisfied, we can assert that $C_+\subset A$, if $e_i\in A \quad \forall i=1,\dots,n$.

Then we prove that $e_1\in A$ (for the rest e_i the proof is similar). Substitute coordinates of e_1 in (12):

$$\frac{(1-np_1{}^2)^2}{r^2(p_1)}+\frac{(np_1p_2)^2}{r^2(p_n)}+\cdots+\frac{(np_1p_n)^2}{r^2(p_n)}=\frac{1-2np_1{}^2+n^2p_1^4}{r^2(p_1)}+n^2p_1^2\left(\frac{p_2^2}{r^2(p_2)}+\cdots+\frac{p_n^2}{r^2(p_n)}\right)\leqslant$$
$$\leqslant\frac{1-2np_1^2+n^2p_1^4}{\frac{n}{p_1^2}+n^3p_1^2}+n^2p_1^2\left(\frac{p_2^2}{\frac{n}{p_2^2}+n^3p_2^2}+\cdots+\frac{p_n^2}{\frac{n}{p_n^2}+n^3p_n^2}\right)\leqslant\frac{p_1^2}{n}+n^2p_1^2\frac{n-1}{n^3}=p_1^2=(e_1,P)^2$$

Hence, $e_j\in A \quad j=1,\dots,n$, thus, $C_+\subset A$. □

Theorem 5.2. *An elliptic cone A defines a reference consistent with stochastic dominance.*

Proof. A preference relation determined by ρ_A is consisted with stochastic dominance if $\forall X,Y:\ F_X\leqslant_1 F_Y$ it is true that $\rho_A(X)\leqslant\rho_A(Y)$.

In our case X and Y are discrete:

$$X=(X^1,X^2,\dots,X^n), \quad Y=(Y^1,Y^2,\dots,Y^n)$$

It means that $F_X(x)\geqslant F_Y(x) \quad \forall x$ is true iff $X^i\leqslant Y^i\ i=1,\dots,n$, that is $X\leqslant Y$.

By Theorem 5.1 A is an acceptance cone and hence the risk measure ρ_A is coherent. Respectively $\rho(X)\leqslant\rho(Y)$ if $X\leqslant Y$. It means that if $F_X\leqslant_1 F_Y$ it is true that $\rho(X)\leqslant\rho(Y)$.

□

6. Some Special Cases of the Elliptic Acceptance Set

Case 1. Consider an elliptic cone A_P such that $P=(p_1,\dots,p_{k-1},0,p_{k+1},\dots,p_n)$.

From Theorem 4.4 it follows that since $X=0\in A_P$ we have $Y=(0,\dots,0,y,0,\dots,0)\in A_P$. Substituting P and Y in (12) we get

$$\frac{y^2}{r^2(0)}\leqslant 0 \quad \forall y\in\mathbf{R}$$

Therefore, we have

$$r(0)=\infty. \tag{16}$$

One can easily see that there are no summands corresponding to $i_1, \dots, i_m$ coordinates in (12) if $p_{i_1} = \dots = p_{i_m} = 0$.

Case 2. Suppose A_P is an elliptic cone corresponding to $P = (p_1, \dots, p_n)$, $p_k = 1$, $p_i = 0, i \neq k$.

From Theorem 4.5 it follows that A_P is specified by $X^k \geqslant 0$. Substitute P in (12) and (13). We obtain

$$\begin{gathered} \frac{(X^k)^2(1-n)^2}{r^2(1)} \leqslant (X^k)^2,\ X^k \geqslant 0 \\ r(1) \geqslant n-1 \end{gathered} \tag{17}$$

