Three Families of Functions of Complexity One

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Three rare families of functions of analytic complexity one were studied. Main results are the description of linear differential equations with solutions of complexity one (Theorem 2), the description of \( L \)-pairs of complexity one (Theorem 5), the description of \( O(2) \)-simple functions (Theorem 7).

Keywords: rare family, analytic complexity.

Introduction

The complexity of analytic functions of several variables has been studied in [1–5]. A method of measuring the complexity of an analytic function in two variables, possibly multivalued, is proposed in [3]. For any analytic function of two variables \( z(x, y) \) one can define its complexity \( N(z) \). It attains values 0, 1, \ldots, \infty and is preserved under any analytic continuation. Functions of one variable have complexity \( N(z) = 0 \). Complexity one have functions \( z(x, y) \) of two variables if they have the form \( z = c(a(x) + b(y)) \), where \( a, b, c \) are nonconstant functions of one variable, and so on. In other words, for a function \( z \) of two variables we write \( N(z) = n \) if \( z \) can be represented in the form \( C(A(x, y) + B(x, y)) \), where \( C \) is a function of one variable, and the complexity of \( A \) and \( B \) is less than \( n \), and there is no such representation with a smaller value of \( n \). This produces an increasing system of classes of functions

\[ C_{l_0} \subset C_{l_1} \subset C_{l_2} \ldots \]

If a function does not belong to any of these classes we write \( N(z) = \infty \). Each of the above classes is defined by differential-algebraic relations. For example, \( C_{l_0} \) is defined by the condition \( z'_{x}z'_{y} = 0 \), and \( C_{l_1} \) by the condition

\[ \delta(z) = z'_{x}z'_{y}(z''_{x}z'_{y} + z''_{x}z'_{x}) + z'_{y}[(z'_{x})^2z''_{y} - (z'_{y})^2z''_{x}] = 0. \]

The differential polynomial \( \delta(z) \) is the numerator of the expression \( \ln(z'_{y}/z'_{x})z''_{xy} \).

1. Linear equations with constant coefficients

Consider the pair of functions \( (z_1 = e^{ax+by}, z_2 = e^{px+qy}) \). If \( ab = pq = 0 \) then we have \( \max(N(z_1), N(z_2)) = 0 \). If it is not so, then \( \max(N(z_1), N(z_2)) = 1 \). What condition on \( (a, b, p, q) \) provides that the complexity of all linear combinations of \( z_1 \) and \( z_2 \) does not exceed one? The answer gives
Lemma 1. Let \((ab, pq) \neq 0\). The complexity of all linear combinations of \(z_1\) and \(z_2\) does not exceed 1 only in three cases: (1) \(p = a\), (2) \(q = b\), (3) \(aq = bp\).

Proof. The condition (1) for \(z = t_1z_1 + t_2z_2\) has the form
\[
(b - q)(a - p)(qa - bp)\left((e^{ax+by})^2 abt_2^2 - (e^{px+qy})^2 pqt_1^2\right)e^{ax+by}e^{px+qy} t_1 t_2 = 0.
\]
So the lemma is proved.

There is a curious corollary from this lemma. Consider a homogeneous linear equation with constant coefficients \(P(D)(z(x, y)) = 0\) and let \(\mathcal{L}\) be the space of its analytic solutions. The complexity \(N(\mathcal{L})\) of the space of solutions \(\mathcal{L}\) is the maximum (finite or infinite) of the solutions’ complexities.

Theorem 2. If \(N(\mathcal{L}) < 1\), then the equation \(P(D)(z(x, y)) = 0\) has one of the forms:
1. \(z_x - Az = 0\), solutions have the form \(z = e^{Ax} b(y)\),
2. \(z_y - Bz = 0\), solutions have the form \(z = e^{By} a(x)\),
3. \(kz_x^2 + t_2z_2 = 0\), solutions have the form \(z = c(x - ky)\),
4. \(z_{xy} = 0\), solutions have the form \(z = a(x) + b(y)\).

Proof. Let \(\chi = \{P(\lambda_1, \lambda_2) = 0\}\) be the characteristic set of this equation and let \((z_1 = e^{ax+by}, z_2 = e^{px+qy})\) be two solutions, i.e. \((a,b), (p,q) \in \chi\). It follows from Lemma 1 that \(\chi\) belongs to a vertical line (case (1)) or to a horizontal line (case (2)), or to a line passing through the origin (case (3)). There is another case (case (4)) outside Lemma 1. In this case \(\chi\) is the coordinate cross and \(N(z_1) = N(z_2) = 0\). The characteristic polynomials have one of the forms: in case (1) \(P(\lambda_1, \lambda_2) = (\lambda_1 - A)^n\), in case (2) \(P(\lambda_1, \lambda_2) = (\lambda_2 - B)^n\), in case (3) \(P(\lambda_1, \lambda_2) = (k\lambda_1 + l\lambda_2)^n\), in case (4) \(P(\lambda_1, \lambda_2) = (\lambda_1\lambda_2)^n\). In all cases it is not difficult to solve these differential equations. The condition \(N(\mathcal{L}) \leq 1\) is true only for \(n_1 = n_2 = n_3 = n_4 = 1\). The theorem is proved.

Note that if the multiplicities \((n_1, n_2 n_3, n_4)\) are arbitrary, then the complexities of the space of solutions are finite but greater than one.

2. L-pairs

A collection of functions forms a linear space if this collection is closed under addition and multiplication by a constant (complex numbers). Multiplication by a nonzero constant does not change the complexity of a function: \(N(\lambda z(x, y)) = N(z(x, y))\). This means that a nonzero function of complexity 1 generates a linear space lying in \(Cl_1\). As for a sum of two functions, if \(N(z_1(x, y))\) and \(N(z_2(x, y))\) do not exceed \(n\) then \(N(z_1(x, y) + z_2(x, y)) \leq (n + 1)\). It can be shown that in ‘general position’ this inequality becomes the equality. There is a simple example: \(N(xy) = 1\), \(N(x^2) = 0\), \(N(xy + x^2) = 2\). But there exist exceptional pairs. For example \(N(xy) = 1\), \(N(x + y) = 1\) and \(N(t_1(xy) + t_2(x + y)) = 1\) for any \((t_1, t_2)\).

Definition. We call a pair of functions \((z_1(x, y), z_2(x, y))\) an L-pair of complexity \(n\) if
\[
N(t_1(z_1(x, y) + t_2z_2(x, y)) \leq \max(N(z_1), N(z_2)) = n \text{ for any } (t_1, t_2).
\]

Here we assume that \(z_1\) and \(z_2\) have analytic germs at the same point. Lemma 1 then becomes a classification of L-pairs of a special form.

Let us formulate several obvious statements.

Statement 3. Two functions \((z_1, z_2)\) is an L-pair of complexity zero if and only if they are functions of the same argument \(x\) or \(y\).
**Statement 4.** The property of being an L-pair is invariant under the action of
(1) the pseudo-group of transformations \(\{(x \to p(x), \ y \to q(y))\}\),
(2) the change \(\{(x \to y, \ y \to x)\}\),
(3) the affine group of transformations of \((z_1, z_2)\)-plane.

The pseudo-group generated by the transformations (1), (2) and (3) we denote by \(\mathcal{G}\). The description of \(L\)-pairs is natural to give up to the \(\mathcal{G}\)-action.

Now let us turn back to Lemma 1. If we assume only that \(N(z_1 + z_2) \leq 1\), we have the same description. Indeed, the condition (1) for \(z = z_1 + z_2\) has the form

\[
(b - q)(a - p)(qa - bp) \left( (e^{ax+by})^2 ab - (e^{ax+qy})^2 pq \right)e^{ax+by}e^{px+qy} = 0,
\]

and it is enough to reach the conclusion of Lemma 1. Taking this into account we modify the definition.

**Definition.** We call a pair \((z_1(x, y), z_2(x, y))\) a pair of complexity \(n\), if \(N(z_1(x, y) + z_2(x, y)) \leq \max(N(z_1), N(z_2)) = n\).

We can strengthen Lemma 1 as follows.

**Lemma 1.** Let \((ab, pq) \neq 0\). The pair \((z_1 = e^{ax+by}, z_2 = e^{px+qy})\) is a pair of complexity one only in three cases (1) \(p = a, (2) q = b, (3) \text{aq = bp}\).

Now we turn to the construction of an arbitrary \(L\)-pair of complexity one. Their description is given in the form of a list of cases that are specified and denoted in the course of exposition.

Let \(z_1\) and \(z_2\) be two functions of complexity not exceeding 1, that is \(z_1 = c_1(a_1(x) + b_1(y)), z_2 = c_2(a_2(x) + b_2(y))\). Assume also that \(\max(N(z_1), N(z_2)) = 1\), i.e one of the functions has complexity one, let it be \(z_2\). Then \(a_2, b_2,\) and \(r\) are non constant and locally invertible at a general point. Replace \(x\) by \(a_2^{-1}(x)\) and \(y\) by \(b_2^{-1}(y)\). The condition takes the form

\[
c(a(x) + b(y)) + t \cdot r(x + y) \in Cl\ 1\ \forall t, \ r' \neq 0.
\]

Let the first term have complexity zero, this is Case (01). Then the first term is a function of one variable, denote it by \(a(x)\). From (1) for \(a(x) + t \cdot r(x + y)\) we get

\[
a_1r_1r_3 = 2a_1r_2^2 - a_2r_1r_2, \quad r_1r_3 = r_2^2.
\]

By lower indices we denote orders of derivatives. If \(r_2 = 0\) then \(r(x + y) = k \cdot (x + y) + l\) and \(a(x)\) is arbitrary. This is Case (01.1). This pair is equivalent to \((a(x), (x + y))\).