The first inequality in (17) is satisfied automatically if the last inequality in (17) is satisfied. Hence, we get one more condition on the function $r(p)$:

$$r(1) \geqslant n-1. \tag{18}$$

Case 3 Consider $P = (\frac{1}{n}, \dots, \frac{1}{n})$. Let the norm in $\mathcal{X}$ be Euclidean ($\|\cdot\| = \|\cdot\|_2$). Substituting P (12), we get the following inequality for A_P:

$$(X^1 - (P,X))^2 + \dots + (X^n - (P,X))^2 \leqslant (P,X)^2 r^2(1/n). \tag{19}$$

By I we denote the unit vector $(1, \dots, 1)$, and by r_0 the following form $r_0 = r\,(1/n)$. Combining this with (19), we obtain

$$\|X - (P,X)I\| \leqslant (P,X) r_0$$

$$\left\| X - \frac{(I,X)}{n} I \right\| \leqslant \frac{(I,X)}{n} r_0, \quad (I,X) \geqslant 0. \tag{20}$$

All sections of the cone by hyperplanes $(I,X) = a$ are n-dimensional spheres of radius $(I,X)r_0/n$.

An acceptance cone is called a **spherical cone** if it is defined by (20).

Risk Aversion for a Spherical Cone

Lemma 6.1. *Let $\mathcal{X}$ be an Euclidean space, $P = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, and let the acceptance cone A be determined by the inequality* $\left\| X - \frac{(I,X)}{n} I \right\| \leqslant \frac{(I,X)}{n} r_0$.

Then $\forall \Delta : E\Delta = 0, \|\Delta\| = 1$

$$\inf_{X \in \partial A} \|X - \Delta\| = \|Z - \Delta\|, \tag{21}$$

where $Z = \dfrac{(I,Z)}{n} I + \dfrac{(I,Z) r_0}{n} \Delta, \quad (I,Z) \geqslant 0.$

Proof. We claim that $(\Delta, I) = 0$. Indeed,

$$E\Delta = \frac{1}{n}\Delta^1 + \dots + \frac{1}{n}\Delta^n = \frac{1}{n}(\Delta, I) = 0 \implies (\Delta, I) = 0$$

1. At first we prove that $Z \in \partial A$:

$$\begin{aligned} \left\| Z - \frac{(I,Z)}{n} I \right\| &= \left\| \frac{(I,Z)}{n} I + \frac{(I,Z) r_0}{n} \Delta - \frac{(I,Z)}{n} I \right\| = \\ &= \left\| \frac{(I,Z) r_0}{n} \Delta \right\| = \frac{(I,Z) r_0}{n} \|\Delta\| = \frac{(I,Z) r_0}{n}, \end{aligned}$$

therefore, $Z \in \partial A$.

2. Then we prove that $\|Z - \Delta\| = \inf_{X \in \partial A} \|X - \Delta\|$. Suppose $\exists\, Y \in \partial A$, then

$$\|Y - \Delta\| \leqslant \|Z - \Delta\|, \tag{22}$$

$Z - \Delta$ and $Y - \Delta$ belong to the same plane. Hence, the triangle, constructed on these vectors, belongs to this plane plane. Therefore, the condition (22) holds iff $\cos\beta \geqslant 0$, where $\beta = (Y - Z)\widehat{\ }(Z - \Delta)$.