If \(r_2\) is not zero then from the second equation we have \(r(t) = \rho \cdot e^{mt} + \bar{\rho}\). And from the first equation we have \(a(x) = \alpha \cdot e^{mt} + \bar{\alpha}\). This pair is equivalent to \((kr, xy)\). We call this Case (01.2)

Consider now Case (11) when both terms have complexity one. This means that \(a', b', c', r'\) are nonconstant functions. From (1) for \(c(a(x) + b(y)) + t \cdot r(x + y)\) we get

\[
a_1^2b_1c_1r_1^2 - a_1b_1^2c_3r_1^2 - a_1^2c_2r_1r_2 - a_1b_2c_2r_1^2 + a_2b_1c_2r_1^2 + \\
b_1^2c_2r_1r_2 - a_1c_1r_1r_3 + 2a_1c_1r_2^2 - a_2c_1r_1r_2 - b_1c_1r_1r_3 - 2b_1c_1r_2^2 + b_2c_1r_1r_2 = 0,
\]

\[
-a_1^3b_1c_1c_3r_1^2 - a_1^3b_1c_2c_1r_1^2 - a_1b_1^3c_3c_1r_1^2 + a_1b_1^3c_2c_2r_1^2 - 2a_1^2b_2c_1c_2r_1^2 + 2a_2b_1^3c_2c_1r_1^2 - \\
-a_1^2c_1^2r_1r_3 + a_1^2c_1^2r_2^2 + 2a_1b_1c_2r_1r_2 - 2a_2b_1c_1^2r_1r_2 + b_1^2c_1r_1r_3 - b_1^2c_1^2r_2^2 = 0,
\]

\[
a_1^3b_1^2c_1c_3r_1 - 2a_1^3b_1^2c_2c_1r_1 - a_1^2b_1^3c_3c_1r_1 + 2a_1^2b_1^3c_2c_2r_1 + a_1^3b_1c_1c_2r_2 - a_1^3b_2c_1c_2r_1 - \\
-a_1b_1^3c_1c_2r_2 + a_2b_1^3c_1c_2r_1 - a_1b_1c_1^2r_3 + a_1b_2c_1^2r_2 + a_1b_1^3c_1^2r_3 - a_2b_1c_1^2r_2 = 0.
\]
Eliminating $c_3$ from the first and second equations and then from the first and third equations, we get two equations. Each of them is a quadratic form in $(c_1, c_2)$ with a common factor $a_1 b_1 r_1 (a_1 - b_1)^2$. In our case this factor can be equal to zero only if $a_1 - b_1 = 0$ (Case (11.1)). This pair has the form $(e(x + y), r(x + y))$.

Assume now $a_1 - b_1 \neq 0$. After dividing by the common factor we get

\[
\begin{align*}
&\quad a_1^2 b_1 c_2^2 r_1^2 + a_1 b_1 c_2^2 r_1^2 - a_1^2 c_1 c_2 r_1 r_2 - 2 a_1 b_1 c_1 c_2 r_1 r_2 + a_2 b_1 c_1 c_2 r_1^2 + \\
&+ a_2 b_1 c_1 c_2 r_1^2 - b_1^2 c_1 c_2 r_1 r_2 + a_1^2 c_2^2 r_1^2 - 2 a_1 c_1 c_2 r_1^2 = 0, \\
&\quad 2a_1^2 b_1 c_2^2 r_1^2 - 2a_1 b_1 c_1 c_2 r_1 r_2 + a_1^2 b_2 c_1 c_2 r_1^2 - 2a_1 b_1 c_1 c_2 r_1 r_2 + \\
&+ a_2 b_1^2 c_1 c_2 r_1^2 + 4a_1 b_1 c_1 c_2 r_1 r_2 - 2a_1 b_2 c_1^2 r_1 r_2 - 2a_2 b_1 c_1^2 r_1 r_2 = 0.
\end{align*}
\]

After elimination of $c_2/c_1$ we have

\[
(a_1 - b_1)^3 a_1 b_1 r_1^6 a_2 b_2 r_2 (a_1^2 b_1 r_2 - a_1^2 b_2 r_1 - a_1 b_1^2 r_2 + a_2 b_1^2 r_1) = 0.
\]

Consider all the possibilities separately.

**Case (11.2).** One of the functions $a'' = 0$ and $b'' = 0$ is linear, let it be $b$, then $b(y) = k \cdot y + l$, where $k \neq 0$. Replace $k \cdot y + l$ by $y$ and $k \cdot x + l$ by $x$, then $r(t)$ becomes $r(t/k)$. The condition (1) for $c(a(x)) + t \cdot r(x + y)$ takes the form

\[
\begin{align*}
&\quad a_1^3 c_1 c_2 r_1^2 + a_1^3 c_1 c_3 r_1 - 2 a_1^3 c_2^2 r_1 - a_1^2 c_1^2 c_2 r_3 - a_1^2 c_1 c_3 r_1 + \\
&+ 2 a_1 c_2^2 r_1 + a_1 c_1^2 c_2 r_3 - a_1 c_1 c_2 r_2 - 2 a_1 c_1^2 r_2 + a_2 c_1 c_2 r_1 = 0, \\
&\quad a_1^3 c_1 c_2 r_1^2 - a_1^3 c_2^2 r_1^2 - a_1^2 c_1^2 c_2 r_3 + a_1^2 c_1 c_3 r_1 + \\
&+ a_1 c_2^2 r_1^2 - 2 a_1 c_2 r_1 r_2 + 2 a_2 c_1 c_2 r_1 + c_1^2 r_1 r_3 - c_1^2 r_2^2 = 0, \\
&\quad -a_1^2 c_2 r_1 r_2 + a_1^2 c_3 r_1^2 - a_1 c_1 r_1 r_3 + 2 a_1 c_1 r_2^2 - 2 a_1 r_1 r_2 - \\
&- a_2 c_1 r_1 r_2 + a_2 c_2 r_1^2 + c_1 r_1 r_3 - 2 c_1 r_2^2 + c_2 r_1 r_3 = 0.
\end{align*}
\]

The expressions for $c_3$ from each of these equations are fractions with the denominators

\[
a_1^2 c_1 r_1 (a_1 - 1), \quad a_1 c_1 r_1^2 (a_1^2 - 1), \quad a_1 r_1^2 (a_1 - 1).
\]

There are two possibilities for vanishing of one of the denominators: $a_1 = 1$ or $a_1 = -1$. If $a_1 \neq b_1$, hence we have only the second possibility $a_1 = -1$, $a(x) = -x + \alpha$. The condition (1) yields

\[
\begin{align*}
&\quad -c_1^2 r_3 - c_1 c_3 r_1 + 2 c_2^2 r_1 = 0, \\
&\quad c_1 r_1 r_3 - 2 c_1 r_2^2 + c_3 r_1^2 = 0,
\end{align*}
\]

where $c$ and $r$ are functions of two independent variables $x - y$ and $x + y$. Separating the variables and solving the differential equations we arrive at Case (11.2.1) : $c(-x + y) = e^{m(-x+y)} + \tilde{c}$, $r(x + y) = e^{\pm m(x+y)} + \tilde{b}$. The pair then has the form $(y/x, xy)$.

If $a_1 \neq \pm 1$, we can eliminate $c_3$ from (5) to get two quadratic form in $(c_1, c_2)$:

\[
\begin{align*}
&\quad (c_2 r_1 - c_1 r_2) (c_2 a_1^2 r_1 + c_2 a_1^2 r_2 - c_1 a_1^2 r_2 - c_1 a_1 r_2 + a_2 r_1) = 0, \\
&\quad (c_2 r_1 - c_1 r_2) (2 c_2 a_1^2 r_1 - 2 c_1 a_1 r_2 + c_2 a_2 r_1) = 0
\end{align*}
\]

with the common factor $(c_2 r_1 - c_2 r_1)$. If this factor is equal to zero (Case (11.2.2)), then we can separate the variables and, taking into account that the Jacobian of the change $(t = a(x) + y; s = x + y)$ does not vanish, we see that both logarithmic derivatives are equal to the
same constant \( m \). From this we get 
\[ z_1 = \gamma e^{m(a(x)+y)} + \gamma, \quad z_2 = \rho e^{m(x+y)} + \rho. \]

The pair has the form \((a(x)y, xy)\).

Otherwise, (Case (11.2.3)), dividing out the common factor and eliminating \( c_2/c_1 \) from two linear forms, we get \( a_1^2 a_2 r_1^2 (a_1 - 1) = 0 \). It vanishes only if \( a_2 = 0 \), \( a_1 = 1 \) is then the constant \( A \). In this case \( Ac_2/c_1 = r_2/r_1 \), and 
\[ z_1 = c(Ax + y) = \gamma e^{\frac{1}{A}(Ax+y)}, \quad z_2 = r(x+y) = \rho e^{m(x+y)}. \]

The pair has the form \((x^by, xy)\).

We see that Cases (11.2.1) and (11.2.3) are subcases of Case (11.2.2). Thus, in Case (11.2) the pair has the form \((a(x)y, xy)\).

In Case (11.3) \( r_2 = 0 \), i.e. \( r(x+y) = \rho(x+y) + \tilde{\rho} \), where \( \rho \neq 0 \). By replacing \( x \) with \( \rho x + \tilde{\rho} \) and \( y \) with \( py \) we obtain \( r(x+y) = x+y \). The condition (1) for \( c(a(x) + b(y)) + (x+y) \) has the form
\[ a_1^3 b_1^2 c_1 c_3 - 2 a_1^3 b_1^2 c_2^2 - a_1^2 b_1^3 c_1 c_3 + 2 a_1^2 b_1^3 c_2^2 - a_1^2 b_1 c_1 c_2 + 2 b_1^3 c_1 c_2 = 0, \]
\[ a_1^3 b_1 c_1 c_3 - a_1^3 b_1 c_2 - a_1^2 b_1^2 c_1 c_3 + a_1 b_1^3 c_2^2 - 2 a_1 b_2 c_1 c_2 + 2 a_1 b_2 c_1 c_2 = 0, \]
\[ a_1^2 b_1 c_3 - a_1 b_1^2 c_3 - a_1 b_2 c_2 + a_2 b_1 c_2 = 0. \]

By eliminating \( c_3 \) and \( c_2/c_1 \), we get
\[ (a_1 - b_1) (a_1^2 b_2 - a_2 b_1^2) = 0. \]

It may vanish only because of the second factor, therefore, separating the variables we get \( a_2/a_1^2 = b_2/b_1^2 = -m \) where \( m \) is a constant. Then
\[ a(x) + b(y) = \frac{1}{m} (\ln(mx + \alpha) + \ln(my + \beta) + \ln(n)), \]
and three equations for \( c(t) \) are
\[ c_3 = mc_2^2, \quad c_3 c_1 = c_2^2, \quad mc_1 c_2 + c_1 c_3 - 2 c_2^2 = 0. \]

Consequently, \( c(t) = \gamma e^{mt} + \tilde{\gamma} \), and the pair has the form \((xy, x+y)\).