Y can be written as

$$Y = \frac{(I, Z)}{n} I + \frac{(I, Z) r_0}{n} \Delta + (Y - Z).$$

Solve the optimization problem

$$\|Y - \Delta\|^2 \to \min_{Y \in \partial A} \tag{23}$$

$$\|Y - \Delta\|^2 = \left\| Y - Z + \tfrac{(I,Z)I}{n} + \left(\tfrac{(I,Z)r_0}{n} - 1\right)\Delta \right\|^2 = \|Y - Z\|^2 + \tfrac{(I,Z)^2}{n^2} I^2 + \left(\tfrac{(I,Z)r_0}{n} - 1\right)^2 \Delta^2 +$$
$$+ 2\left(Y - Z, \tfrac{(I,Z)}{n} I\right) + 2\left(Y - Z, \left(\tfrac{(I,Z)r_0}{n} - 1\right)\Delta\right) = \|Y - Z\|^2 + \tfrac{(I,Z)^2}{n^2} + \left(\tfrac{(I,Z)r_0}{n} - 1\right)^2 +$$
$$+ 2\left(Y - Z, \tfrac{(I,Z)}{n} I + \tfrac{(I,Z)}{n}\Delta - \Delta\right) = \|Y - Z\|^2 + \tfrac{(I,Z)^2}{n^2} + \left(\tfrac{(I,Z)r_0}{n} - 1\right)^2 + 2(Y - Z, Z - \Delta)$$

The optimization problem (23) is equivalent to the problem

$$b = \|Y - Z\|^2 + 2(Y - Z, Z - \Delta) \;\to\; \min_{Y}$$

$$b = \|Y - Z\|^2 + 2\|Y - Z\| \cdot \|Z - \Delta\| \cos\beta \geqslant 0$$

$$b = 0 \quad \text{if } Y = Z$$

Finally, we obtain $\inf_{X \in \partial A} \|X - \Delta\| = \|Z - \Delta\|$. □

Theorem 6.1. *Let $\mathcal{X}$ be an Euclidean space. The cone A is determined by $\left\| X - \frac{(I,X)}{n} I \right\| \leqslant \dfrac{(I, X)}{n} r_0$.*

Then $\forall \Delta : E\Delta = 0, \|\Delta\| = 1$ is true that

$$c_{0,\Delta} = \frac{1}{\sqrt{n + r_0^2}} \tag{24}$$

Proof. Since $Z \perp (\Delta - Z)$, by Pythagorean theorem we obtain

$$\|Z\|^2 + \|Z - \Delta\|^2 = \|\Delta\|^2$$

$$\|Z - \Delta\|^2 = 1 - \|Z\|^2 \tag{25}$$

On the other hand, $\dfrac{(I, Z)}{n} I \perp (Z - \dfrac{(I, Z)}{n} I)$, therefore, we get

$$\|Z\|^2 = \left\| \frac{(I, Z)}{n} I \right\|^2 + \left\| Z - \frac{(I, Z)}{n} I \right\| = \frac{(I, Z)^2}{n^2} \|I\|^2 + \frac{(I, Z)^2 r_0^2}{n^2} = \frac{(I, Z)^2}{n} \left(\frac{r_0^2}{n} + 1 \right) =$$
$$= \frac{\|I\|^2 \|Z\|^2 \cos^2\alpha}{n} \left(1 + \frac{r_0^2}{n} \right), \quad \text{where } \alpha = Z\widehat{\ }I$$

$$\|Z\|^2 = \frac{\|I\|^2 \|Z\|^2 \cos^2\alpha}{n} \left(1 + \frac{r_0^2}{n} \right)$$

$$\cos^2\alpha = \frac{n}{\|I\|^2\left(1+\frac{r_0^2}{n}\right)} = \frac{n}{n+r_0^2}$$

By Lemma 6.1 vector Z belongs to same two-dimensional plane as the vectors I and Δ.

Since $I \perp \Delta$, we have $\gamma = Z^\wedge\Delta = \dfrac{\pi}{2} - \alpha$.

$$\|Z\|^2 = \|\Delta\|^2\cos^2\beta = 1-\cos^2\alpha = \frac{r_0^2}{n+r_0^2}$$

Substituting this result in (25), we get

$$\|Z-\Delta\| = \frac{n}{n+r_0^2} \quad \forall\Delta : E\Delta = 0,\ \|\Delta\| = 1.$$

$$c_{0,\Delta}^2 = \frac{f^2(\Delta)}{f^2(-I)} = \frac{n}{(n+r_0^2)n} = \frac{1}{n+r_0^2},$$

It now follows that (24) holds. □

From Theorem 6.1 we get the following condition on $r(p)$:

$$r\left(\frac{1}{n}\right) = \frac{\sqrt{1-nc_{0,\Delta}^2}}{c_{0,\Delta}} \tag{26}$$

7. Some Classes of Axial Functions

It follows from (12) that the function $r(p)$ is a parameter of a cone which determines an individual attitude to risk. It also determines how the attitude to risk changes depending on the changes of the probability measure.