**Case (11.4)**
\[ a_1^2 b_1 r_2 - a_1^2 b_2 r_1 - a_1 b_1^2 r_2 + a_2 b_1^2 r_1 = 0. \]  
(6)

From this we get
\[ \frac{r_2}{r_1} = \frac{a_1^2 b_2 - a_2 b_1^2}{a_1 b_1 (a_1 - b_1)} \]  
(7)

(the denominator is not zero). The condition that \( \frac{r_2}{r_1} \) is a function of \( x+y \), namely the equality of its derivatives with respect to \( x \) and \( y \), is
\[ -a_1^4 b_1 b_3 + a_1^3 b_2^2 + a_1^3 b_2 b_1^2 - 2 a_1^3 b_2 b_2^2 - a_1^2 a_3 b_1^3 + 2 a_1 a_2 b_1^3 + a_1 a_3 b_1^4 - a_2 b_1^4 = 0 \]  
(8)

We can decrease the order of equation (8) twice. First, putting \( a_1 = a'(x) = A, \ b_1 = b'(y) = B \). Second, introducing \( a_2 = a''(x) = P(A), \ b_2 = b''(y) = Q(B) \). In this notation we have \( a_3 = a'''(x) = P'(A) P(A), \ b_3 = b'''(y) = Q'(B) Q(B) \) and we can write (8) as
\[-A^4 B \left( \frac{d}{dB} G(B) \right) G(B) + A^4 (G(B))^2 + A^3 B^2 \left( \frac{d}{dB} G(B) \right) G(B) - 2A^3 B (G(B))^2 - A^2 \left( \frac{d}{dA} F(A) \right) F(A) B^3 + 2 A (F(A))^2 B^3 + A \left( \frac{d}{dA} F(A) \right) F(A) B^4 - (F(A))^2 B^4 = 0. \]

\[ -420 - \]
After the substitution \( f(A) = \sqrt{F(A)} \), \( g(B) = \sqrt{G(B)} \) we previous equation becomes linear

\[
-A^4 B \frac{d}{dB} g(B) + 2 A^4 g(B) + A^3 B^2 \frac{d}{dB} g(B) - 4 A^3 B g(B) - A^2 B^3 \frac{d}{dA} f(A) + 4 A f(A) B^3 + AB^4 \frac{d}{dA} f(A) - 2 f(A) B^4 = 0.
\]

From this we find \( \frac{d}{dB} g(B) \) and write the condition of its independence from \( A \):

\[
-A^4 B^2 \frac{d^2}{dA^2} f(A) + 2 A^3 B^3 \frac{d^2}{dA^2} f(A) - A^2 B^4 \frac{d^2}{dA^2} f(A) + 6 A^3 B^2 \frac{d}{dA} f(A) - 10 A^2 B^3 \frac{d}{dA} f(A) + 4 AB^4 \frac{d}{dA} f(A) + 2 A^4 g(B) - 12 A^2 B^2 f(A) + 16 Af(A) B^3 - 6 f(A) B^4 = 0.
\]

Now we express \( g(B) \) and write the condition of its independence from \( A \):

\[
A^3 \frac{d^3}{dA^3} f(A) - 6 A^2 \frac{d^2}{dA^2} f(A) + 18 A \frac{d}{dA} f(A) - 24 f(A) = 0.
\]

By looking for solutions of the form \( f(A) = A^m \), we get the equation

\[
m(m-1)(m-2) - 6m(m-1) + 18m - 24 = (m-2)(m-3)(m-4).
\]

Hence, a general solution to (9) is \( f(A) = l_1 A^4 + m_1 A^3 + n_1 A^2 \). By eliminating \( f(A) \) from (9), we obtain \( g(B) = l_2 B^4 + m_2 B^3 + n_2 B^2 \). Substituting these \( f(A) \) and \( g(B) \) in (9), we get \( l_1 = l_2 \), \( m_1 = m_2 \), \( n_1 = n_2 \). Finally, \( f(A) = l A^4 + m A^3 + n A^2 \), \( g(B) = l B^4 + m B^3 + n B^2 \). We see that \( \alpha(x) = a'(x) \) and \( \beta(y) = b'(y) \) satisfy the same differential equation

\[
\frac{d\alpha}{dx} = \sqrt{l} \alpha^4 + ma^3 + na^2, \quad \frac{d\beta}{dy} = \sqrt{l} \beta^4 + mb^3 + nb^2.
\]

Since \( a \) and \( b \) are not linear, we may assume that the constants \( l, m, \) and \( n \) are not zeros simultaneously. Thus, if \( l = n = 0 \) and \( m \neq 0 \) (Case (11.4.1)), then

\[
\int \frac{dt}{t \sqrt{mt}} = -\frac{2}{\sqrt{l}}.
\]

Therefore

\[
a'(x) = \alpha(x) = \frac{4}{m(x+C)^2}, \quad a(x) = -\frac{4}{m(x+C)} + \hat{C}, \quad a''(x) = \frac{-8}{m(x+C)^3}.
\]

Analogously,

\[
b'(x) = \beta(x) = \frac{4}{m(y+D)^2}, \quad b(y) = -\frac{4}{m(y+D)} + \hat{D}, \quad b''(y) = \frac{-8}{m(y+D)^3}.
\]

Now, from (7) we get

\[
\frac{r_2}{r_1} = \frac{a_1^2 b_2 - a_2 b_1^2}{a_1 b_1 (a_1 - b_1)},
\]

and then we have \( r(t) = -\frac{\rho}{t+C+D} \). Computing \( c_2/c_1 \) from any of (4) and substituting the expression for \( r_2/r_1 \), we get

\[
\frac{c_2}{c_1} = \frac{a_1 b_2 - a_2 b_1}{a_1 b_1 (a_1 - b_1)}
\]

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and
\[ c(a(x) + b(y)) = \frac{1}{x + c} + \frac{1}{y + c}. \]

Thus, the pair has the form
\[ (z_1 = \frac{xy}{x + y}, \quad z_2 = \frac{1}{x + y}). \]

If \( l \neq 0 \) or \( n \neq 0 \) (Case (11.4.2)), then
\[ \int \frac{dt}{t\sqrt{lt^2 + mt + n}} = -\frac{2}{\sqrt{lt^2 + n}} \arctgh \left( \frac{\sqrt{lt^2 + mt + n}}{\sqrt{lt^2 + n}} \right), \]
and we get \( a'(x) = \alpha(x) \) and \( b'(y) = \beta(y) \) as inversion of the integrals, and \( a(x) \) and \( b(y) \) by one more integration. As in the previous case, from (7) we get \( r(t) \) and \( c(t) \) from any relation of (4).

Finally, we have the theorem.

**Theorem 5.** Let \( z_1(x, y) \) and \( z_2(x, y) \) is an \( L \)-pair of complexity one, then this pair up \( G \)-action has the form

For \( N(z_1) = 0 \), \( N(z_2) = 1 \)

(01.1) \( z_1 = a(x), \quad z_2 = x + y, \quad a \) is arbitrary,

(01.2) \( z_1 = x, \quad z_2 = xy, \)

For \( N(z_1) = N(z_2) = 1 \)

(11.1) \( z_1 = c(x + y), \quad z_2 = r(x + y), \) where \( c \) and \( r \) are arbitrary,

(11.2) \( z_1 = a(x)y, \quad z_2 = xy, \quad a \) is arbitrary,

(11.3) \( z_1 = xy, \quad z_2 = x + y, \)

(11.4.1) \( z_1 = \frac{xy}{x + y}, \quad z_2 = \frac{1}{x + y}, \)

(11.4.2) In this case there are no explicit expressions for the pair \( z_1 = c(a(x) + b(y)), \quad z_2 = r(x + y) \). The four functions \( (a, b, c, r) \) are constructed as described above. In particular, they can be expressed by quadratures.

As shown above, all pairs in this list are \( L \)-pairs. In Cases (01.1), (01.2), (11.1), (11.2) it is obvious. In Case (11.3) we can also see it easily: \( z = xy + tx(x + y) = (x + t)(y + t) - t^2 \). In Case (11.4.1) it is not that clear. We need to check that
\[ z = \frac{xy}{x + y} + t \frac{1}{x + y} = \frac{t + xy}{x + y} \in C1 \) for all \( t \).

After the change \( t \) by \( t^2 \) we get
\[ z = \frac{t^2 + xy}{x + y}. \]

By replacing \( x \) with \( tx \), \( y \) with \( ty \), and \( z \) with \( t/z \), we get
\[ z = \frac{x + y}{1 + xy}. \]

Now, we replace \( x \) with \( th(x) \), \( y \) with \( th(y) \), and \( z \) with \( th(z) \) and use the addition formula
\[ th(x + y) = \frac{th(x) + th(y)}{1 + th(x) \ th(y)} \]
to get \( z = x + y \). Since all the transformations here do not change complexity, this proves that the complexity of the original function is 1.

For Case (11.4.2) the author does not know a similar reasoning. The open question is what mysterious relations are behind that fact.

The set of pairs of complexity one is certainly wider than the set of \( L \)-pairs of complexity one. This is another open problem: to describe all pairs of complexity one.
3. $O(2)$-simplicity

The standard action of the $O(2)$ on the $(x, y)$-plane is

$$g_\phi = (x \rightarrow \cos(\phi)x - \sin(\phi)y, \ y \rightarrow \sin(\phi)x + \cos(\phi)y)$$

where $\phi \in \mathbb{C}$. This action induces an action on functions

$$z(x, y) \rightarrow g_\phi(z)(x, y) = z(\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y).$$

Denote $t = \tan(\phi/2)$, then we have another form for this action

$$g_t = \left( x \rightarrow \frac{1 - t^2}{1 + t^2} x - \frac{2}{1 + t^2} y, \ y \rightarrow \frac{1 - t^2}{1 + t^2} y + \frac{2}{1 + t^2} x \right).$$

If $N(z(x, y)) = n$, then $N(z(\lambda x, \lambda y)) = n$ also, therefore we can replace $g_t(x, y)$ with $h_t(x, y) = (1 + t^2)g_t(x, y)$.