We have shown that the function satisfies the conditions (16), (18), (26). Is is also reasonable to assume that $r(p)$ is monotonically decreasing on $[0,1]$.

One of the ways to define an individual preference by an elliptic acceptance cone is to take $r(p)$ from some class of one-parameter functions (that satisfy (16), (18), (26)) and evaluate the parameter according to the previous decisions of the individual.

8. An Example of an Axial Function

Assume that $r_m(p) = \dfrac{\sqrt{n+n^3}}{p^m}$, $m \geqslant 1$ is an axial function. It obviously satisfies (16), (18), (26). Consider two representatives of this class: $r_1(p) = \dfrac{\sqrt{n+n^3}}{p}$, $r_2(p) = \dfrac{\sqrt{n+n^3}}{p^2}$. As $r_1(p) \leqslant r_2(p)\ \forall p \in [0,1]$ we get that an individuum whose preferences are determined by the function $r_1(p)$ is more cautious than an individuum whose preferences are determined by $r_2(p)$.

On Figure 1 we see acceptance cones for the case of two-dimensional space and $P = (\frac{1}{2}, \frac{1}{2})$. A_1 is an elliptic acceptance cone corresponded to r_1, and A_2 corresponds to r_2. Note that $A_1 \subset A_2$.

Conclusion

The generalized coherent risk measures afford the opportunity to value risk according to individual preferences. The elliptic acceptance cone introduced in the paper is an instrument for constructing such measures. As the parameter of the elliptic cone determines an individual attitude to risk it should be studied more detail.

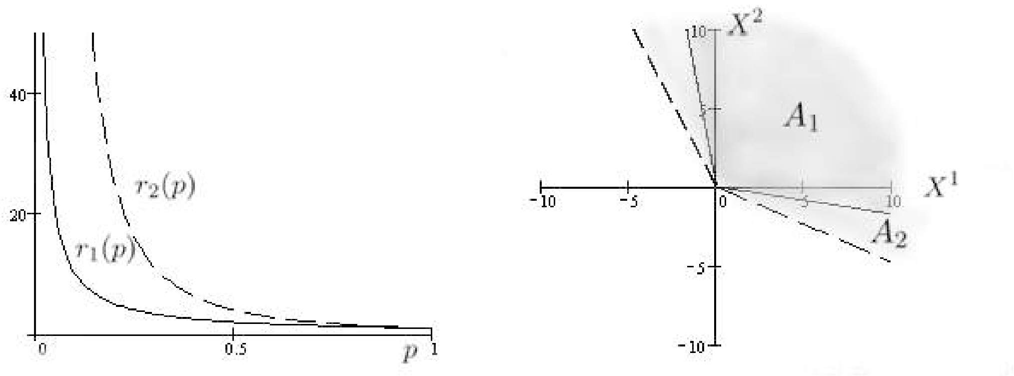

Figure 1. Axial functions $r_1(p)$ and $r_2(p)$ and acceptance cones corresponded to them

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Описание предпочтений на множестве рисков с помощью обобщенных когерентных мер риска

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В работе рассматривается модель обобщенных когерентных мер риска. В рамках этой модели изучаются свойства множеств приемлемых рисков. Вводится понятие эллиптического конуса приемлемых рисков. Рассматриваются его свойства, в частности взаимосвязь между формой конуса и величиной неприятия риска.

Ключевые слова: отношение предпочтения, стохастическое доминирование, мера риска, неприятие риска, обобщенные когерентные меры риска, множество приемлемых рисков, эллиптический конус.