If $N(z) \leq n$, then $N(g_\phi(z)) \leq n + 1$, and for arbitrary $z$ and $\phi$ there is no reason to expect that $N(g_\phi(z)) \leq n$. For example, let $z = xy$, then $N(z) = 1$. For $\delta(h_t(z))$ we have

$$4t \left( x^2 + y^2 \right) (t - 1) (t + 1) \left( t^2 + 2t - 1 \right) \left( t^2 - 2t - 1 \right) \left( t^2 + 1 \right)^4.$$ 

We see that $N(h_t(xy)) = 1$ only for 9 values of $t$, namely $t = 0, \pm 1, \pm i, \pm 1 \pm \sqrt{2}$. The corresponding functions are proportional to

$$xy, \ x^2 - y^2, \ (x \pm iy)^2.$$ 

For another values $t$ the complexity $N(h_t(xy))$ is equal to two.

**Definition.** A function $z(x, y)$ is called $O(2)$-simple if $N(g_t(z)) \leq 1$ for all $t$.

All linear functions are, of course, $O(2)$-simple. Now, we want to describe all $O(2)$-simple functions. It is clear that for such functions $N(z) \leq 1$, then $z = c(a(x) + b(y))$. If one of the functions $(a, b, c)$ is constant, then $N(z) = 0$, and $z$ depends only on one variable or a constant. Any such function is $O(2)$-simple (Case 0). Assume that $N(z) = 1$, i.e. $a, b, c$ are not constant.

**Statement 6.** (1) $z$ is $O(2)$-simple if and only if $\delta(g_t(z)) = 0$ for all $(x, y, t)$. (2) $c(a(x) + b(y))$ is $O(2)$-simple if and only if $a(x) + b(y)$ is $O(2)$-simple. (3) $z(x, y)$ is $O(2)$-simple if and only if $z(y, x)$ is $O(2)$-simple.

The proof is obvious.

Let $a(x) + b(y)$ is $O(2)$-simple, then, in particular,

$$\frac{d}{dt}\delta(g_t(a(x) + b(y)))|_{t=0} = 0, \quad (10)$$

in index notation for derivatives we have

$$-a_1^2a_2b_2 - a_1^2b_1b_3 + a_1^2b_2^2 - a_1a_3b_1^2 + a_2^2b_1^2 - a_2b_1^2b_2 = 0. \quad (11)$$

We can decrease the order of equation (11) twice. First, putting $a_1 = a'(x) = A, \ b_1 = b'(y) = B$. Second, introducing $P(A) = a_2 = a''(x), \ Q(B) = b_2 = b''(y)$. In this notation we have $a_3 = a'''(x) = P'(A)P(A), \ b_3 = b'''(y) = Q'(B)Q(B)$ and we can write (11) as

$$-QA^2P - BQ_1QA^2 + Q^2A^2 - B^2AP_1P + B^2P^2 - B^2QP = 0 \quad (12)$$
By differentiating (12) with respect to $A$, we get
\[-2 QAP - QA^2 P_1 - 2 ABQ_1 Q + 2 Q^2 A + B^2 P_1 P - B^2 AP_1 P - B^2 AP_2 P - B^2 QP = 0. \tag{13}\]

The relations (12) and (13) are a system of linear equations in $Q(B)$ and $Q'(B)$, its determinant is equal to
\[-B PA \left( A^3 P_1 + B^2 AP_1 - 2 B^2 P \right).\]

This determinant is identically equal to zero only if $P(A) = 0 \text{ (Case 1)}$. The solution to the system for $Q(B)$ is
\[Q(B) = -\frac{B^2 (A^2 P_2 P + A^2 P_1^2 - 3 AP P_1 + 2 P^2)}{A^3 P_1 + B^2 AP_1 - 2 B^2 P}.\]

The condition of independence $Q$ from $A$ is
\[-A^3 P_3 PP_1 + A^3 P_2^2 P - 2 A^3 P_2 P_1^2 - AB^2 P_3 PP_1 + + A^2 P_2 P_1^2 + 2 A^2 P_2 P_1 + 4 A^2 P_1^3 + + 2 B^2 P_3 P^2 + 3 B^2 P_2 PP_1 + 2 AP_2 P^2 - 10 AP P_1^2 + 6 P^2 P_1 = 0, \tag{14}\]

which splits into two relations: terms free of $B$ and terms with the factor $B^2$. Eliminating $P'''(A)$ from them, we get
\[P \left( AP_1 - 2 P \right) \left( AP_2 P - 2 AP_1^2 + 3 P_1 P \right) \left( A^2 PP_2 + A^2 P_1^2 - 3 AP P_1 + 2 P^2 \right) = 0.\]

The case $P = 0$ (Case 1) has been considered above. Now we turn to the remaining cases.

\[\begin{align*}
(AP_1 - 2 P) &= 0 \text{ (Case 2)}, \\
(AP_2 P - 2 AP_1^2 + 3 P_1 P) &= 0 \text{ (Case 3)}, \\
(A^2 PP_2 + A^2 P_1^2 - 3 AP P_1 + 2 P^2) &= 0 \text{ (Case 4)}. 
\end{align*}\]

The solutions to the corresponding differential equations are
\[P(A) = 0 \text{ (Case 1)},\]
\[P(A) = CA^2 \text{ (Case 2)},\]
\[P(A) = \frac{A^2}{A^2 C_1 + C_2} \text{ (Case 3)},\]
\[P(A) = A \sqrt{C_1 \ln(A) + C_2} \text{ (Case 4)}.\]

To find $Q(B)$ corresponding to $P(A)$, we substitute these solutions in (13).

In Case 1 $P(A) = 0$, $Q(B) = CB$.

In Case 2 $P(A) = CA^2$, $Q(B) = -CB^2$.

In Case 3 $P(A) = A^2/cA^2 + d$ and for $Q(B)$ we have
\[-A^6 BQQ_1 c^3 + A^6 Q^2 c^3 - 3 A^4 BQQ_1 c^2 d - 6 A^6 Q^2 c^2 + 3 A^4 QQ_1 c^2 d - 3 A^2 BQQ_1 cd^2 + B^2 A^4 c - 2 A^4 Qc - 3 A^4 QQ_1 c^2 d - BQQ_1 d^2 - A^2 B^2 d - A^2 Qd^2 - B^2 Qd^2 + Q^2 d^2 = 0, \tag{15}\]

which is a polynomial in $A^2$ and splits into four differential equations of first order on $Q(B)$ (the coefficients at 1, $A^2$, $A^4$, $A^6$). These equations yield $d = 0$, and $P(A) = Q(B) = C = \text{const}$.

In Case 4 we have $P(A) = A \sqrt{c \ln(A) + d}$ and
\[\begin{align*}
-2 QA^2 \sqrt{c \ln(A) + d} - B^2 Ac - 2 ABQQ_1 - 2 B^2 Q \sqrt{c \ln(A) + d} + 2 AQ^2 &= 0. 
\end{align*}\]
The functions

\[ \sqrt{c \ln (A) + d}, \quad A, \quad A^2 \sqrt{c \ln (A) + d} \]

are linearly independent, hence \( Q(B) = 0 \) and \( c = 0 \). So the answer in Case 4 coincides with the answer in Case 1 after replacing \( A \rightarrow B \).

Now we can return to equations in \( a(x) \) and \( b(y) \) and find the answers:

In Case 1: \( P(A) = 0 \) means \( a''(x) = 0 \) and \( a(x) = \alpha_1 x + \alpha_0 \), then \( Q(B) = CB \) means \( b''(y) = Cb'(y) \) and \( b(y) = \beta_1 e^{Cy} + \beta_0 \). Then we write the \( O(2) \)-simplicity condition \( \delta(y(z)) = 0 \) for \( a + b \) and see that it holds only for \( \alpha_1 \beta_1 = 0 \). The same goes in Case 4.

In Case 2: \( P(A) = CA^2 \) means \( a''(x) = C(a'(x))^2 \) and \( a(x) = -\ln(\alpha_1 x + \alpha_0)/C \), then from \( Q(B) = CB^2 \) we get \( b(y) = \ln(\beta_1 y + \beta_0)/C \). Since

\[ a(x) + b(y) = \frac{1}{C} \ln \left( \frac{\beta_1 y + \beta_0}{\alpha_1 x + \alpha_0} \right), \]

it is enough to check the \( O(2) \)-simplicity condition only for

\[ z = \frac{\beta_1 y + \beta_0}{\alpha_1 x + \alpha_0}. \]

It is easy to see that the condition \( \delta(y(z)) = 0 \) holds.

In Case 3: \( P(A) = C \) means \( a''(x) = C \) and \( a(x) = Cx^2 + \alpha_1 x + \alpha_0 \), then from \( Q(B) = C \) we get \( b(y) = Cy^2 + \beta_1 y + \beta_0 \). We see that the \( O(2) \)-simplicity condition for \( a + b \) holds.

Thus, we have the theorem.

**Theorem 7.** The complete list of \( O(2) \)-simple functions up to transformations \(( z(x, y) \rightarrow f(z(x, y)) \) and \(( z(x, y) \rightarrow z(y, x) \)) is

\[ z = \frac{\beta_1 y + \beta_0}{\alpha_1 x + \alpha_0}, \]

\[ z = (x^2 + y^2) + \alpha x + \beta y, \]

\[ z = \alpha x + \beta y. \]

**Corollary 8.** Any \( O(2) \)-simple function is a rational function, up to a transformation \(( z(x, y) \rightarrow f(z(x, y)) \).

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**References**


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Three Families of Functions Complexity One


Три семейства функций сложности один

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В работе описаны некоторые семейства функций двух переменных аналитической сложности единица, обладающие некоторыми редкими свойствами. Во-первых, классифицированы линейные уравнения с постоянными коэффициентами, т.ч. все их аналитические решения имеют сложность не выше единицы (теорема 2). Во-вторых, классифицированы пары аналитических функций, таких что любая их линейная комбинация имеет сложность не выше единицы (теорема 5). В-третьих, дано явное описание функций, т.ч. их орбиты под действием группы $O(2)$ состоят из функций сложности не выше единицы (теорема 7).

Ключевые слова: редкие семейства, аналитическая сложность.